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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Jeffrey Heninger **Email/Phone:** jeffrey.heninger@yahoo.com

Speaker's Name: Abed Bounemoura

Talk Title: KAM Theory for Ultra-Differentiable Hamiltonians

Date: 11 / 29 / 2018 **Time:** 2 : 00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: In the KAM theorem, the frequencies for which the invariant tori persist depend on how smooth the Hamiltonian is. Proves a new Diophantine-like condition for the torus to persist for classes of Hamiltonians that are smoother than C^∞ .

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
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- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

N1

KAM Theory for Ultra-Differentiable Hamiltonians

Abed Bounemoura   Notes by Jeffrey Heninger

$n \geq 2$, $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, $B \subseteq \mathbb{R}^n$ ball around 0 [joint with J. Féjoz]

Hamiltonian $H : \mathbb{T}^n \times B \to \mathbb{R}$ of the form:

$H(\theta, I) = h(I) + f(\theta, I)$ $\varepsilon := |f|$ what norm will be specified later

I: action, in B; θ: angle, in $\mathbb{T}^n$ $\nabla h(0) =: \omega \in \mathbb{R}^n$

assume $\nabla^2 h(0)$ non-degenerate

For $\varepsilon = 0$, $T_0 = \mathbb{T}^n \times \{0\}$ is a torus embedded in phase space invariant by $H = h$ and quasi-periodic with frequency ω.

Question: "persistence" of T_0 when ε is small enough?

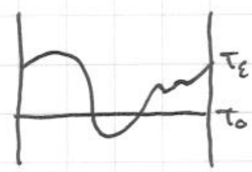
(1) Kolmogorov: Assume

(i) H is real-analytic   norm is C^k, with $k > 2\tau + 2$

(ii) ω is Diophantine: $\exists \gamma > 0$, $\tau \geq n-1$ s.t. $\forall k \in \mathbb{Z}^n \setminus \{0\}$, $|k \cdot \omega| \geq \gamma / |k|^\tau$

ω is far from rational.

Then for all ε small enough, there exists a torus T_ε, invariant by H, quasiperiodic with frequency ω such that $T_\varepsilon \xrightarrow[\varepsilon \to 0]{} T_0$



(2) Moser: Assume

(i) H is C^∞

(ii) ω is Diophantine

Then the torus persists (as before), with $T_\varepsilon \to T_0$ in C^∞

Rmk (ii) is "optimal" for (i)

(3) Rüssmann Assume:

(i) H is real-analytic

(ii) ω is BR (Bruno-Rüssmann) – the thm works for more ω than Kolmogorov

$$\int_1^\infty \frac{\ln \Psi_\omega(Q)}{Q^2} \, dQ < +\infty, \quad |k \cdot \omega| \geq \frac{1}{\Psi_\omega(|k|)}$$

Then the torus persists

Rmk (ii) is "optimal" for (i) if $n = 2$,   [Yoccoz]
but probably not for $n \geq 3$ [open]

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(4) Popov : Assume

(i) H is α-Gevrey , $\alpha \geq 1$

always smooth, $\alpha = 1 \Leftrightarrow$ real analytic , $\alpha \to \infty \Leftrightarrow C^\infty$

(ii) ω is Diophantine

Then the torus persists (in α-Gevrey space)

How can we find an α- Bruno-Rüssmann criterion?

Ultra-Differentiable Functions

$m \geq 1$, $B \subseteq \mathbb{R}^m$ ball , $F : B \to \mathbb{R}$

Fix a positive real sequence $M = (M_\ell)_{\ell \in \mathbb{N}}$. Normalize $M_0 = M_1 = 1$.
and fix $s > 0$.

F is (M, s) u-diff if

(i) F is C^∞

(ii) $\exists\, C > 0$ such that $\forall\, k \in \mathbb{N}^m$, $\forall\, x \in B$ $\qquad |k| = k_1 + k_2 + \cdots + k_m$

$$|\partial^k F(x)| \leq C\, s^{-|k|} M_{|k|}$$

Ex Take $M_\ell = \ell!$ $\Leftrightarrow$ real analytic and s is (up to a factor) the radius of convergence

$$|\partial^k F(x)| \leq C\, s^{-|k|}\, |k|! \Leftrightarrow \underbrace{\frac{1}{|k|!} |\partial^k F(x)|}_{\text{terms of Taylor series}} \leq C\, s^{-|k|}$$

Ex $M_\ell = (\ell!)^\alpha$, $\alpha \geq 1$ $\overset{\text{def}}{\Longleftrightarrow}$ α-Gevrey

Define a norm: $$|F|_s = \sup_{x \in B} \sup_{k \in \mathbb{N}^m} \frac{\partial^k F(x)\, s^{|k|}}{M_{|k|}}$$

We obtain a Banach space $U_{M,s}$

$$U_M = \bigcup_{s > 0} U_{M,s}$$

Associated to M, we have other sequences μ, N, ν

$$\mu_\ell = \frac{M_{\ell+1}}{M_\ell}, \quad N_\ell = \frac{M_\ell}{\ell!}, \quad \nu_\ell = \frac{N_{\ell+1}}{N_\ell}$$

Impose conditions on the ~~seq~~ allowed sequences:

(H1) N is log-convex, i.e. ν is non-decreasing

(H2) μ is sub-exponential , $\lim_{\ell \to +\infty} \frac{\ln(\mu_\ell)}{\ell}$

Ex. $M_\ell = (\ell!)^{1/2}$ does not satisfy (H1)

$F(x) = e^{x^2}$, not stable by composition

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$(H2) \Rightarrow \forall \quad 0 < \sigma \leq 1 \ , \quad C(\sigma) = \sup_{\ell \in \mathbb{N}} \mu_\ell e^{-\sigma \ell} < \infty$

C is like a Cauchy function

This defines $C :]0,1] \to [1, +\infty[$

continuous, decreasing close to zero, i.e. on $]0, \bar{\sigma}]$

thus C has an inverse C^*

Ex $M_\ell = (\ell!)^\alpha$, $\alpha \geq 1$, $\mu_\ell = (\ell+1)^\alpha$

(H1), (H2) satisfied, $C(\sigma) \sim \frac{1}{\sigma^\alpha}$, $C^*(y) \sim \frac{1}{y^{1/\alpha}}$

Arithmetic Condition $\quad \omega \in \mathbb{R}^n$

$$\Psi_\omega(Q) = \sup \{ |k \cdot \omega|^{-1} \mid k \in \mathbb{Z}^n, \ 0 < |k| \leq Q \}$$

$$\Delta_\omega(Q) = Q \, \Psi_\omega(Q)$$

$$\Delta_\omega^*(x) = \sup \{ Q \geq 1 \mid \Delta_\omega(Q) \leq x \}$$

$\omega \in BR_M$ if for any $c > 0$,

$$\int_{\Delta^*(1/c)}^{+\infty} C^* \left(c \Delta_\omega^*(x) \right) \frac{dx}{x} < \infty$$

Thm: Assume

(i) $H \in \mathcal{U}_M$ i.e. M satisfies (H1), (H2)

(ii) $\omega \in BR_M$

then the torus persists

Ex $M_\ell = (\ell!)^\alpha \qquad \omega \in BR_M$

$$\Longleftrightarrow \int_1^{+\infty} \frac{\ln \Psi_\omega(Q)}{Q^{1+1/\alpha}} \, dQ < \infty$$

Ex $\mu_\ell = \exp(\ell^{1/2})$ then $BR_M = \emptyset$

so the theorem is an empty statement

when you get away from good cases, C no longer controls Fourier coefficients

so this statement becomes a lot weaker