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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Jeffrey Heninger **Email/Phone:** jeffrey.heninger@yahoo.com

Speaker's Name: Martin Berz

Talk Title: Rigorous High-Order Methods in the Description of Large Particle Accelerators

Date: 11 / 26 / 2018 **Time:** 3 : 30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: Described the history of particle accelerators. In a large synchrotron, particles contact the fields an extremely large number of times, so we need to check that the motion of the beam is stable. Choose strips around Taylor approximations of the invariant manifolds of a fixed point so that they map into themselves. This gives us rigorous symbolic dynamics, bounds for the topological entropy, and Nekhoroshev - type estimates for the worst error after a certain number of turns.

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(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
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- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
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- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Rigorous High-Order Methods in the Description of Large Particle Accelerators

Martin Berz, MSU

and

Kyoko Makino, Johannes Grote, Alex Wittig, ...

Brief History of Particle Accelerators 1

- Much of the early work was done here in Berkeley, starting with Ernest O. Lawrence.
- First step: R. J. Van de Graaf developed a very powerful voltage source. It can deliver 1.5 Million Volts.

Beams and Beam Physics 13

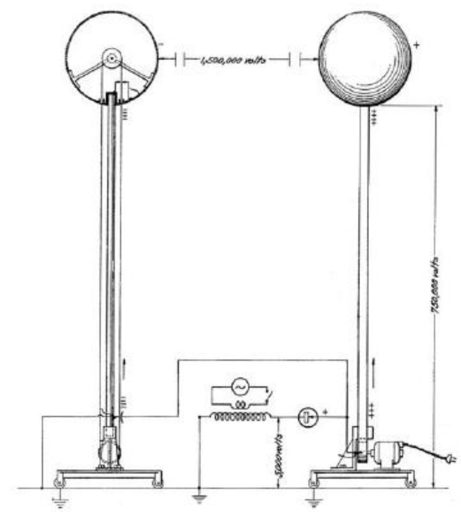


FIGURE 1.8: Design sketch of the Van de Graaff high voltage generator. (From R. J. Van de Graaff, US Patent 1,991,236, 1931 [18].)

Brief History of Particle Accelerators 1

- Much of the early work was done here in Berkeley, starting with Ernest O. Lawrence.
- First step: R. J. Van de Graaf developed a very powerful voltage source. It can deliver 1.5 Million Volts.
- Second step: Use this voltage difference to accelerate charged particles.
- Put experimenters and their equipment into a cage, where the accelerated ions are studied
- A big apparatus, but a somewhat inefficient method.

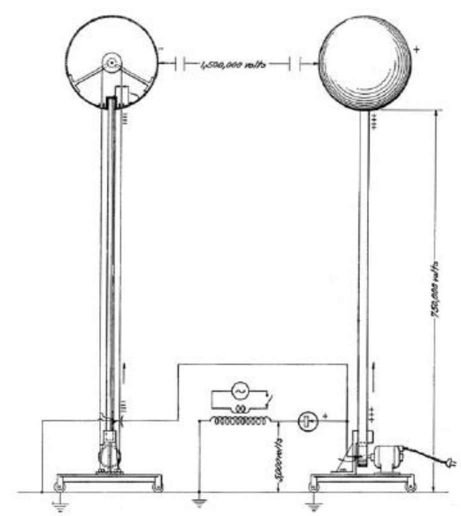


FIGURE 1.8: Design sketch of the Van de Graaff high voltage generator. (From R. J. Van de Graaff, US Patent 1,991,236, 1931 [18].)

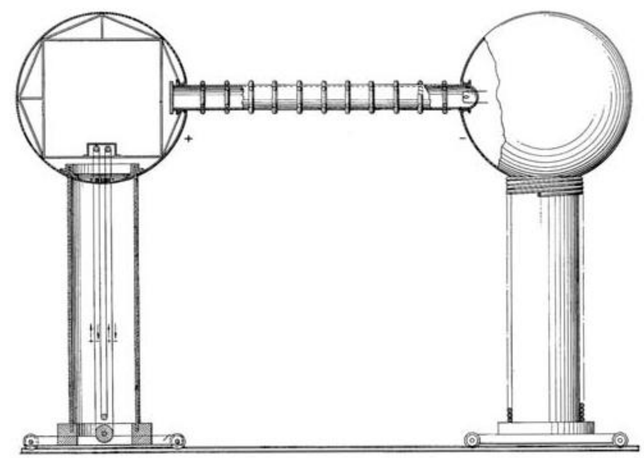


FIGURE 1.9: Design sketch of the use of the Van de Graaff generator as a particle accelerator. (From R. J. Van de Graaff, US Patent 1,991,236, 1931 [18].)

Brief History of Particle Accelerators 2

- Third step: Start at voltage 0, end at voltage 0, and gain twice the energy.
- (How can this be done – sounds like a slight of hand)
- Tandem Van de Graaf: begin with negative ion, accelerate towards a chamber under positive voltage.

Brief History of Particle Accelerators 2

- Third step: Start at voltage 0, end at voltage 0, and gain twice the energy.
- (How can this be done – sounds like a slight of hand)
- Tandem Van de Graaf: begin with negative ion, accelerate towards a chamber under positive voltage.
- Inside the chamber, send the negative ions through a thin stripping foil. This strips off electrons, resulting in positive ions.
- The positive ions are now accelerated by going back from positive voltage to 0.

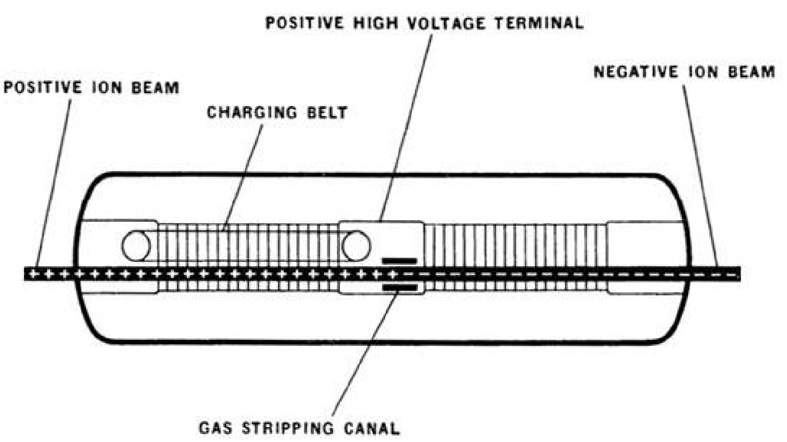


FIGURE 1.10: The principle of the tandem Van de Graaff accelerator. (Reprinted from *Nucl. Instrum. Methods*, v. 8, R. J. Van de Graaff, Tandem electrostatic accelerators, p. 195–202, Copyright (1960), with permission from Elsevier [19].)

Brief History of Particle Accelerators 3

- Enter E. O. Lawrence:
- As perhaps an early pioneer of the green spirit of Berkeley, Lawrence, together with Sloan, re-cycles the voltage, which leads to the so-called linear accelerator.
- A radiofrequency oscillator is pulsed to feed voltage at just the right time when ions are coming through. Voltage is re-used repeatedly.

Beams and Beam Physics 17

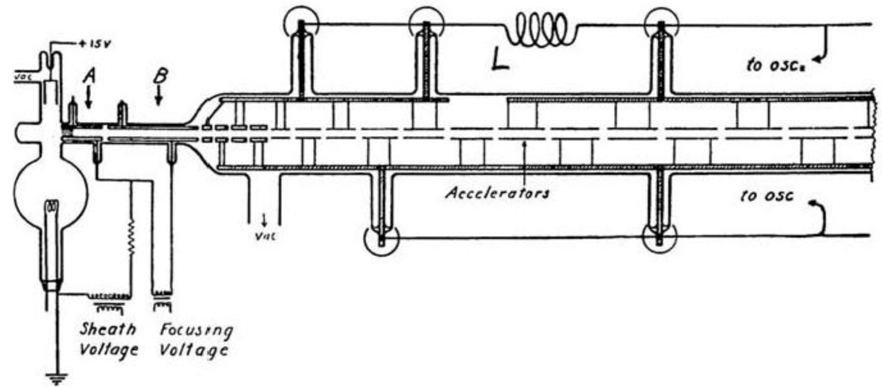


FIGURE 1.12: Illustration of a linear accelerator, designed by E. O. Lawrence and H. D. Sloan. (Reprinted the middle picture of Fig. 1 with permission from [64] as follows: D. H. Sloan and E. O. Lawrence, *Phys. Rev.*, 38, 2021, 1931. Copyright (1931) by the American Physical Society.)

Brief History of Particle Accelerators 3

- Enter E. O. Lawrence:
- As perhaps an early pioneer of the green spirit of Berkeley, Lawrence, together with Sloan, re-cycles the voltage, which leads to the so-called linear accelerator.
- A radiofrequency oscillator is pulsed to feed voltage at just the right time when ions are coming through. Voltage is re-used repeatedly.
- Showing even more green spirit, Lawrence also re-cycles the space.
- Using a magnet to force particles on curved orbits leads to the cyclotron.
- Crucial for this is that the needed frequency does not depend on velocity.

Beams and Beam Physics 17

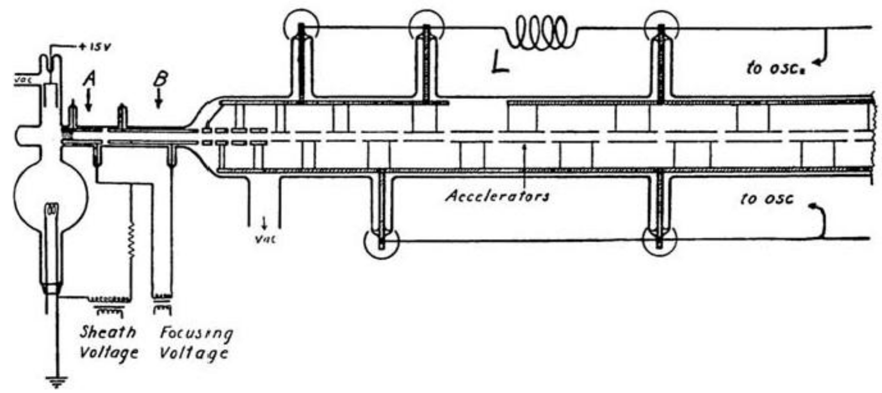


FIGURE 1.12: Illustration of a linear accelerator, designed by E. O. Lawrence and H. D. Sloan. (Reprinted the middle picture of Fig. 1 with permission from [64] as follows: D. H. Sloan and E. O. Lawrence, *Phys. Rev.*, 38, 2021, 1931. Copyright (1931) by the American Physical Society.)

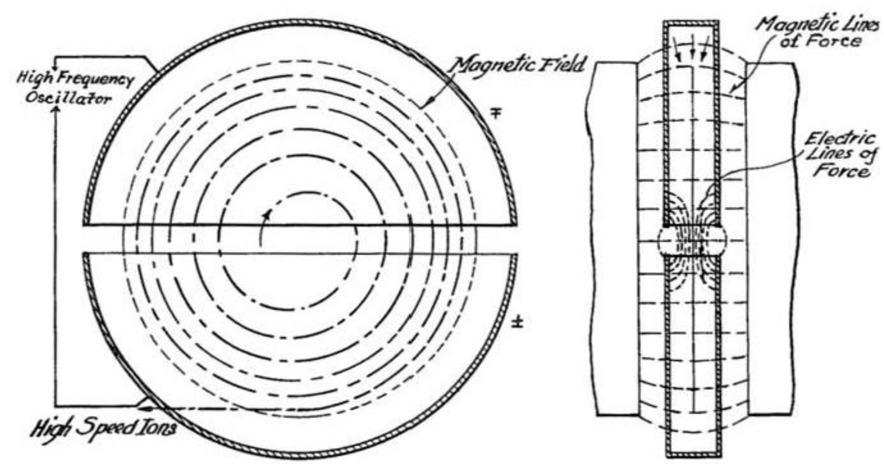


FIGURE 1.18: The principle of the cyclotron, with top view on the left and side view on the right. (From E. O. Lawrence, US Patent 1,948,384, 1932 [40].)

Brief History of Particle Accelerators 4

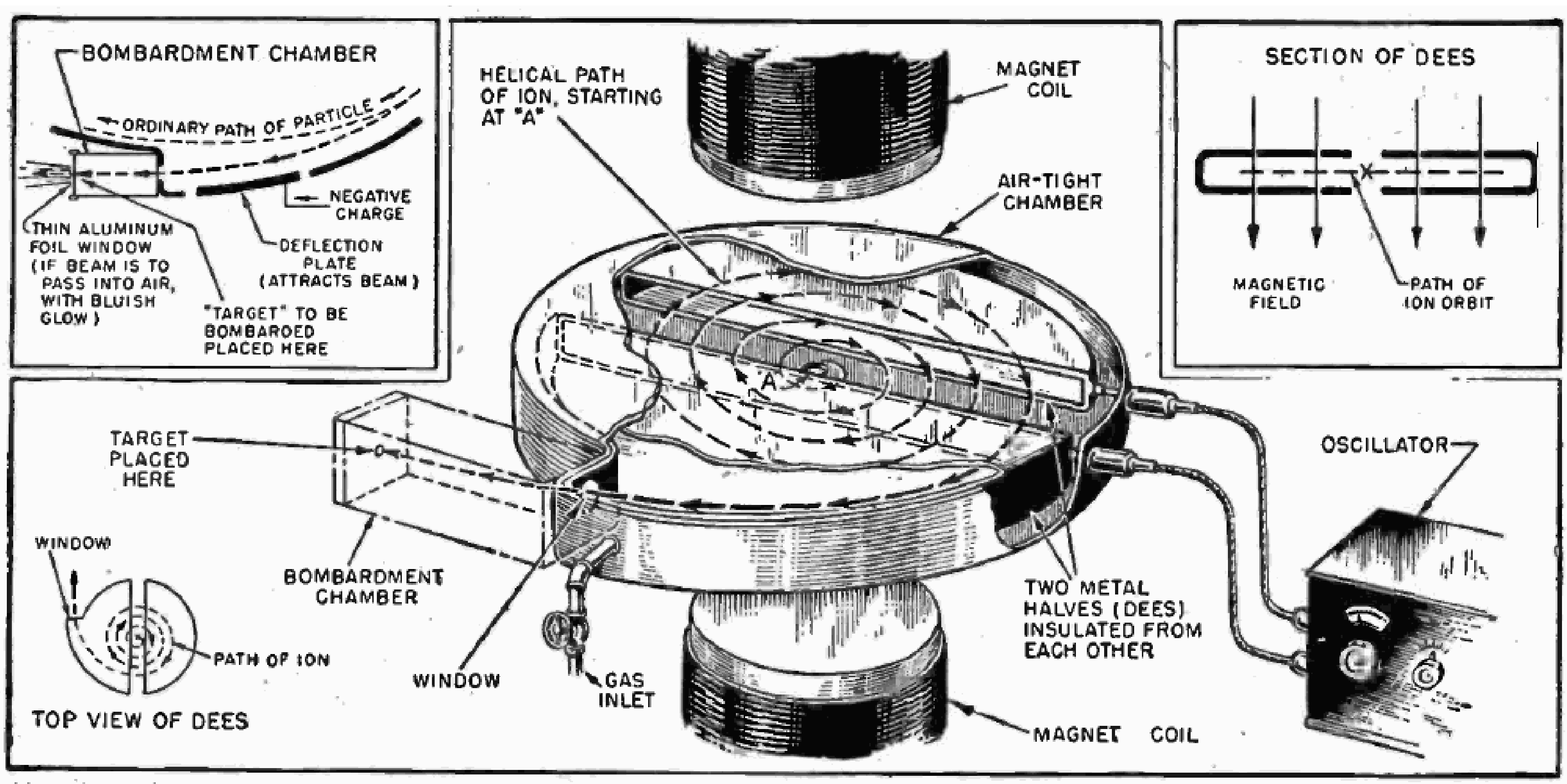
- A Lawrence 37” Cyclotron, in front of Lawrence Hall of Science.
- You can see the poles of the magnets, as well as the large support structure keeping everything in place.

Brief History of Particle Accelerators 4

- A Lawrence 37” Cyclotron, in front of Lawrence Hall of Science.
- You can see the poles of the magnets, as well as the large support structure keeping everything in place.
- Limitations: more energy requires stronger magnetic field - hard to do, nowadays use superconducting magnets in cyclotrons
- More energy requires larger radii – gets expensive quickly.

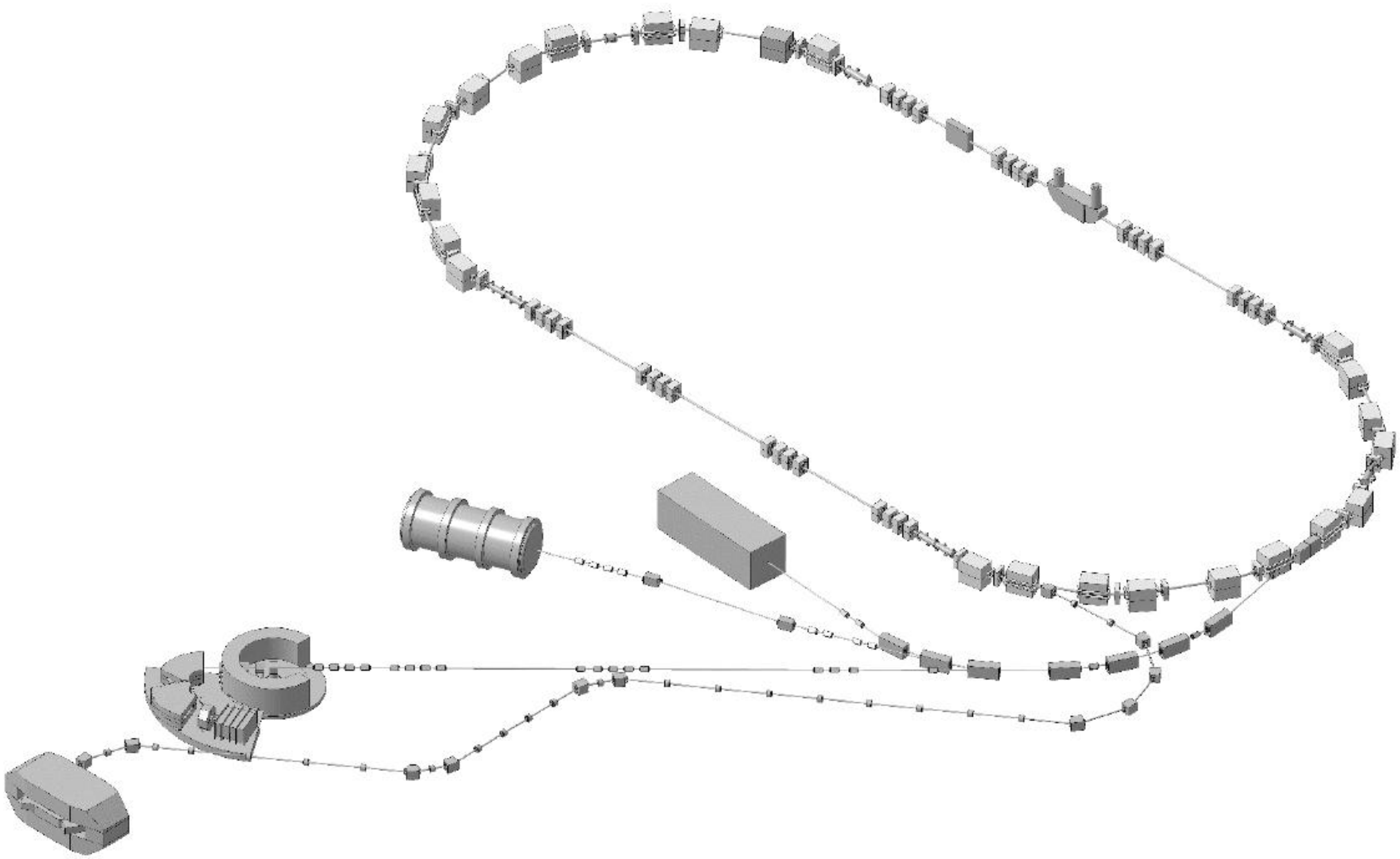
Brief History of Particle Accelerators 5

- A schematic diagram of a cyclotron, courtesy Wikipedia
- We see the magnet coil, the oscillator "Dees", the oscillator, and the extraction system



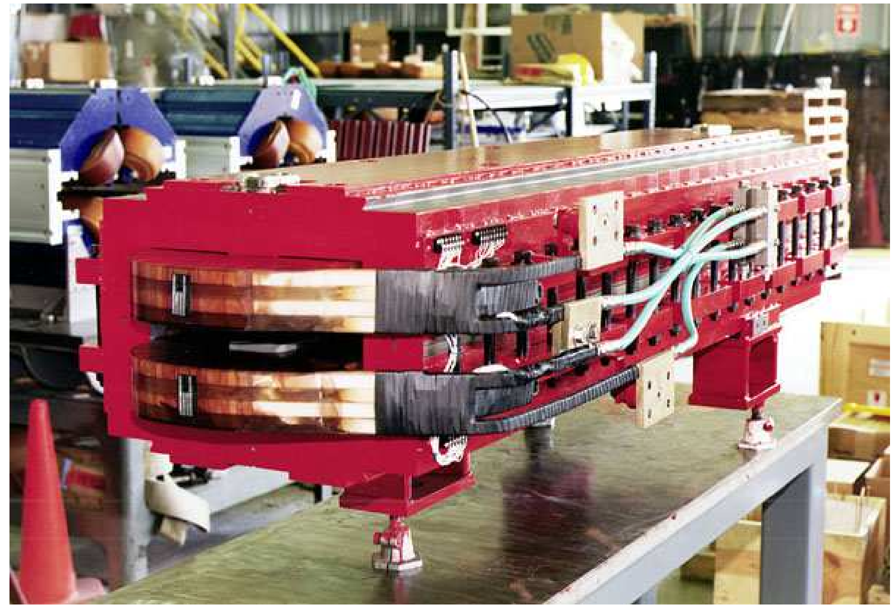
Brief History of Particle Accelerators 6

- Further economization: don't let the particles occupy a lot of space, rather adjust the magnet strength as particles gain energy.
- This results in the synchrotron. A small example is the COSY ring in Jülich, Germany.
- In addition to bending magnets, also need focusing elements



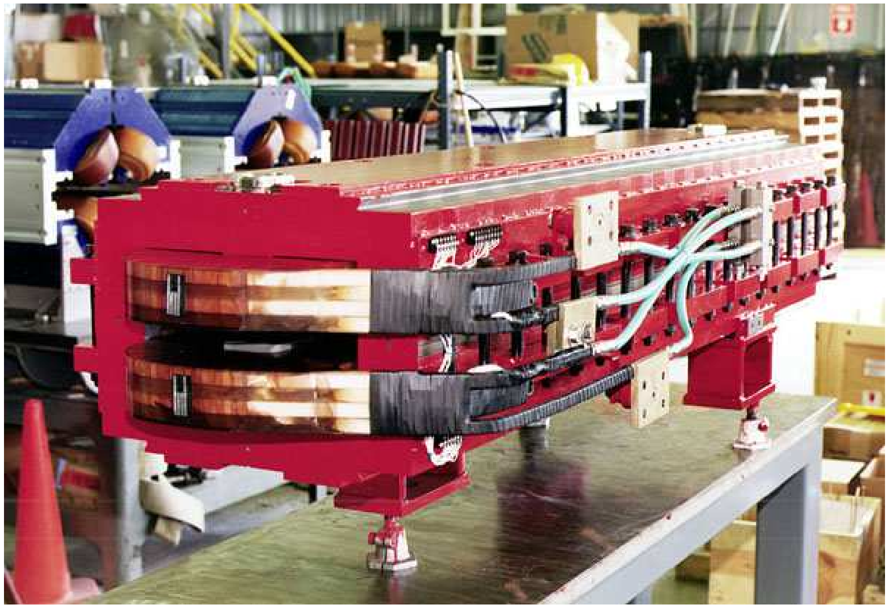
Brief History of Particle Accelerators 7

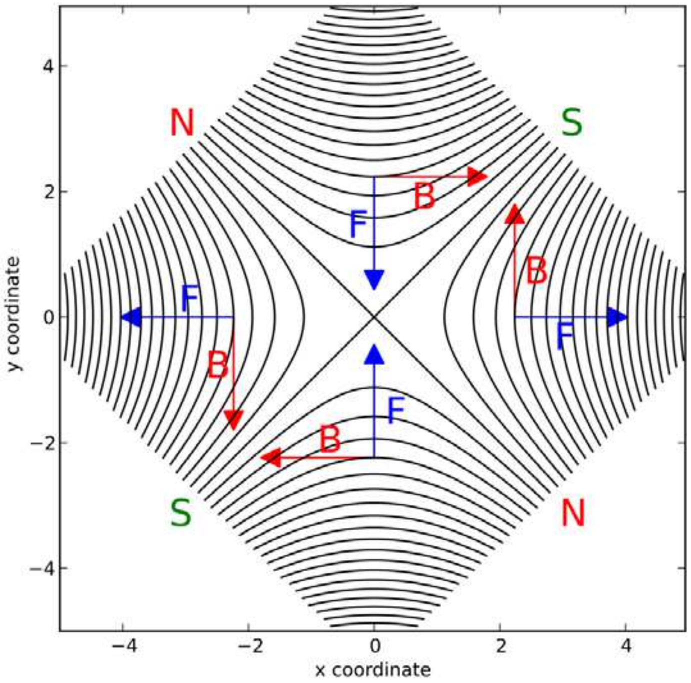
- Bending magnets: “Dipoles”
- Use strong vertical magnetic fields



Brief History of Particle Accelerators 7

- Bending magnets: “Dipoles”
- Use strong vertical magnetic fields
- Focusing magnets: “Quadrupoles”.
- Quads focus in one direction, defocus in the other. Clever combination leads to overall focusing.,
- There are also nonlinear elements, sextupoles, octupoles, etc

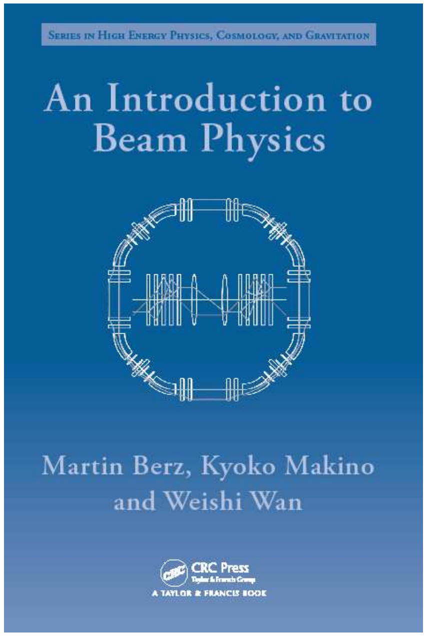




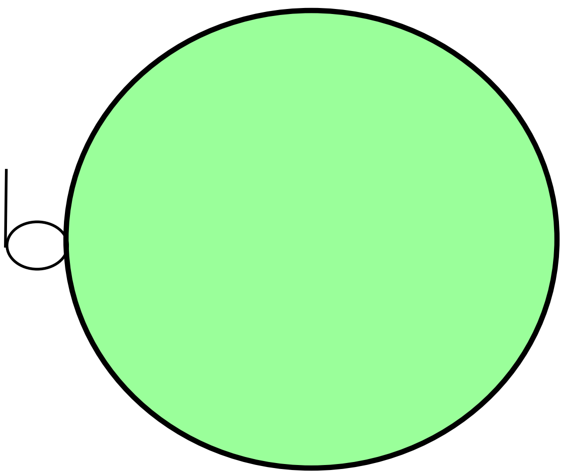
Brief History of Particle Accelerators 8

- Large Synchrotrons: Fermilab's Tevatron (see above, about 6km circumference, and CERN's LHC, 30km)

- (If you are interested, we have an introductory text book about this and more)



Motion in the Tevatron



- Speed of Light: $3x10^8$ m/sec
- Circumference: $6.28x10^3$ m

→ $4x10^4$ revs/sec.

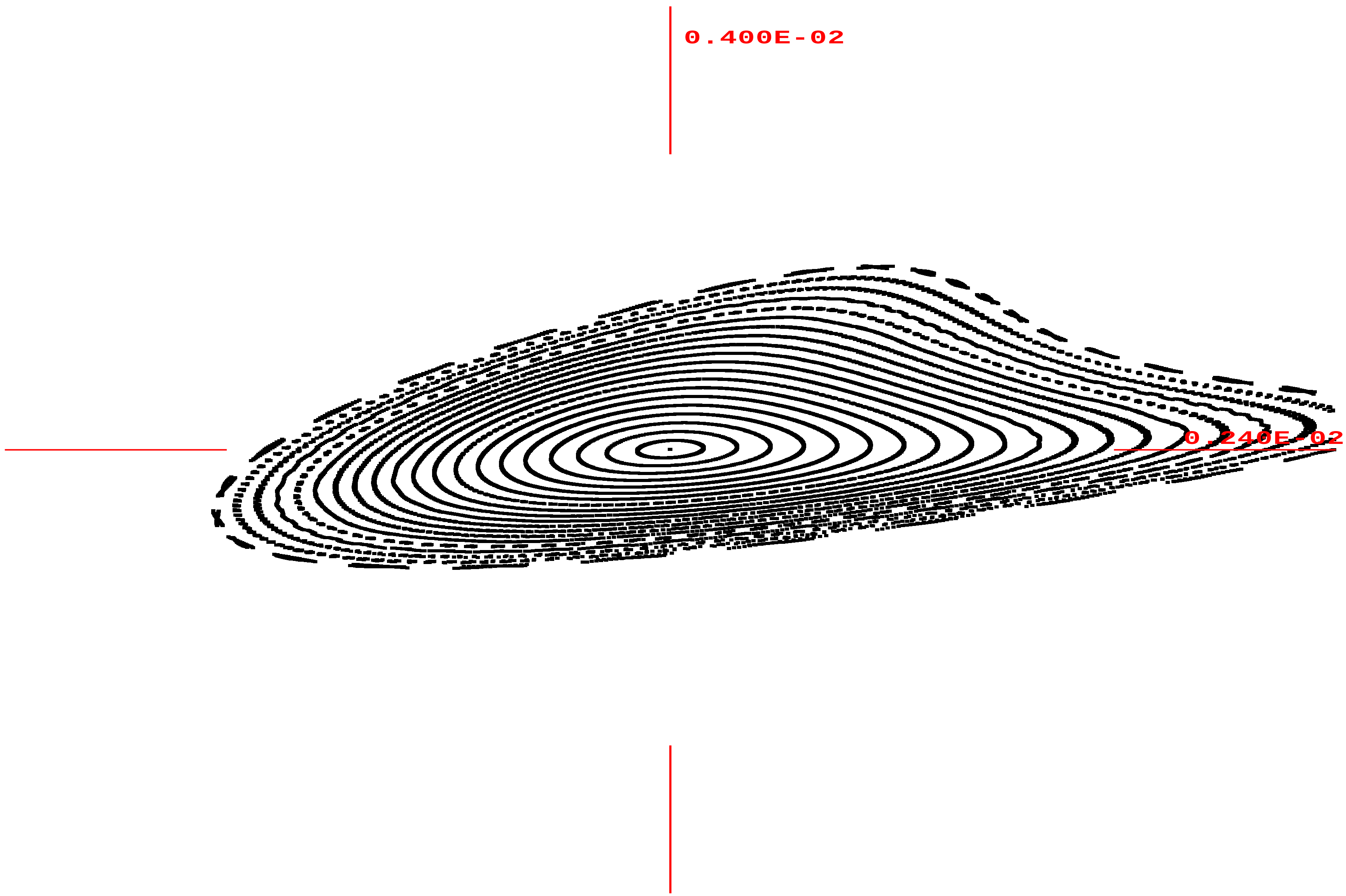
- Need to store about 10 hours, or $4x10^5$ sec

→ $2x10^{10}$ revolutions total.

- 10,000 magnets in ring

→ $2x10^{14}$ contacts with fields!

- Extremely challenging computationally
- Need for several State-Of-The-Art Methods:
  - Phase Space Maps
  - Perturbation Theory
  - Lyapunov- and other Stability Theories
  - High-Performance Verified Methods


0.400E-02
0.240E-02

The Particle Optical Equations of Motion

$$x' = a \cdot (1 + hx) \cdot \frac{p_0}{p_z}$$

$$y' = b \cdot (1 + hx) \cdot \frac{p_0}{p_z}$$

$$l' = (1 + \delta_m) \cdot (1 + hx) \cdot \frac{1 + \eta}{1 + \eta_0} \cdot \frac{p_0}{p_z}$$

$$a' = \left((1 + \delta_m) \cdot \frac{1 + \eta}{1 + \eta_0} \cdot \frac{p_0}{p_z} \cdot \frac{E_x}{\chi_{E0}} \; - \; \frac{B_y}{\chi_{M0}} \; + \; b \cdot \frac{p_0}{p_z} \cdot \frac{B_z}{\chi_{M0}} \right) \cdot (1 + hx) \cdot (1 + \delta_z) + h \cdot \frac{p_z}{p_0}$$

$$b' = \left((1 + \delta_m) \cdot \frac{1 + \eta}{1 + \eta_0} \cdot \frac{p_0}{p_z} \cdot \frac{E_y}{\chi_{E0}} \; + \; \frac{B_x}{\chi_{M0}} \; - \; a \cdot \frac{p_0}{p_z} \cdot \frac{B_z}{\chi_{M0}} \right) \cdot (1 + hx) \cdot (1 + \delta_z)$$

Here the following abbreviations are used:

$$\chi_{E0} = \frac{p_0 \cdot v_0}{z_0 e} \;,\;\; \chi_{M0} = \frac{p_0}{z_0 e}$$

$$\eta = \left(\frac{K_0 \cdot (1+\delta_k) - z_0 \cdot e \cdot (1+\delta_z) \cdot V(x,y,s)}{m_0 c^2 \cdot (1+\delta_m)} \right)$$

$$\frac{p_z}{p_0} = \sqrt{(1+\delta_m)^2 \cdot \frac{\eta(2+\eta)}{\eta_0(2+\eta_0)} - a^2 - b^2}$$

Equations are expressed in **curvilinear coordinates,** an orthogonal system attached to a **reference orbit.** From earliest times, these have proven to be advantageous in practice for numerical stability.

History of Higher Order Optics

	Light Optics (Round Lenses)	**Electron Optics** (Round Lenses)	**Particle Optics** (Non-Round Lenses)
1	Gauss 1841		?
2	(Gauss 1841)		Brown 1959
3	Petzval 1840 Seidel 1856	Scherzer 1936	Matsuda, Wollnik 1965
4			
5	Kohlschütter, Schwarzschild 1905 Rabinovich 1946		M.B. 1985

```
      SUBROUTINE elmm(L,Z,K01,K02,K03,K04,K05,K27,K32,NORDER,NG,ND)
*     **************************************************************
*
*     Subroutine to Compute Aberration Equations Equations
*     Magnetic Multipole to Fifth Order
*     Computer Generated by Program HAMILTON (C) M. Berz 1985
*
      IMPLICIT DOUBLE PRECISION (A - Z)
*
      INTEGER NORDER, NG, ND
*
      DOUBLE PRECISION L(0:461,7)
*
      K30    = 1./(1+K32)
      K31    = 1./(1+K32/2.)
*
      FX2 = -K01*K27
*
      FY2 = +K01*K27
*
      IF(FX2.LT.-1.D-8) THEN
         AFX = SQRT(-FX2)
         CX  = COS(AFX*Z)
         SX  = SIN(AFX*Z)/AFX
      ELSEIF(FX2.GT.1.D-8) THEN
         AFX = SQRT(FX2)
         EX  = EXP(AFX*Z)
         EEX = 1.D0/EX
         CX  = (EX + EEX)/2.D0
         SX  = (EX - EEX)/2.D0/AFX
      ELSE
         CX  = 1.D0
         SX  = Z
         FX2 = 1.D-8
      ENDIF
*
      IF(FY2.LT.-1.D-8) THEN
         AFY = SQRT(-FY2)
         CY  = COS(AFY*Z)
         SY  = SIN(AFY*Z)/AFY
      ELSEIF(FY2.GT.1.D-8) THEN
         AFY = SQRT(FY2)
         EY  = EXP(AFY*Z)
         EEY = 1.D0/EY
         CY  = (EY + EEY)/2.D0
```

```
         SY  = (EY - EEY)/2.D0/AFY
      ELSE
         CY  = 1.D0
         SY  = Z
         FY2 = 1.D-8
      ENDIF
*
*
      CS2 = CX
      CS3 = SX
      CS4 = CY
      CS5 = SY
      CS6 = Z
      KK2 = K31*K32
      KK3 = K30*K32
      FF2 = FX2
      TT2 = CS2
      TT3 = CS3
      TT4 = CS3*FF2
      TT5 = CS4
      TT6 = CS5
      TT7 = CS5*FF2
      TT8 = CS6
      TT9 = CS6*KK2
      TT10 = CS6*KK3
      L(1,1) = (+TT2)
      L(2,1) = (+TT3)
      L(1,2) = (+TT4)
      L(2,2) = (+TT2)
      L(3,3) = (+TT5)
      L(4,3) = (+TT6)
      L(3,4) = (-TT7)
      L(4,4) = (+TT5)
*
      IF(ND.EQ.0.AND.NG.EQ.0) GOTO 100
*
      L(6,6) = (+1)
      L(6,7) = (-0.5D+00*TT8-0.25D+00*TT9+TT10)
*
      IF(NG.EQ.0) GOTO 100
*
      L(5,5) = (+1)
      L(5,7) = (+0.5D+00*TT8+0.25D+00*TT9-TT10)
*
  100 IF(NORDER.EQ.1) GOTO 1000
*
      CS7 = CS3*CX
```

```
CS8 = CS3*SX
CS9 = CS4*CX
CS10 = CS4*SX
CS11 = CS5*CX
CS12 = CS5*SX
CS13 = CS5*CY
CS14 = CS5*SY
CS15 = CS6*CX
CS16 = CS6*SX
CS17 = CS6*CY
CS18 = CS6*SY
KK4 = KK2*K31*K32
KK5 = KK3*K31*K32
KK6 = K02*K27
FF3 = 1/FX2/FX2
FF4 = FF3*FX2
TT11 = KK6*FF4
TT12 = CS2*KK6*FF4
TT13 = CS8*KK6
TT14 = CS3*KK6*FF4
TT15 = CS7*KK6*FF4
TT16 = KK6*FF3
TT17 = CS2*KK6*FF3
TT18 = CS8*KK6*FF4
TT19 = CS14*KK6
TT20 = CS13*KK6*FF4
TT21 = CS14*KK6*FF4
TT22 = CS16*FF2
TT23 = CS16*KK2*FF2
TT24 = CS3*KK2
TT25 = CS15
TT26 = CS15*KK2
TT27 = CS3*KK6
TT28 = CS7*KK6
TT29 = CS13*KK6
TT30 = CS3*KK2*FF2
TT31 = CS15*FF2
TT32 = CS15*KK2*FF2
TT33 = CS4*KK6*FF4
TT34 = CS9*KK6*FF4
TT35 = CS12*KK6
TT36 = CS10*KK6*FF4
TT37 = CS5*KK6*FF4
TT38 = CS11*KK6*FF4
TT39 = CS4*KK6*FF3
TT40 = CS9*KK6*FF3
TT41 = CS12*KK6*FF4
```

```
      TT42 = CS18*FF2
      TT43 = CS18*KK2*FF2
      TT44 = CS5*KK2
      TT45 = CS17
      TT46 = CS17*KK2
      TT47 = CS10*KK6
      TT48 = CS5*KK6
      TT49 = CS11*KK6
      TT50 = CS5*KK2*FF2
      TT51 = CS17*FF2
      TT52 = CS17*KK2*FF2
      TT53 = CS7*FF2
      TT54 = CS6*FF2
      TT55 = CS8*FF2
      TT56 = CS7
      TT57 = CS13*FF2
      TT58 = CS14*FF2
      TT59 = CS13
      TT60 = CS6*KK4
      TT61 = CS6*KK5
      L(7,1) = (+0.33333334327D+00*(+TT11-TT12-TT13))
      L(8,1) = (+0.66666668654D+00*(+TT14-TT15))
      L(13,1) = (+0.66666668654D+00*(-TT16+TT17)-0.33333333333D+00*TT
     *  18)
      L(18,1) = (+0.60000002384D+00*(-TT11+TT12)+0.2D+00*TT19)
      L(19,1) = (+0.40000000596D+00*(+TT14-TT20))
      L(22,1) = (+0.40000000596D+00*(-TT16+TT17)-0.2D+00*TT21)
      L(7,2) = (-0.33333333333D+00*TT27-0.66666666667D+00*TT28)
      L(8,2) = (+0.66666668654D+00*(-TT11+TT12)-0.13333333333D+01*TT13)
      L(13,2) = (+0.66666668654D+00*(+TT14-TT15))
      L(18,2) = (+0.6D+00*TT27+0.4D+00*TT29)
      L(19,2) = (+0.40000000596D+00*(-TT11+TT12)+0.8D+00*TT19)
      L(22,2) = (+0.40000000596D+00*(+TT14-TT20))
      L(9,3) = (+0.40000000596D+00*(-TT33+TT34)+0.8D+00*TT35)
      L(14,3) = (+0.4D+00*TT36-0.12D+01*TT37+0.8D+00*TT38)
      L(10,3) = (-0.8D+00*TT36+0.40000000596D+00*(+TT37+TT38))
      L(15,3) = (+0.80000001192D+00*(+TT39-TT40)+0.4D+00*TT41)
      L(9,4) = (+0.12D+01*TT47+0.40000000596D+00*(+TT48+TT49))
      L(14,4) = (+0.12000000477D+01*(-TT33+TT34)+0.4D+00*TT35)
      L(10,4) = (+0.40000000596D+00*(+TT33-TT34)+0.12D+01*TT35)
      L(15,4) = (-0.4D+00*TT36-0.8D+00*TT37+0.12D+01*TT38)
      L(7,7) = (+0.25D+00*(+TT53-TT54))
      L(8,7) = (+0.5D+00*TT55)
      L(13,7) = (+0.25D+00*(+TT56+TT8))
      L(18,7) = (+0.25D+00*(-TT57+TT54))
      L(19,7) = (-0.5D+00*TT58)
      L(22,7) = (+0.25D+00*(+TT59+TT8))
```

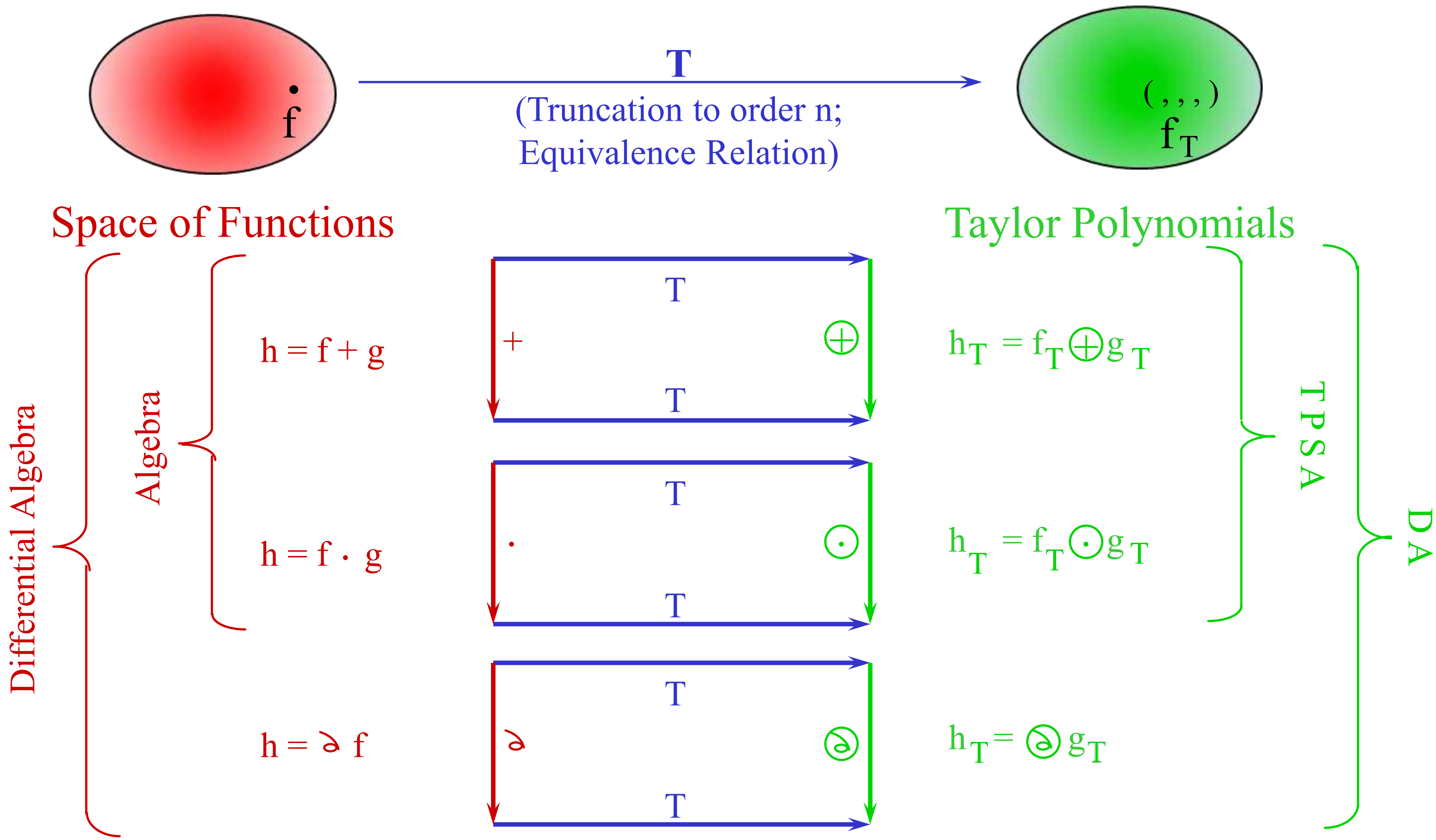
27,000 lines further down:

```
      L(449,7) = (-0.87890625D-01*TT59-0.244140625D-01*TT347
     *  +0.2197265625D-01*TT1723+0.14282226563D+00*TT6924+0.390625D-01*(
     *  +TT348-TT1724)-0.263671875D+00*TT6925-0.380859375D+00*TT8
     *  -0.1162109375D+00*TT9+0.9521484375D-01*TT60+0.26733398438D+00*TT
     *  351+0.1484375D+00*(+TT10-TT61)-0.439453125D+00*TT352
     *  +0.29296875D+00*TT349+0.91796875D-01*TT350-0.732421875D-01*TT
     *  1725-0.12451171875D+00*TT6926+0.109375D+00*(-TT1726+TT1727)
     *  +0.17578125D+00*TT6927+0.140625D+00*TT1728+0.546875D-01*TT1729
     *  -0.3515625D-01*TT1730-0.29296875D-01*TT6928+0.3125D-01*(-TT
     *  6929+TT6930)+0.234375D-01*TT6931+0.78125D-02*(+TT6892-TT6933)
     *  +0.390625D-02*(+TT6893+TT6934)+0.1953125D-02*(-TT6894+TT6935)
     *  -0.9765625D-03*TT6895-0.15625D-01*TT6932)
      L(458,7) = (+0.234375D-01*TT8-0.390625D-02*TT9-0.8203125D-01*TT
     *  60+0.380859375D+00*TT351-0.32470703125D+00*TT1731
     *  +0.76904296875D-01*TT6936+0.15625D-01*TT10+0.21875D+00*TT61
     *  -0.9140625D+00*TT352+0.7421875D+00*TT1732-0.1708984375D+00*TT
     *  6937)
      L(459,7) = (+0.390625D-01*TT8+0.390625D-02*TT9-0.1171875D-01*TT
     *  60-0.68359375D-01*TT351+0.18798828125D+00*TT1731
     *  -0.76904296875D-01*TT6936-0.15625D-01*TT10+0.3125D-01*TT61
     *  +0.1640625D+00*TT352-0.4296875D+00*TT1732+0.1708984375D+00*TT
     *  6937)
      L(460,7) = (+0.13671875D+00*TT8+0.29296875D-01*TT9+0.5859375D-02
     *  *TT60-0.48828125D-02*TT351-0.25634765625D-01*TT1731
     *  +0.38452148438D-01*TT6936-0.1171875D+00*TT10-0.15625D-01*TT61
     *  +0.1171875D-01*TT352+0.5859375D-01*TT1732-0.8544921875D-01*TT
     *  6937)
*
  500 IF(NORDER.EQ.5) GOTO 1000
*
 1000 CONTINUE
*
      RETURN
      END
```

History of Higher Order Optics

	Light Optics (Round Lenses)	**Electron Optics** (Round Lenses)	**Particle Optics** (Non-Round Lenses)
1	Gauss 1841		?
2	(Gauss 1841)		Brown 1959
3	Petzval 1840 Seidel 1856	Scherzer 1936	Matsuda, Wollnik 1965
4			
5	Kohlschütter, Schwarzschild 1905 Rabinovich 1946		M.B. 1985

Function Algebras



Differential Algebra
(also want "exp", "sin" etc: Banach DA)

Diagrams commute exactly

T: Extracts information considered relevant

Differential Algebra
(even Banach DA)

DX 1, NO = 11, NV = 6, INA = 27
****

I	COEFFICIENT	ORDER	EXPONENTS					
1	-.11235361224856E-27	0	0	0	0	0	0	0
2	-2.4577764619843	1	1	0	0	0	0	0
3	382.75567438967	1	0	1	0	0	0	0
4	-2.9058143083356	1	0	0	0	0	1	0
5	-.18527560925181E-01	2	2	0	0	0	0	0
6	-2.0449053795051	2	1	1	0	0	0	0
7	230.43212330444	2	0	2	0	0	0	0
8	0.10745263897241E-01	2	0	0	2	0	0	0
9	-.40104008916025	2	0	0	1	1	0	0
10	0.10748538944562E-01	2	1	0	0	0	1	0
11	-2.3077398243793	2	0	1	0	0	1	0
12	4.4690872366630	2	0	0	0	2	0	0
13	0.61271339425922E-02	2	0	0	0	0	2	0
14	0.42422562884505E-03	3	3	0	0	0	0	0
15	-.25018158222129	3	2	1	0	0	0	0
16	34.164428767158	3	1	2	0	0	0	0
17	-2131.5807609885	3	0	3	0	0	0	0
18	-.42666140335568E-03	3	1	0	2	0	0	0
19	0.61867753199362	3	0	1	2	0	0	0
20	-.11349802973152	3	1	0	1	1	0	0
21	56.549952804501	3	0	1	1	1	0	0
22	0.15244016913989E-02	3	2	0	0	0	1	0
23	-.46319864237315	3	1	1	0	0	1	0
24	40.581998425553	3	0	2	0	0	1	0
25	-.25450130254498E-02	3	0	0	2	0	1	0
26	-4.6823358538539	3	1	0	0	2	0	0
27	1456.5235333461	3	0	1	0	2	0	0
28	-.25643847152791	3	0	0	1	1	1	0
29	0.15479910051681E-02	3	1	0	0	0	2	0
30	-.26430052248601	3	0	1	0	0	2	0
31	-7.1898343490889	3	0	0	0	2	1	0
32	0.58340280013060E-03	3	0	0	0	0	3	0
33	-.54394288125262E-04	4	4	0	0	0	0	0
34	0.38915493503025E-01	4	3	1	0	0	0	0
35	-10.856955574871	4	2	2	0	0	0	0
36	1254.1599009949	4	1	3	0	0	0	0
37	-54750.339715548	4	0	4	0	0	0	0
38	0.42865921904244E-03	4	2	0	2	0	0	0
39	-.10129872906961	4	1	1	2	0	0	0
40	16.600177401064	4	0	2	2	0	0	0
41	-.83438533353867E-03	4	0	0	4	0	0	0
42	0.44656225936067E-01	4	2	0	1	1	0	0

43	-14.242733931933	4	1	1	1	1	0	0
44	1812.0210497771	4	0	2	1	1	0	0
45	-.14401693756858	4	0	0	3	1	0	0
46	-.26812285838556E-03	4	3	0	0	0	1	0
47	0.14433601146177	4	2	1	0	0	1	0
48	-25.002121384519	4	1	2	0	0	1	0
49	1422.4392191357	4	0	3	0	0	1	0
50	0.67315613953893E-03	4	1	0	2	0	1	0
51	-.16340868932716	4	0	1	2	0	1	0
52	1.3257841503873	4	2	0	0	2	0	0
53	-472.13503057908	4	1	1	0	2	0	0
54	51681.544792611	4	0	2	0	2	0	0
55	-9.7160870866478	4	0	0	2	2	0	0
56	0.88334103087622E-01	4	1	0	1	1	1	0
57	-19.022409240455	4	0	1	1	1	1	0
58	-.48689687612391E-03	4	2	0	0	0	2	0
59	0.16755301265834	4	1	1	0	0	2	0
60	-14.106243085175	4	0	2	0	0	2	0
61	0.43660754939067E-03	4	0	0	2	0	2	0
62	-300.75757764658	4	0	0	1	3	0	0
63	2.9012366747428	4	1	0	0	2	1	0
64	-572.73159967434	4	0	1	0	2	1	0
65	0.53258960203668E-01	4	0	0	1	1	2	0
66	-.37768280924720E-03	4	1	0	0	0	3	0
67	0.62989087030710E-01	4	0	1	0	0	3	0
68	-3573.0798614067	4	0	0	0	4	0	0
69	1.6716112876871	4	0	0	0	2	2	0
70	-.10658118377282E-03	4	0	0	0	0	4	0
71	-.25414137178589E-05	5	5	0	0	0	0	0
72	0.14017294168757E-02	5	4	1	0	0	0	0
73	-.22915066598219	5	3	2	0	0	0	0
74	9.3046379608457	5	2	3	0	0	0	0
75	711.33599214748	5	1	4	0	0	0	0
76	-36864.079818068	5	0	5	0	0	0	0
77	0.76846052404647E-05	5	3	0	2	0	0	0
78	-.24398280565609E-02	5	2	1	2	0	0	0
79	0.49036157764730E-01	5	1	2	2	0	0	0
80	10.611563244730	5	0	3	2	0	0	0
81	-.69006293687322E-05	5	1	0	4	0	0	0
82	-.58227683391493E-03	5	0	1	4	0	0	0
83	0.62878426765274E-03	5	3	0	1	1	0	0
84	-.93993449228250E-01	5	2	1	1	1	0	0

1100 lines further down ...

```
 2198  -14060.942901990      11     1 1  0 6  3 0
 2199   1653837.7385130      11     0 2  0 6  3 0
 2200  -257.59556830741      11     0 0  2 6  3 0
 2201  -.35544757510218      11     1 0  1 5  4 0
 2202  -79.173081098476      11     0 1  1 5  4 0
 2203  0.51087261295152E-01  11     2 0  0 4  5 0
 2204  -18.716238122641      11     1 1  0 4  5 0
 2205   1542.8309211899      11     0 2  0 4  5 0
 2206  -.28054439599319E-01  11     0 0  2 4  5 0
 2207  0.22645955750085E-02  11     1 0  1 3  6 0
 2208  -.38909056207237      11     0 1  1 3  6 0
 2209  -.24095385693697E-04  11     2 0  0 2  7 0
 2210  0.88753236469477E-02  11     1 1  0 2  7 0
 2211  -.78021595715990      11     0 2  0 2  7 0
 2212  0.12444200845638E-04  11     0 0  2 2  7 0
 2213  -.25672923569253E-06  11     1 0  1 1  8 0
 2214  0.46190659792613E-04  11     0 1  1 1  8 0
 2215  0.85194029561067E-09  11     2 0  0 0  9 0
 2216  -.30147170356411E-06  11     1 1  0 0  9 0
 2217  0.25626592777412E-04  11     0 2  0 0  9 0
 2218  -.27294471690391E-09  11     0 0  2 0  9 0
 2219  -116911999.30448      11     1 0  010  0 0
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 2221  -19548517.253335      11     0 0  1 9  1 0
 2222  -48961.786425583      11     1 0  0 8  2 0
 2223   8224946.4473064      11     0 1  0 8  2 0
 2224  -3422.0920883930      11     0 0  1 7  3 0
 2225   17.260507499664      11     1 0  0 6  4 0
 2226  -4159.1942930668      11     0 1  0 6  4 0
 2227  -.53695898362788E-01  11     0 0  1 5  5 0
 2228  0.21124533035838E-01  11     1 0  0 4  6 0
 2229  -3.4758111999671      11     0 1  0 4  6 0
 2230  0.37040986757764E-03  11     0 0  1 3  7 0
 2231  -.72548521359208E-05  11     1 0  0 2  8 0
 2232  0.12676479473598E-02  11     0 1  0 2  8 0
 2233  -.33276602342639E-07  11     0 0  1 1  9 0
 2234  0.20264295213287E-09  11     1 0  0 0 10 0
 2235  -.34321706228840E-07  11     0 1  0 0 10 0
 2236  -97662091.940232      11     0 0  010  1 0
 2237  -17547.398707127      11     0 0  0 8  3 0
 2238   4.1188522337990      11     0 0  0 6  5 0
 2239  0.33673939625236E-02  11     0 0  0 4  7 0
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 2241  0.20981553537075E-10  11     0 0  0 0 11 0
```
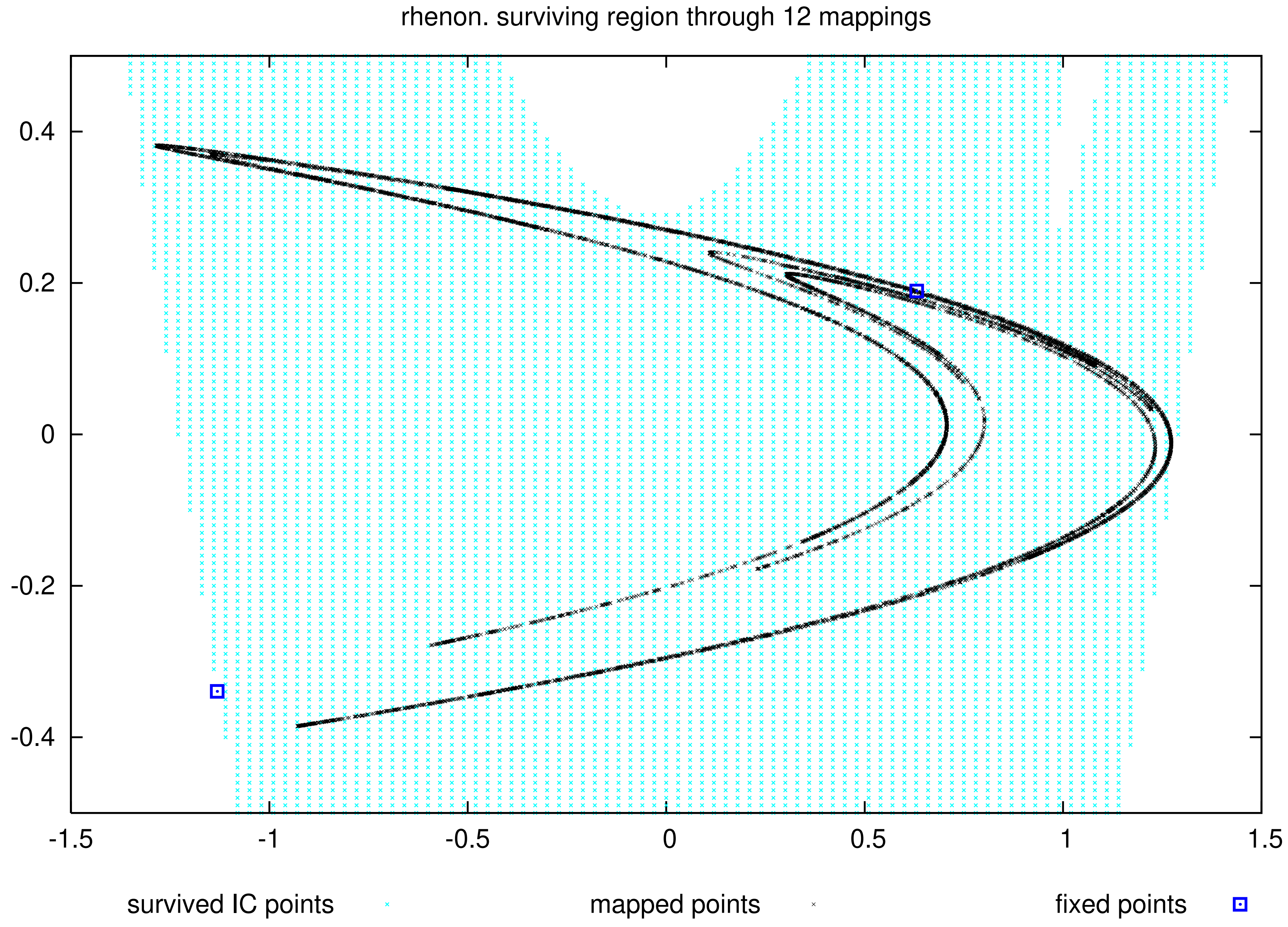
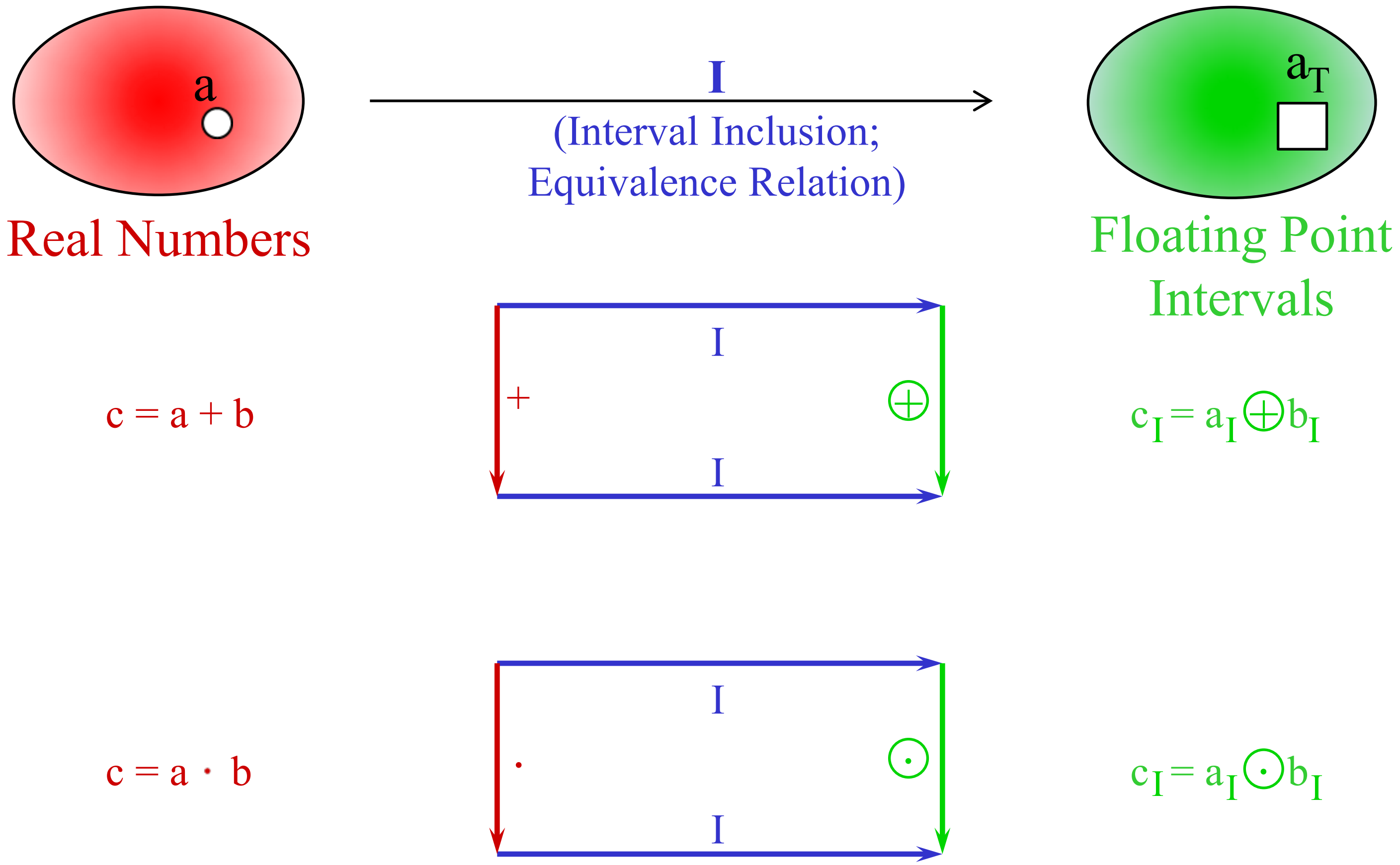



rhenon. surviving region through 12 mappings
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5
survived IC points
mapped points
fixed points

Set Inclusions (Intervals)

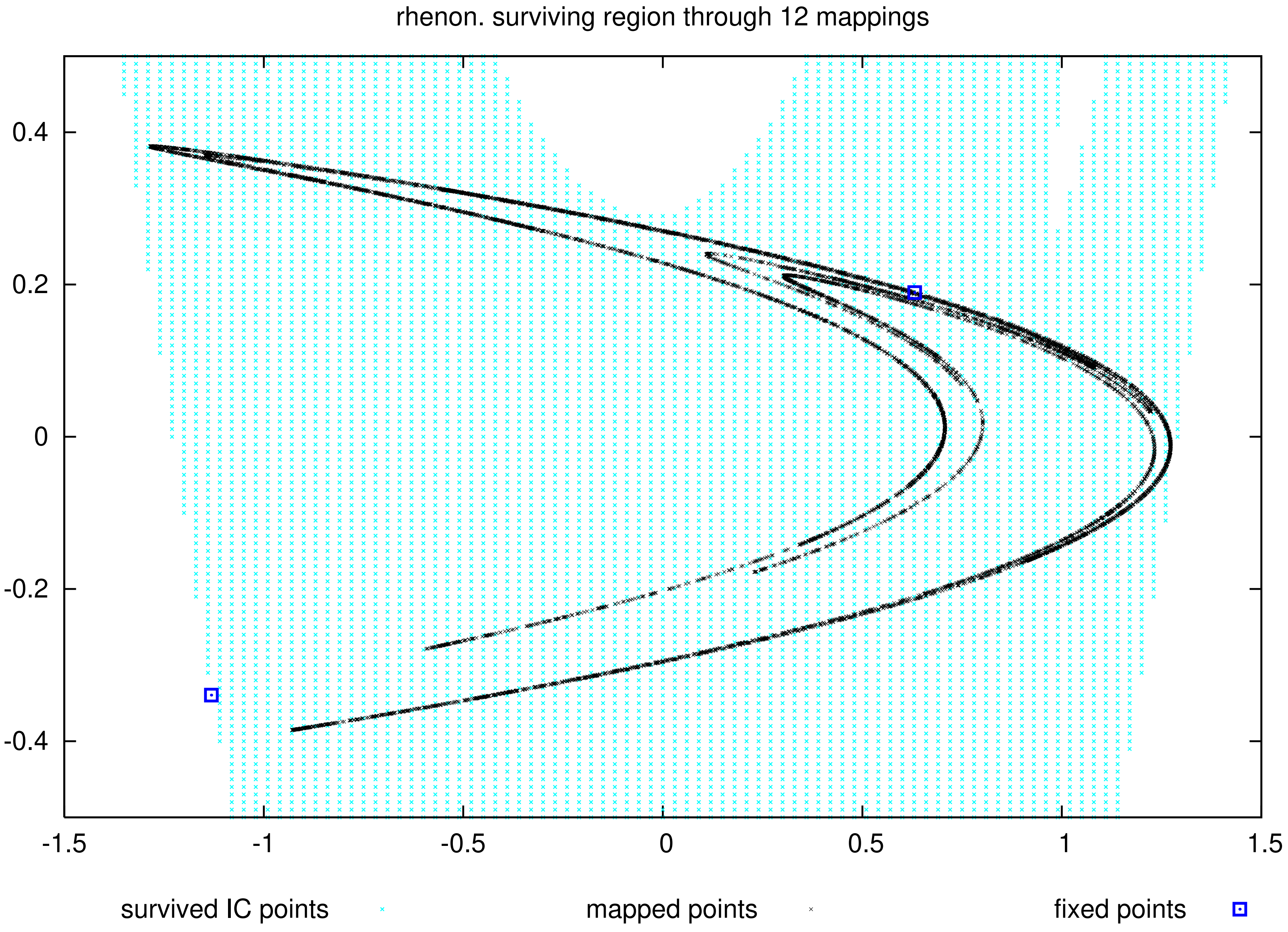


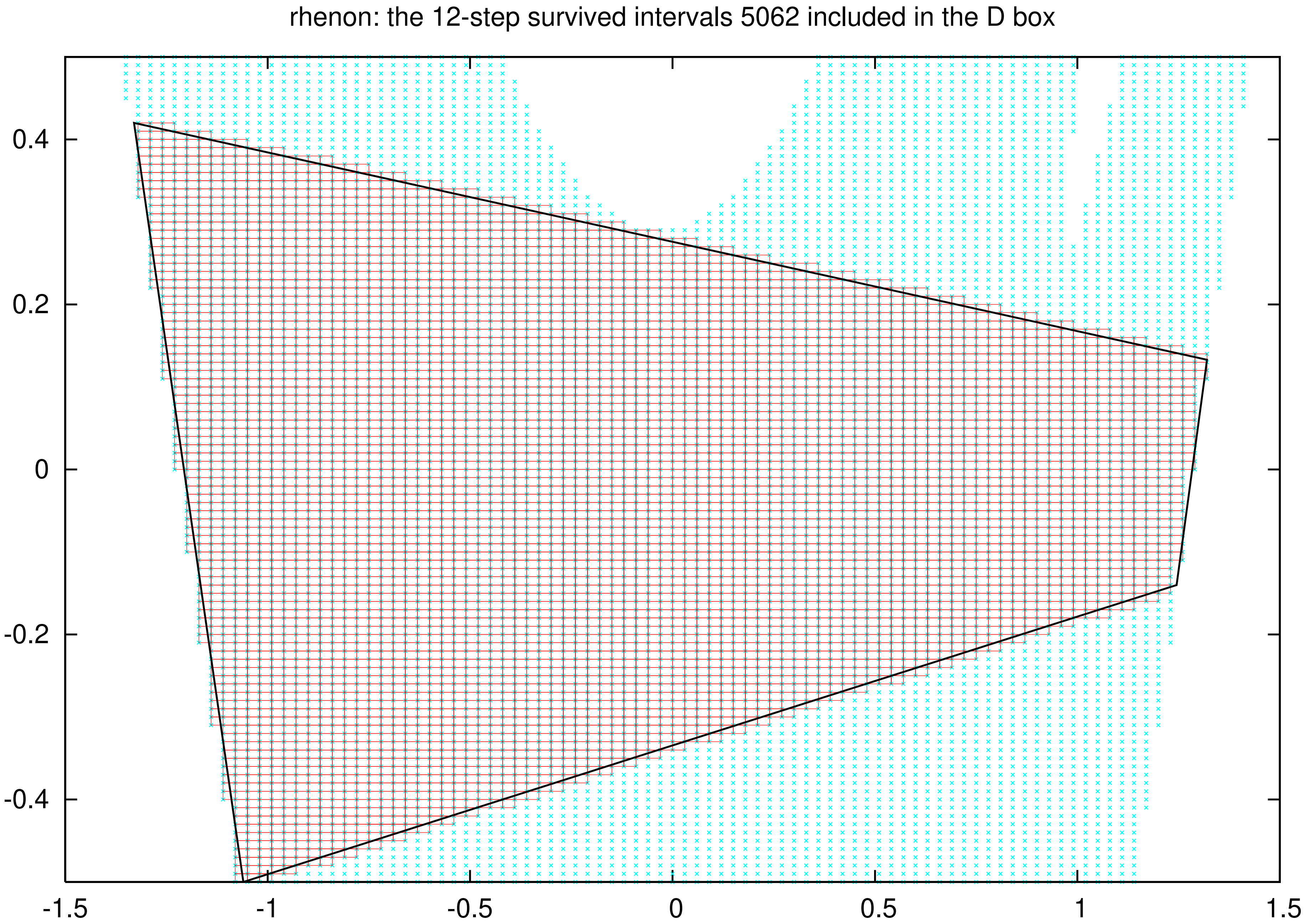
Field
(Also want “exp”, “sin”
etc: Banach Field)

Diagrams commute
exactly!

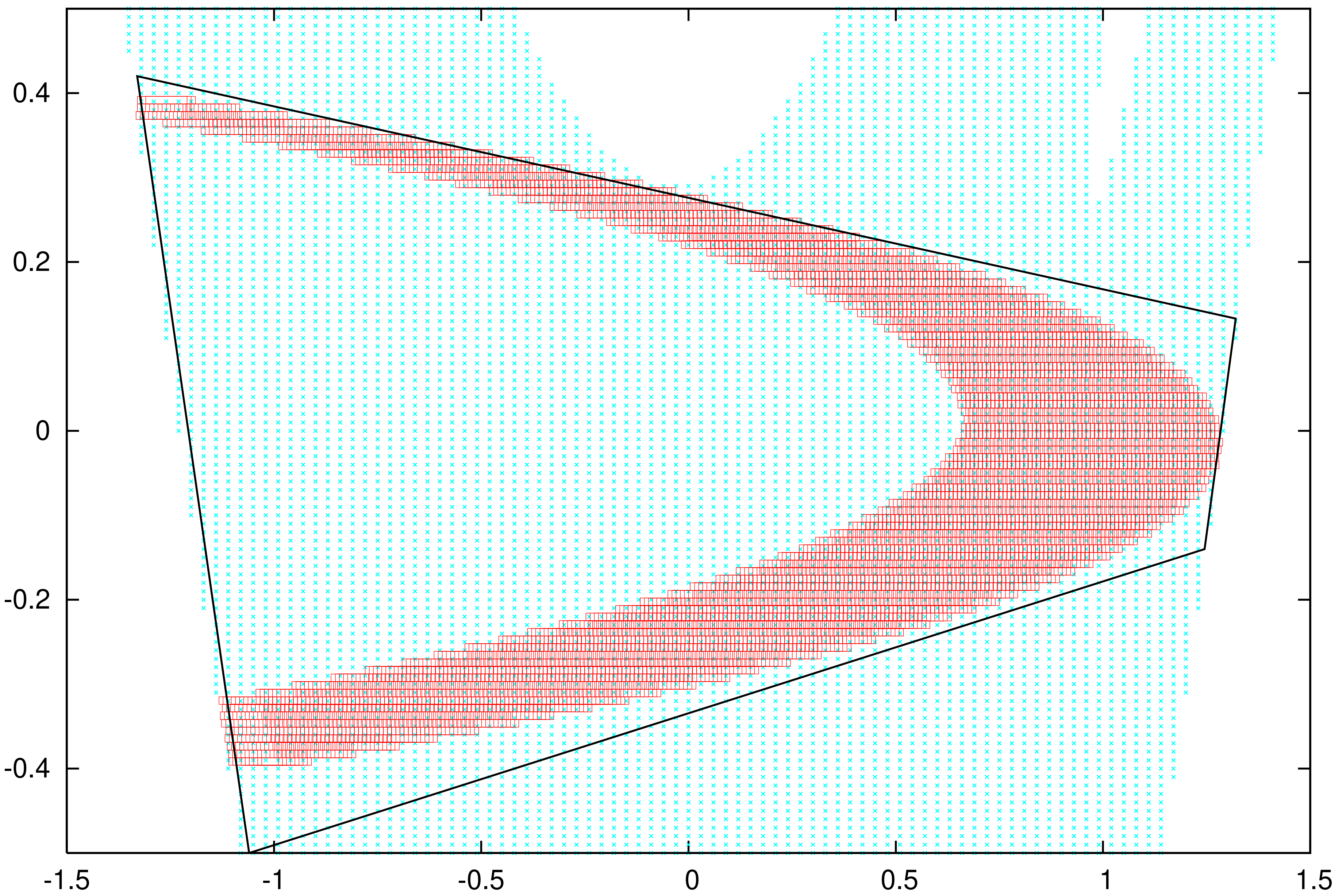
**Little Algebraic
Structure**

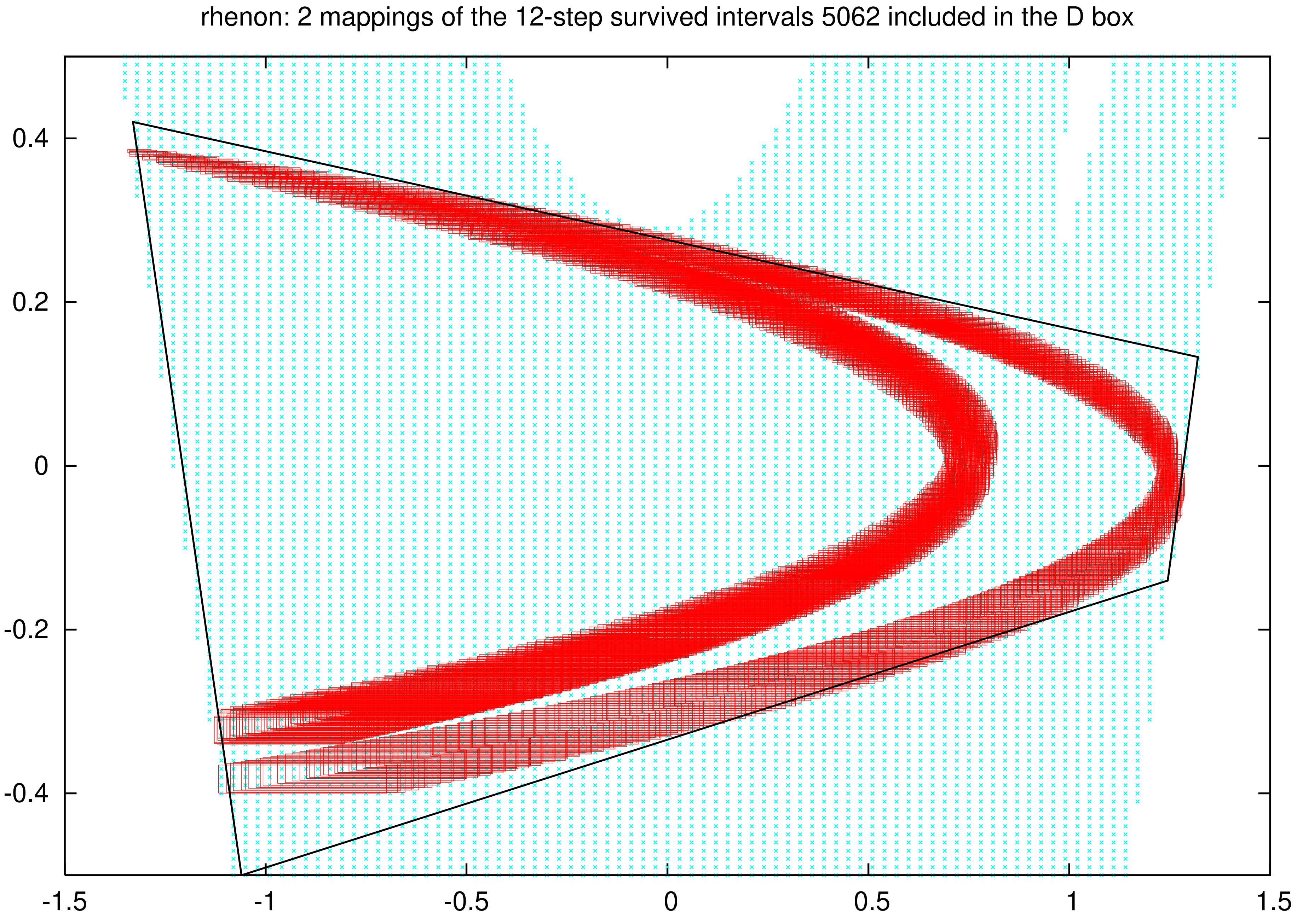
I: Extracts information
considered relevant


rhenon. surviving region through 12 mappings
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5
survived IC points
mapped points
fixed points

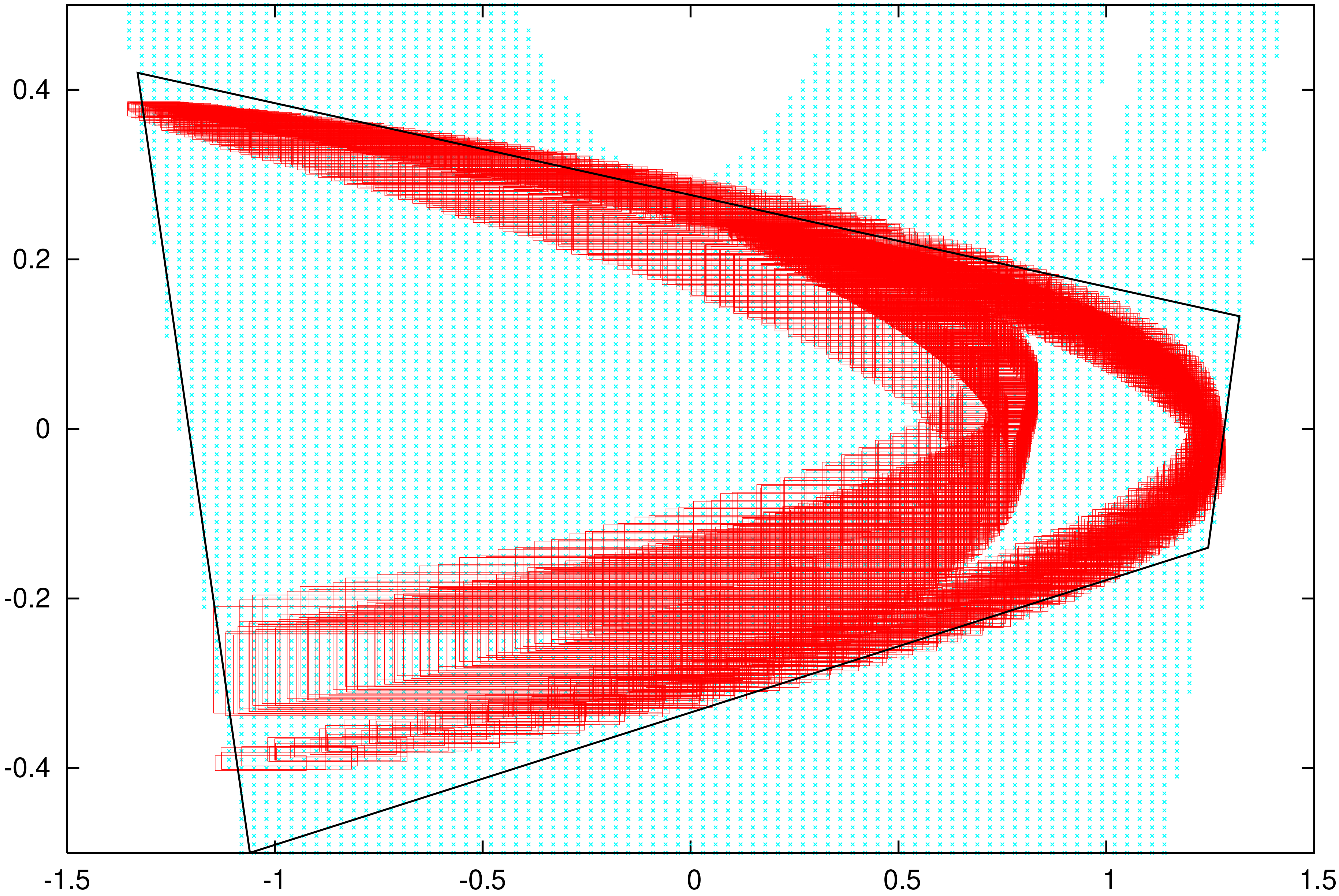

rhenon: the 12-step survived intervals 5062 included in the D box
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5

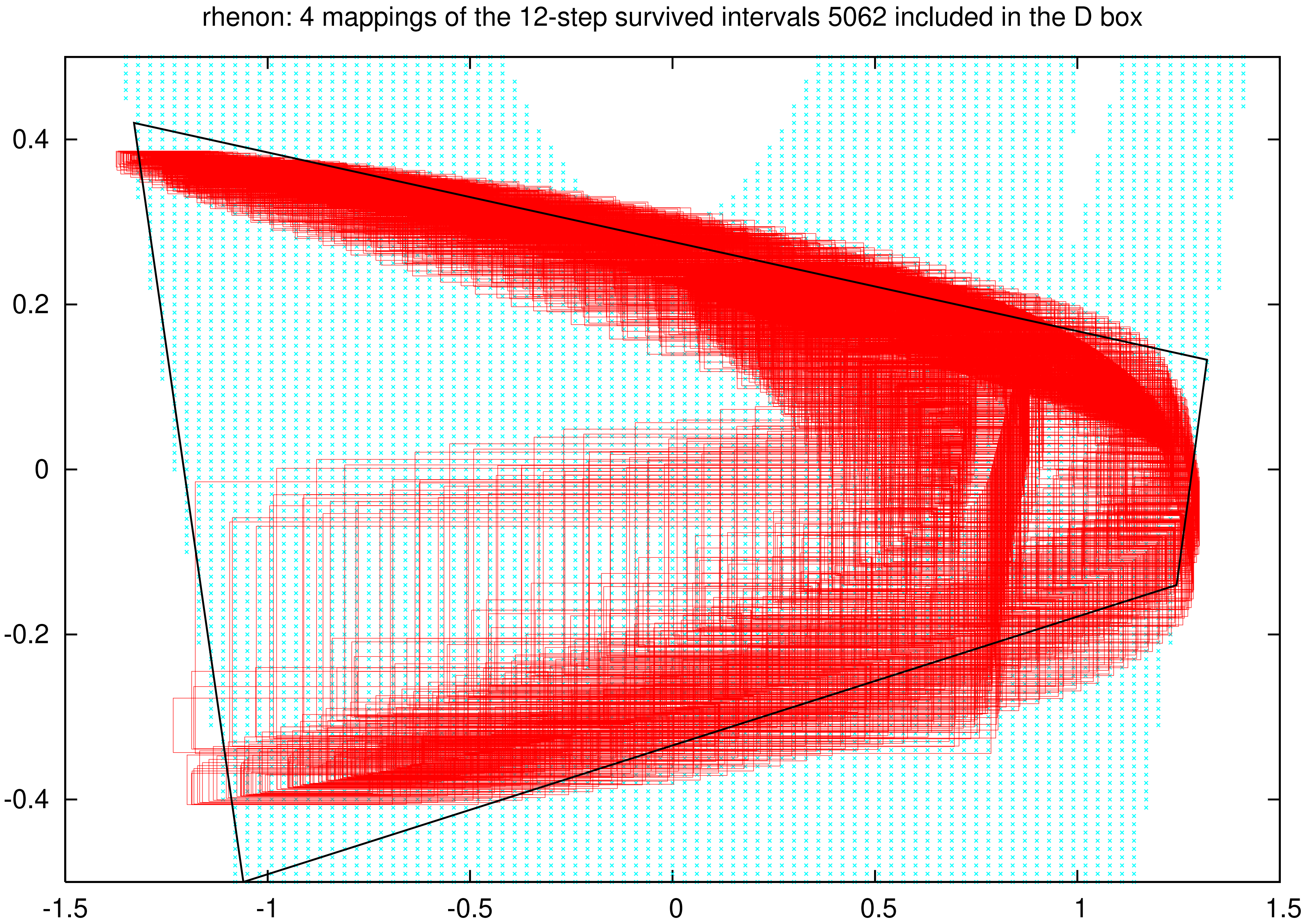
rhenon: 1 mapping of the 12-step survived intervals 5062 included in the D box




rhenon: 2 mappings of the 12-step survived intervals 5062 included in the D box
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5

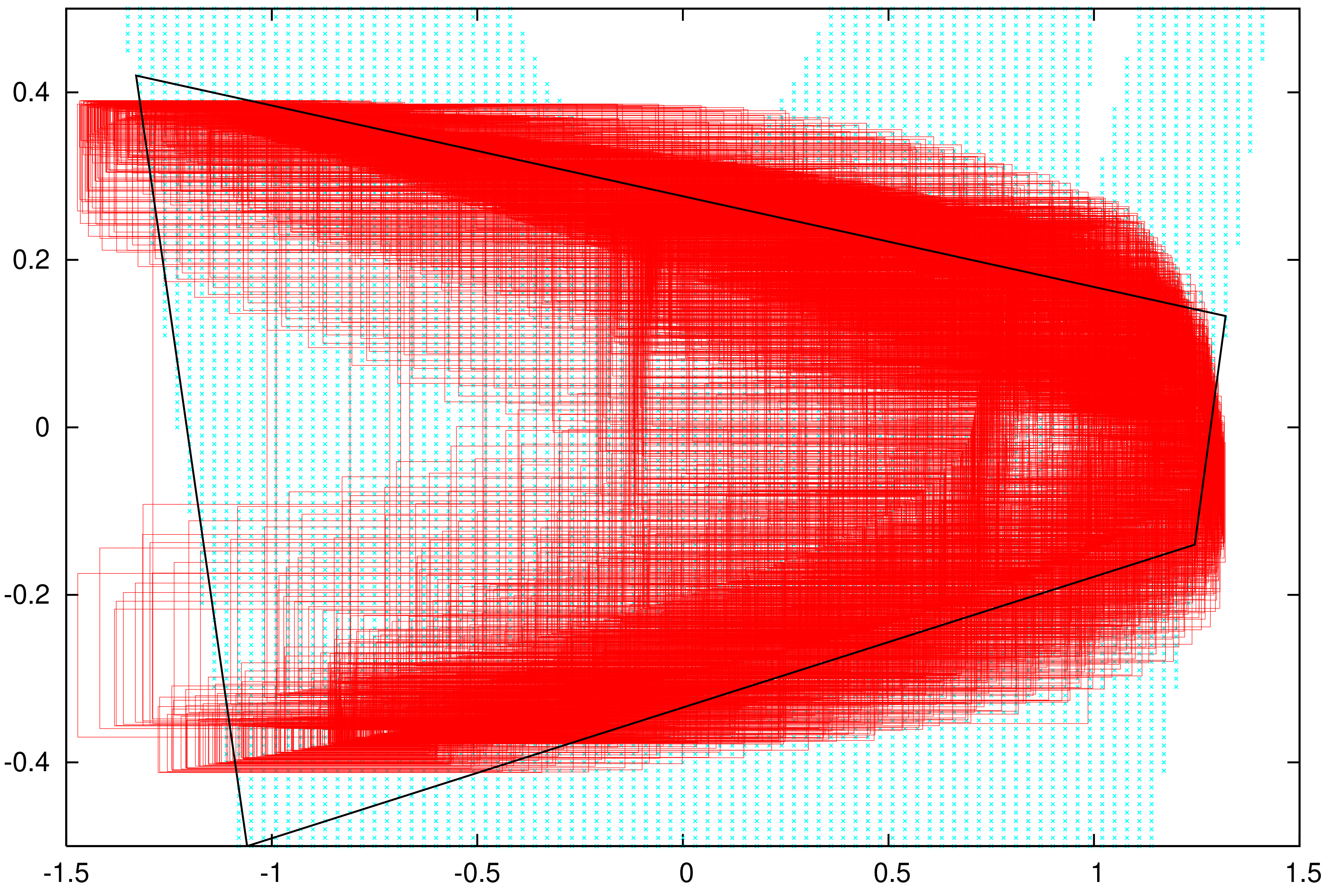
rhenon: 3 mappings of the 12-step survived intervals 5062 included in the D box



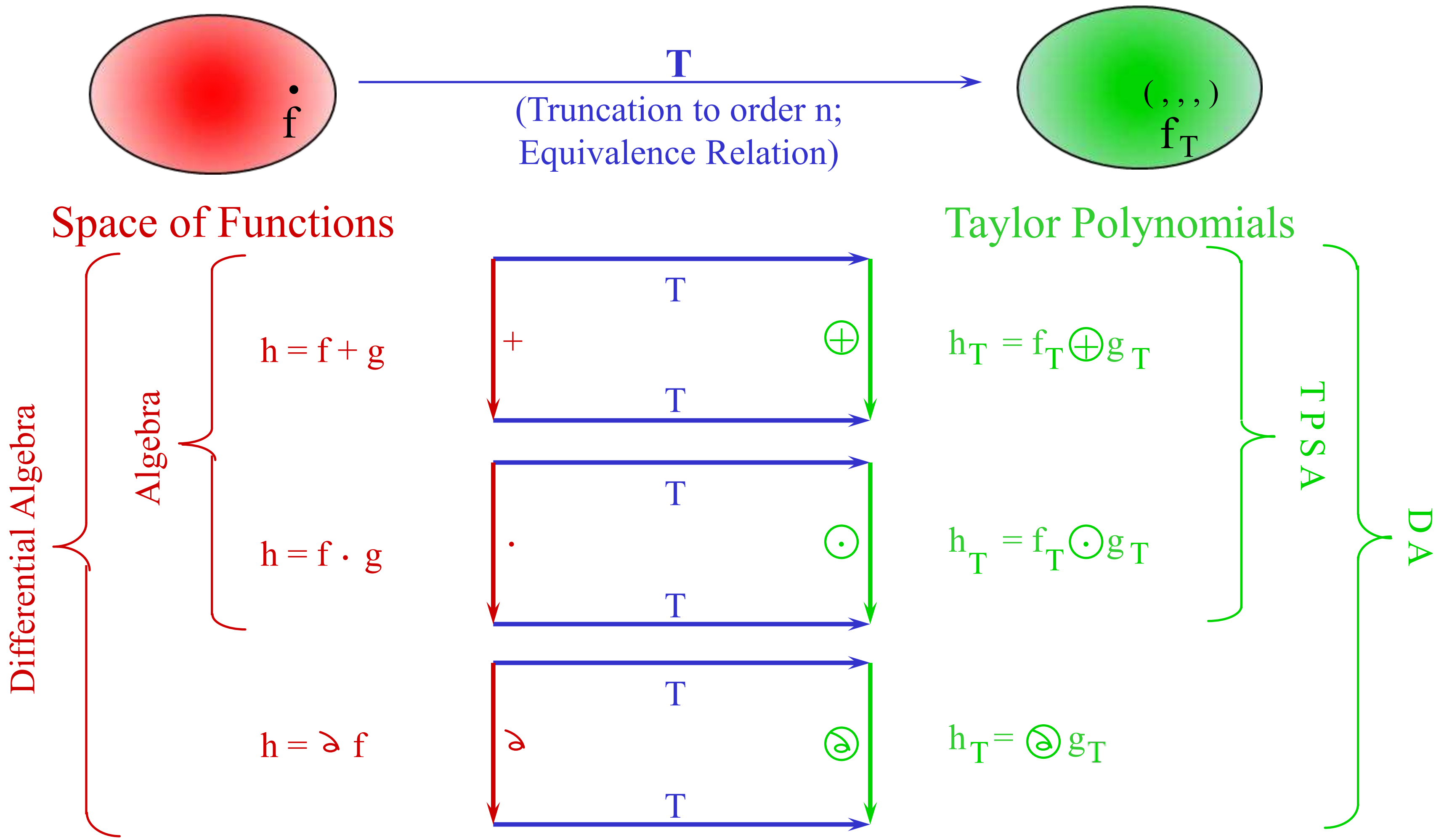


rhenon: 4 mappings of the 12-step survived intervals 5062 included in the D box
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5

rhenon: 5 mappings of the 12-step survived intervals 5062 included in the D box



Function Algebras



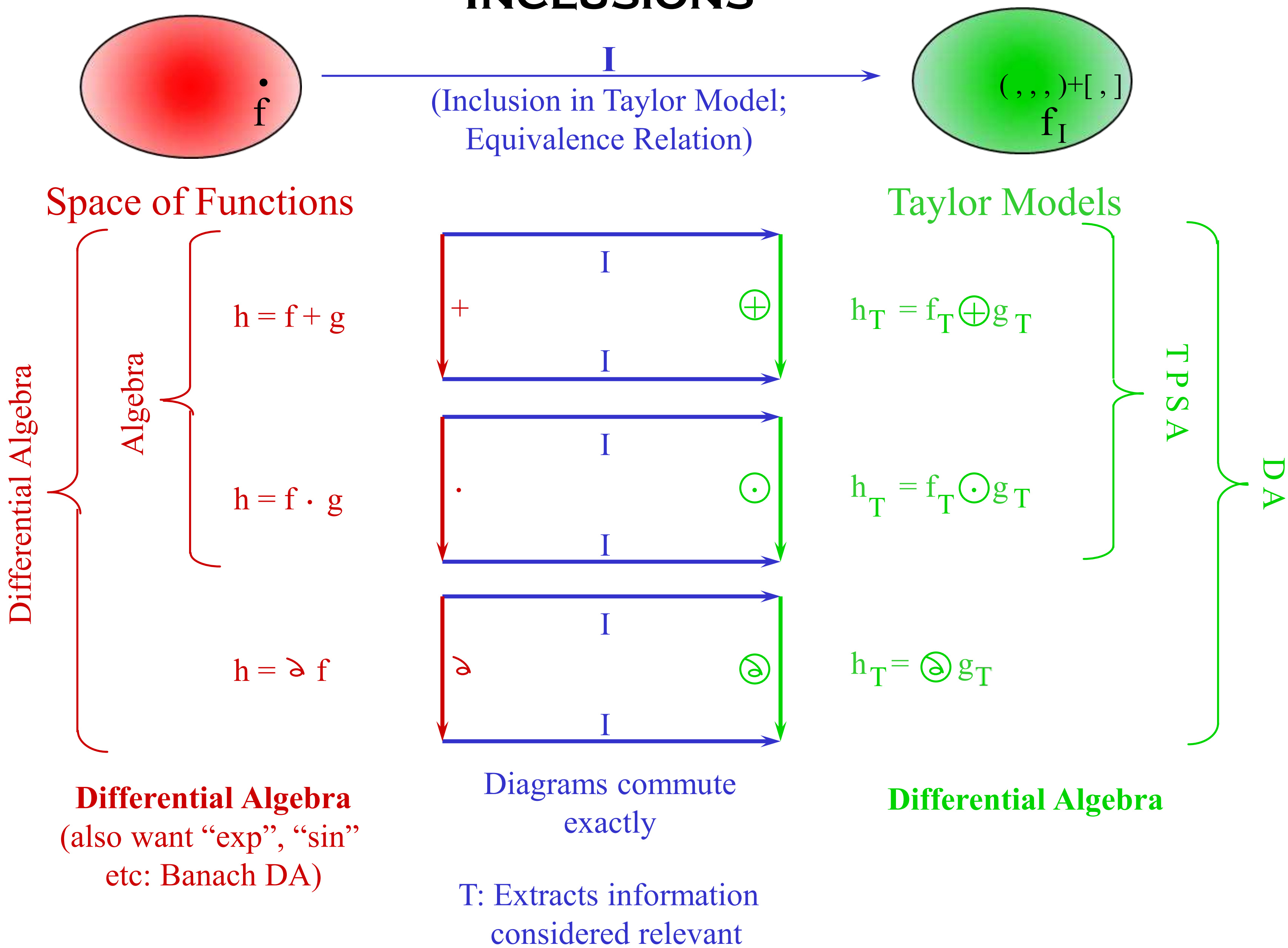
Differential Algebra
(also want "exp", "sin" etc: Banach DA)

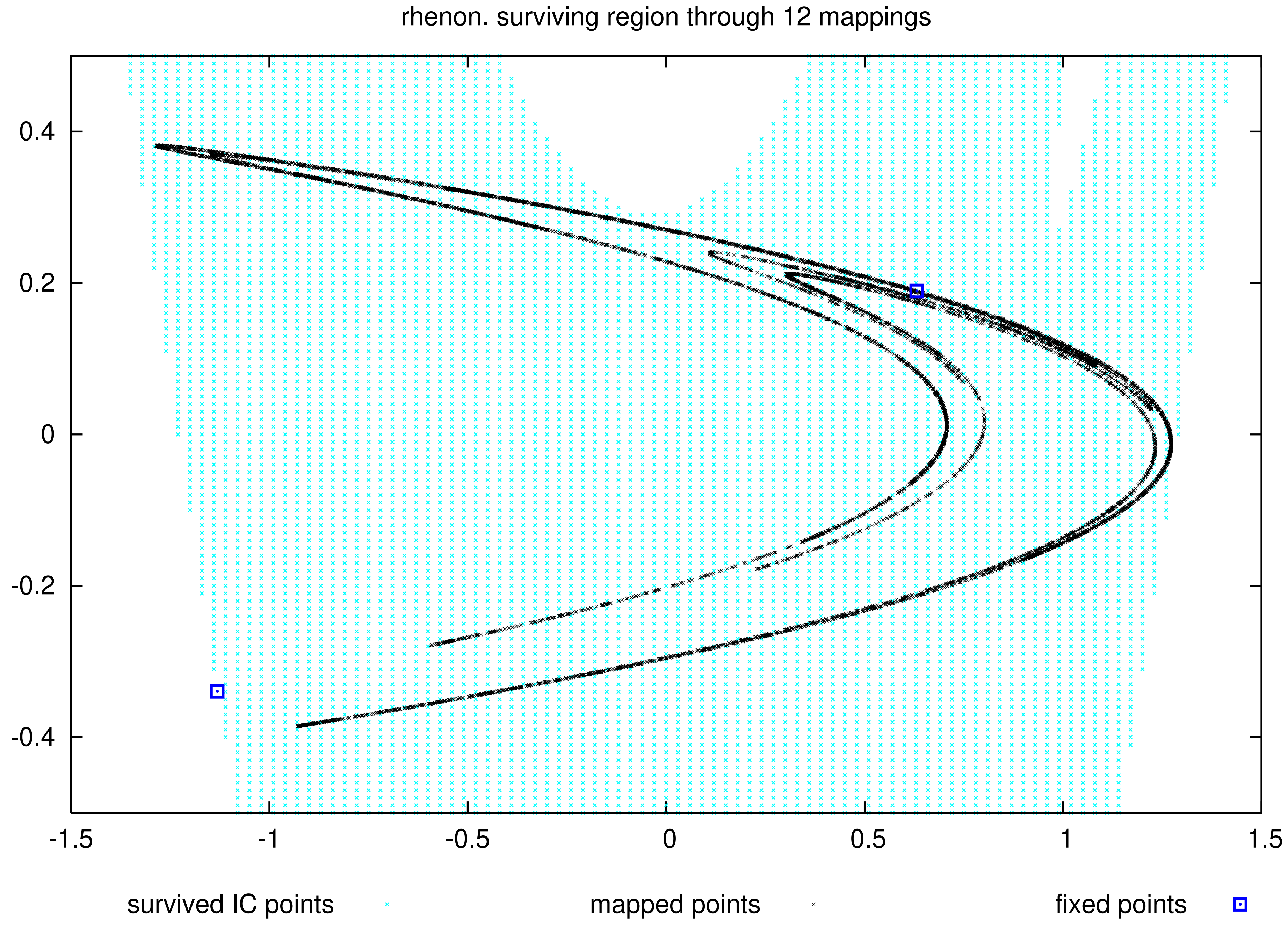
Diagrams commute exactly

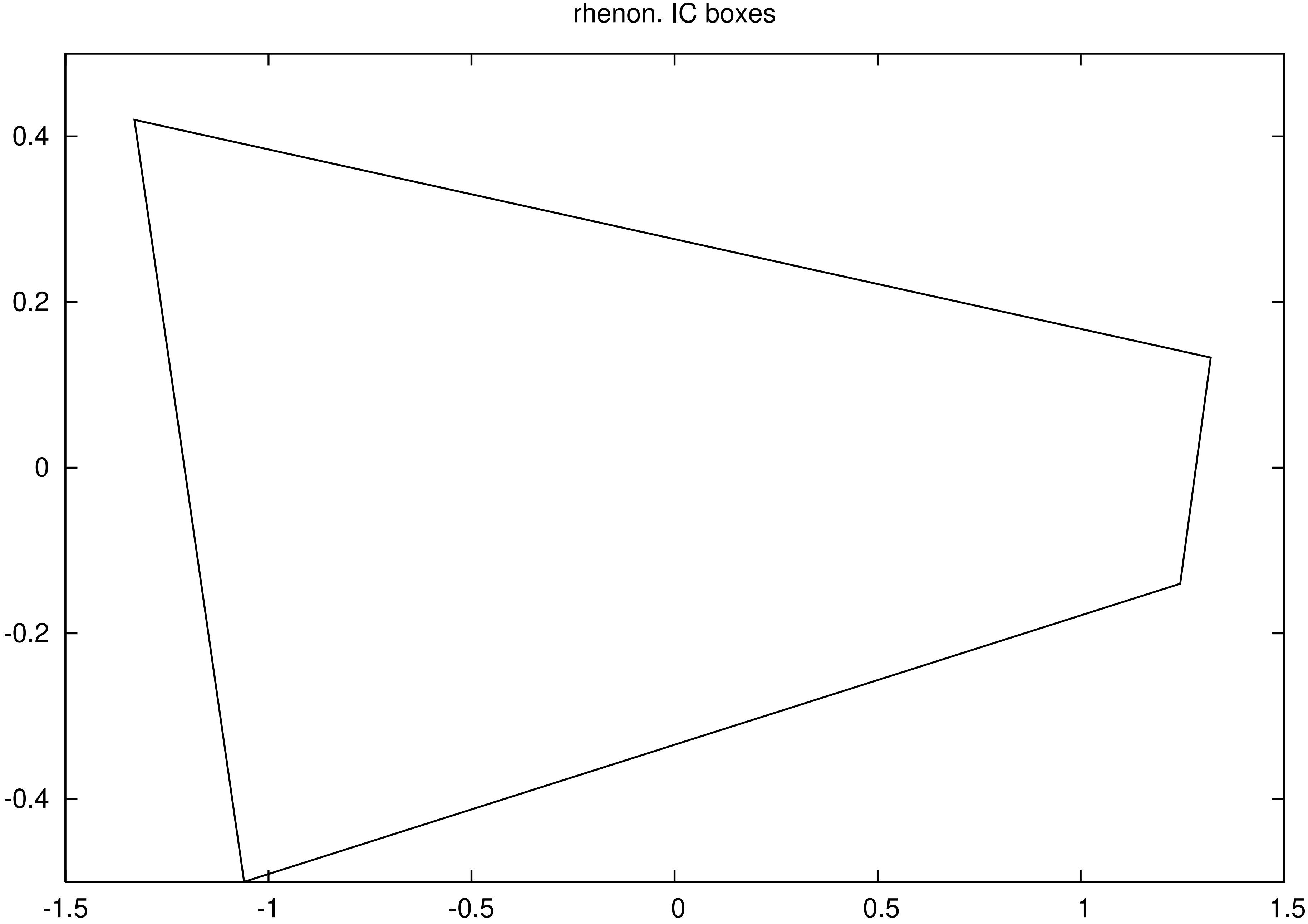
T: Extracts information considered relevant

Differential Algebra
(even Banach DA)

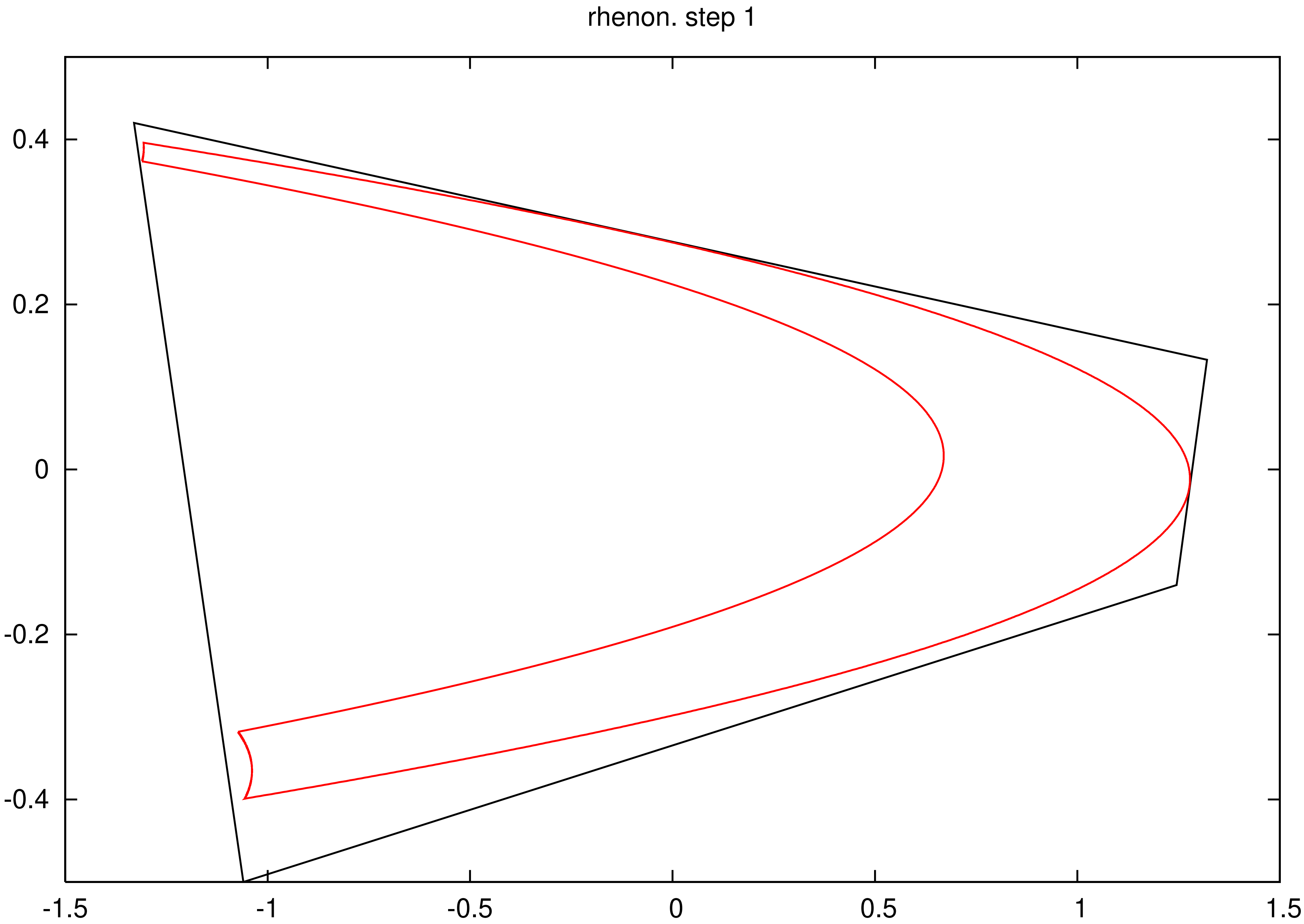
Function Algebra Inclusions



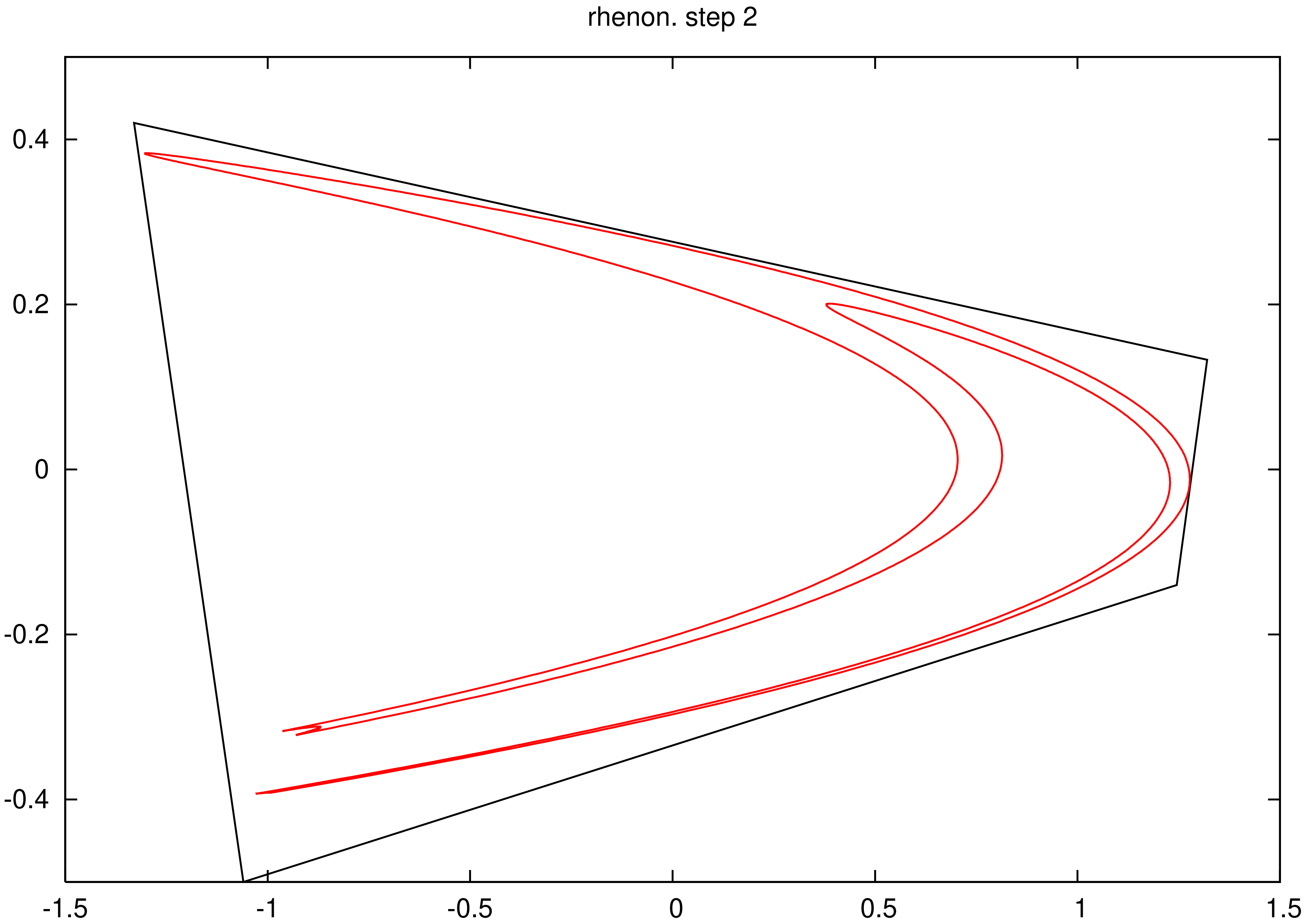

rhenon. surviving region through 12 mappings
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5
survived IC points
mapped points
fixed points

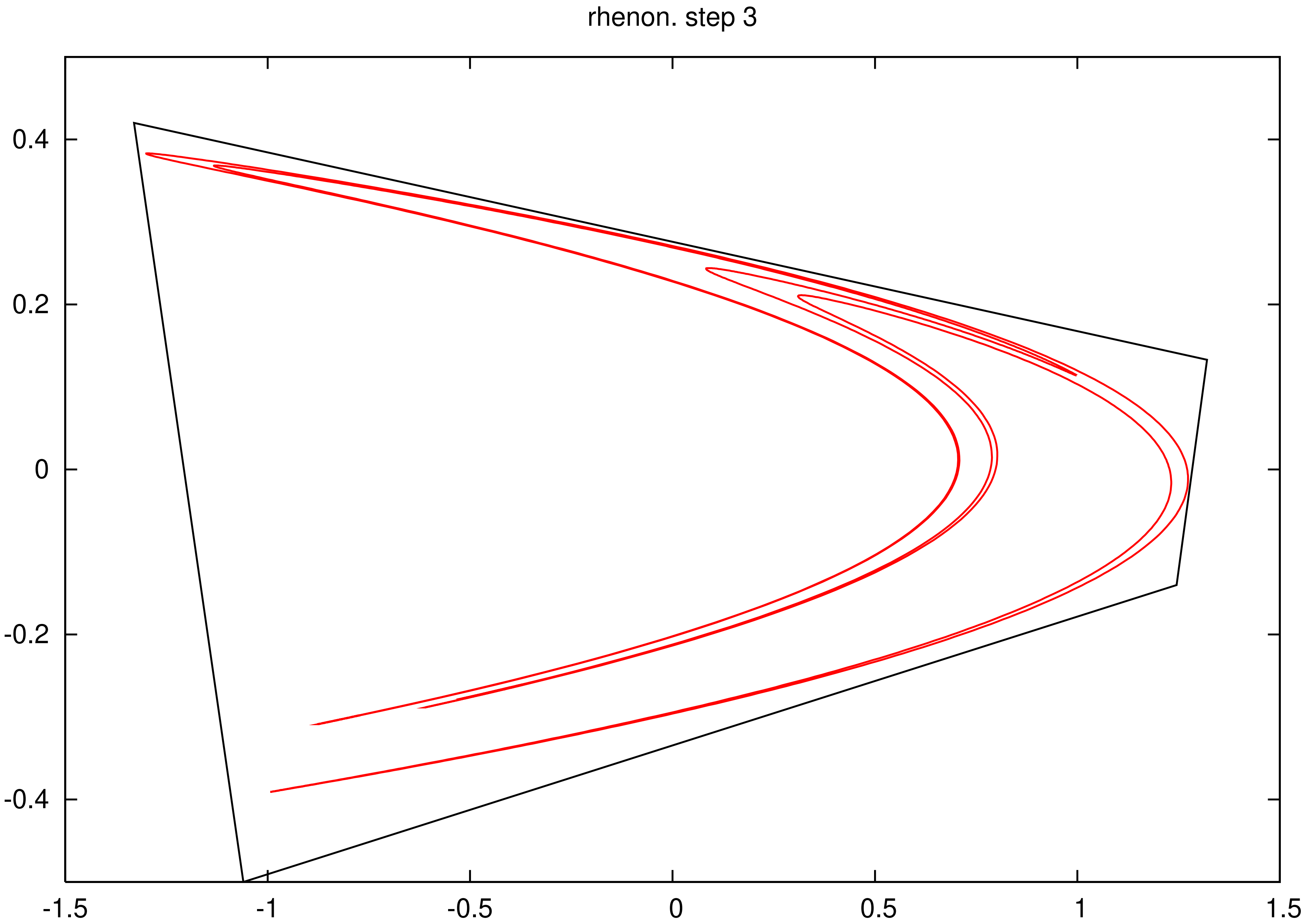


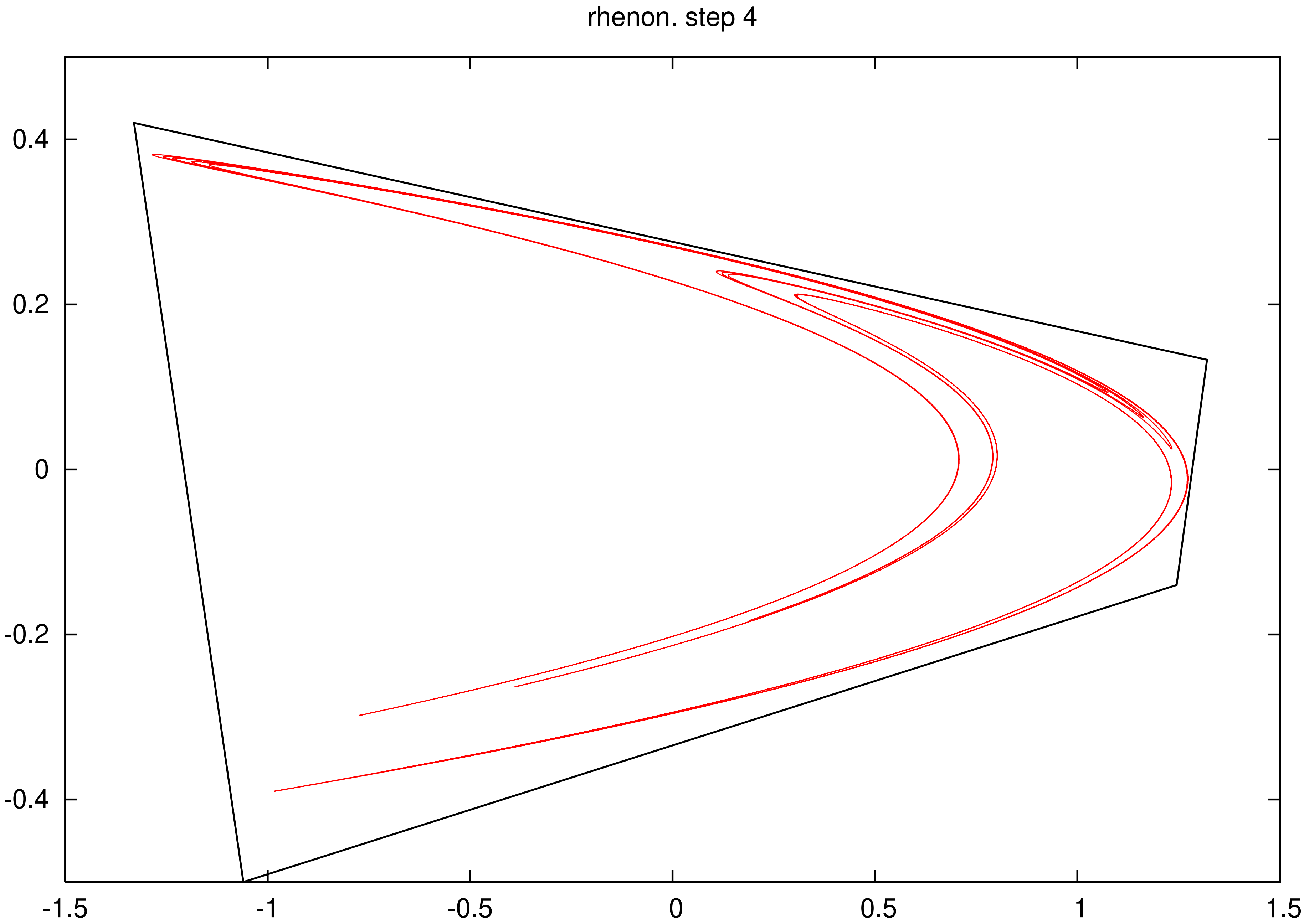
rhenon. IC boxes
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5

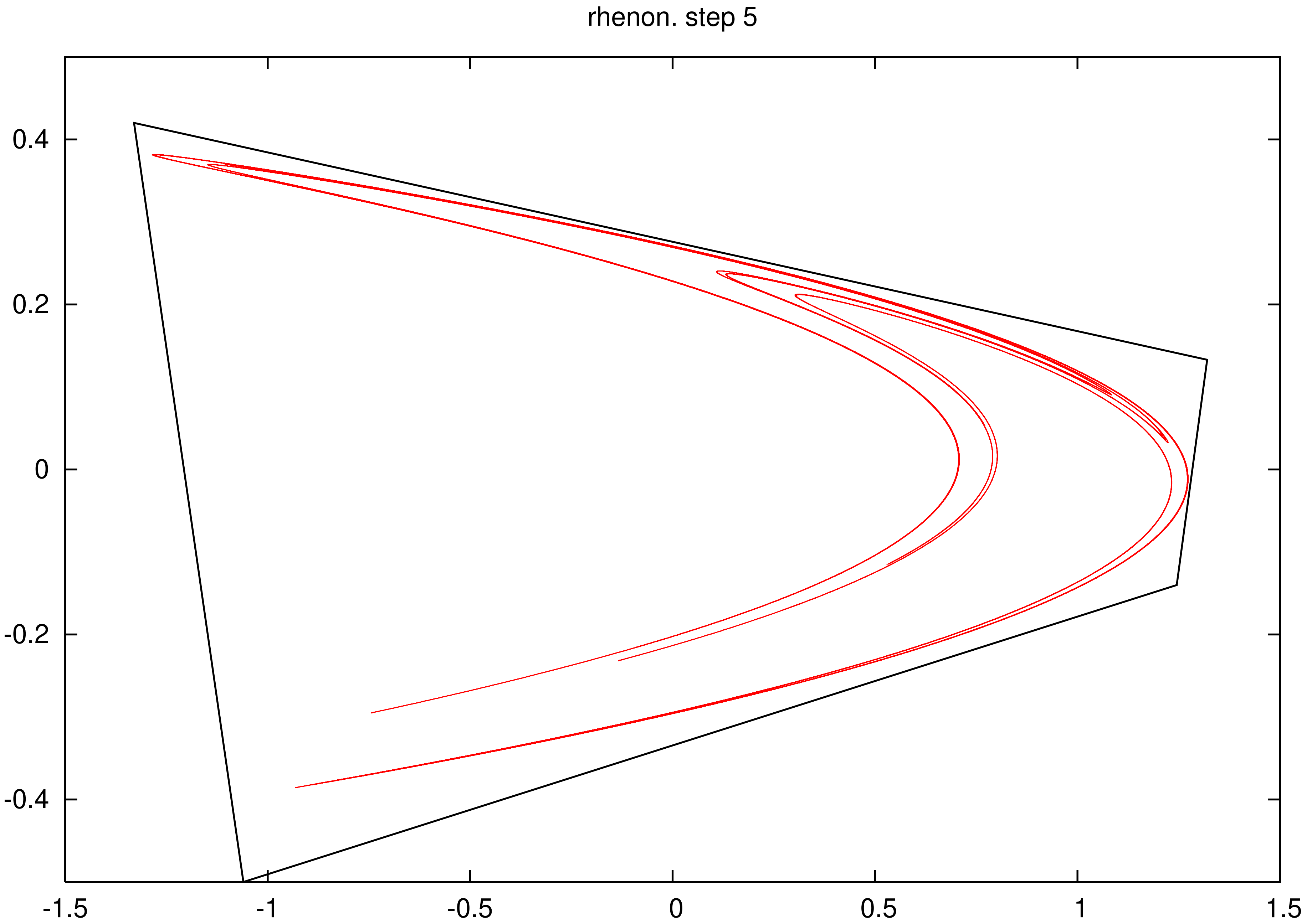


rhenon. step 1
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5


rhenon. step 2
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5

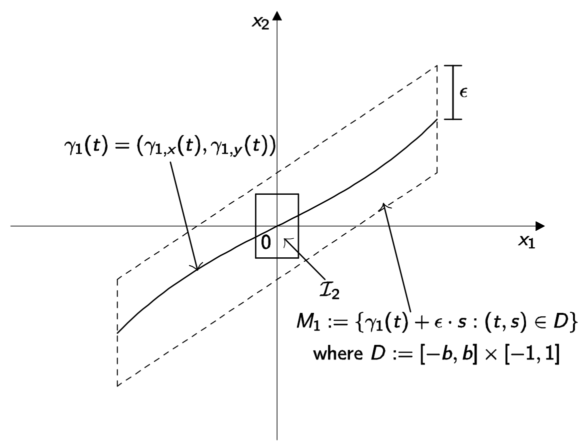

rhenon. step 3
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5


rhenon. step 4
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5


rhenon. step 5
0.4
0.2
0
-0.2
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5

Rigorous Unstable Manifold Enclosures I

Goal: Find collection of hopefully very narrow Taylor models that contain a hopefully long stretch of unstable manifold.



Begin with unstable manifold near fixed point:

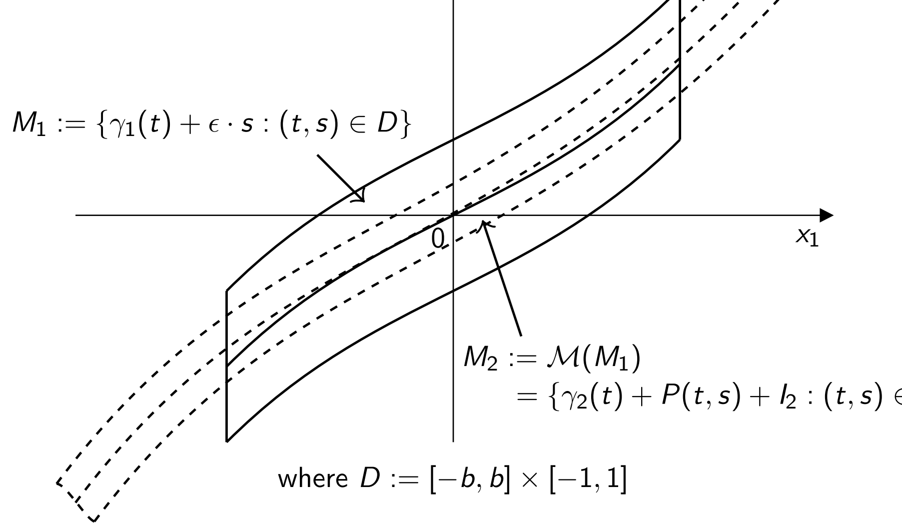
- Obtain approximate polynomial path $\gamma(t)$ as image of normal form $\vec{e}_1$ axis
- Put "test tube" around $\gamma(t)$ to get $\gamma(t) + \varepsilon \cdot s \cdot \vec{e}_2$. Practical choice: $\varepsilon = 10^{-14}$

Rigorous Unstable Manifold Enclosures II

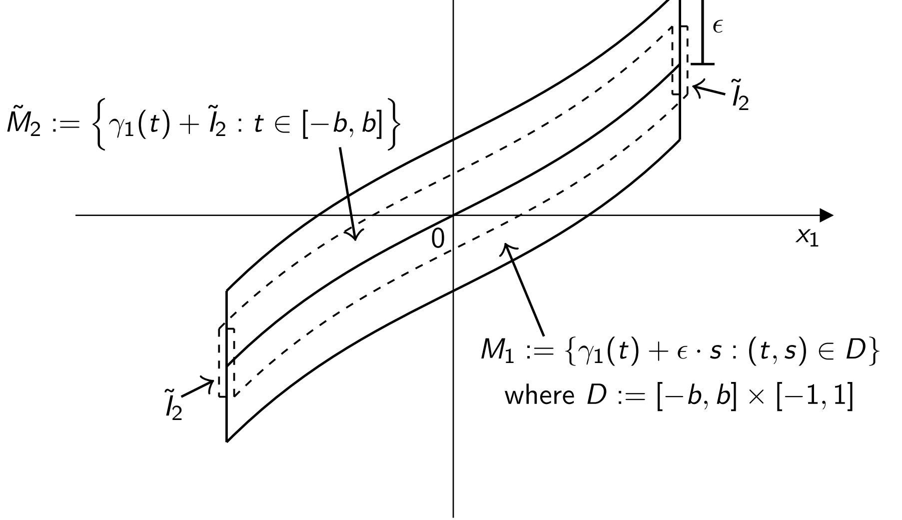
- Verify that $M(\gamma(t) + \varepsilon \cdot s \cdot \vec{e}_2)$ leaves "test tube" only at ends. Very useful for that:
  1. $M(\gamma(t)) =_n \gamma(\lambda_1 \cdot t)$, so orbit of γ is reproduced to order n
  2. M is contracting with λ_2 perpendicular to γ
  3. $\gamma(t)+\varepsilon\cdot s\cdot\vec{e}_2$ and its image under M can be treated rigorously in Taylor model arithmetic

After these steps, it is assured that

- The unstable manifold does NOT leave $\gamma(t) + \varepsilon \cdot s \cdot \vec{e}_2$ at top or bottom
- The unstable manifold DOES leave $\gamma(t) + \varepsilon \cdot s \cdot \vec{e}_2$ at the sides (easy to show)



$M_1 := \{\gamma_1(t) + \epsilon \cdot s : (t,s) \in D\}$
0
x_1
$M_2 := \mathcal{M}(M_1)$
$= \{\gamma_2(t) + P(t,s) + l_2 : (t,s) \in$
where $D := [-b,b] \times [-1,1]$



ϵ
$\tilde{I}_2$
$\tilde{M}_2 := \left\{ \gamma_1(t) + \tilde{I}_2 : t \in [-b, b] \right\}$
0
x_1
$M_1 := \{ \gamma_1(t) + \epsilon \cdot s : (t, s) \in D \}$
where $D := [-b, b] \times [-1, 1]$
$\tilde{I}_2$

Rigorous Unstable Manifold Enclosures III

Unstable manifold can be drawn as far as desired by

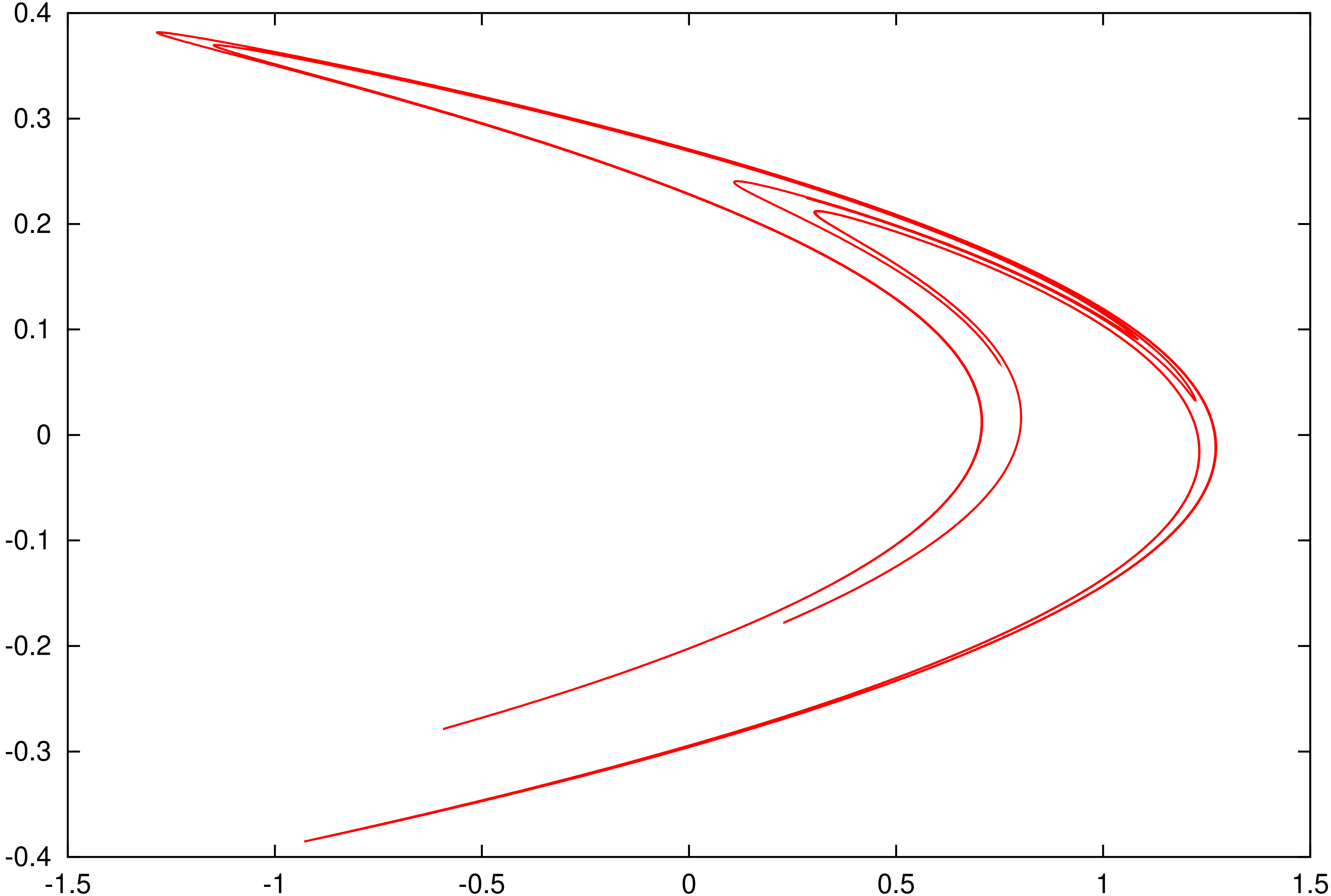
- Mapping $\gamma(t) + \varepsilon \cdot s \cdot \vec{e}_2$ through M repeatedly
- Splitting result if length $>$ tolerance

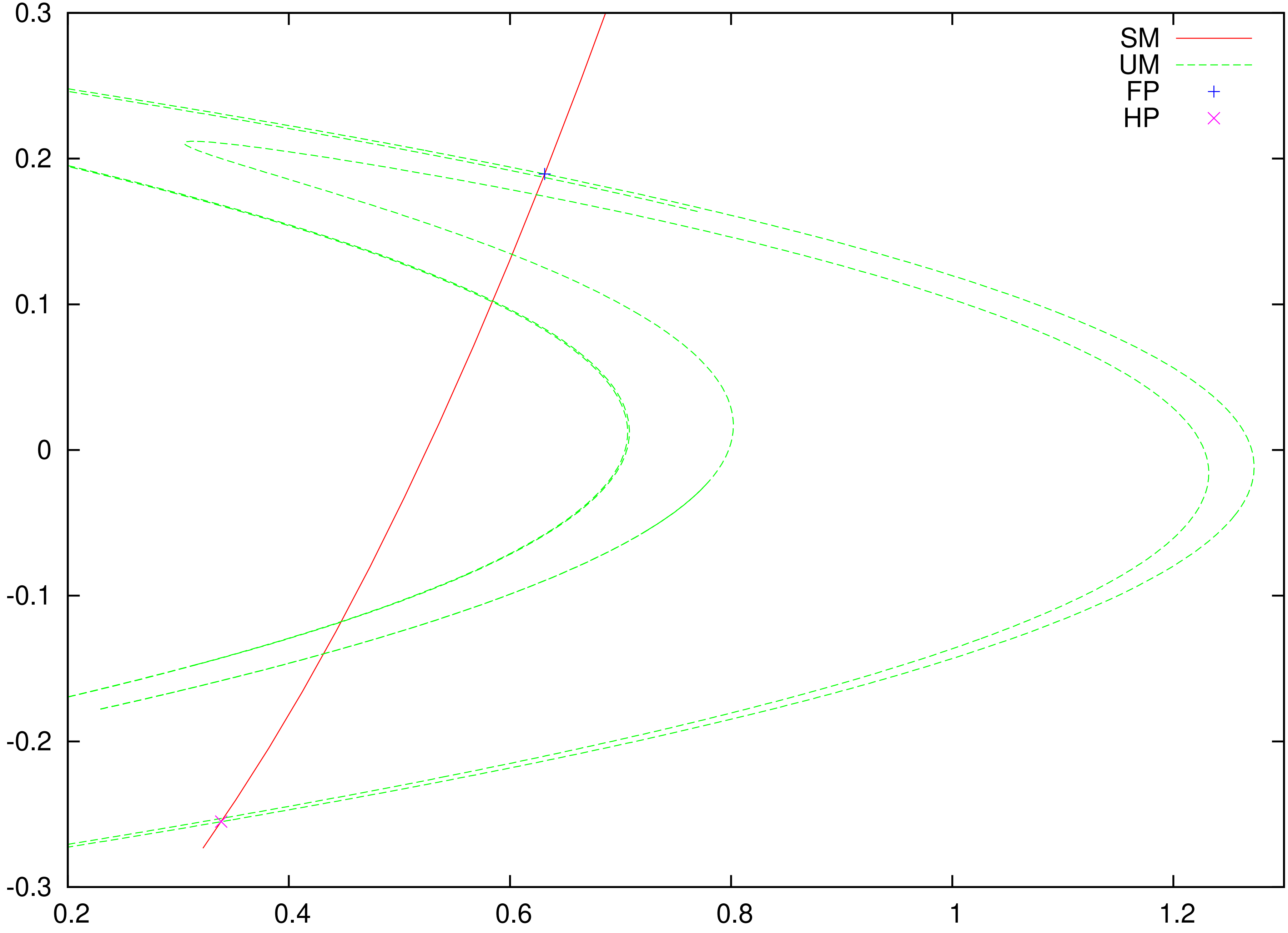
As a result, we obtain a collection of as many Taylor model as we wish, each of which

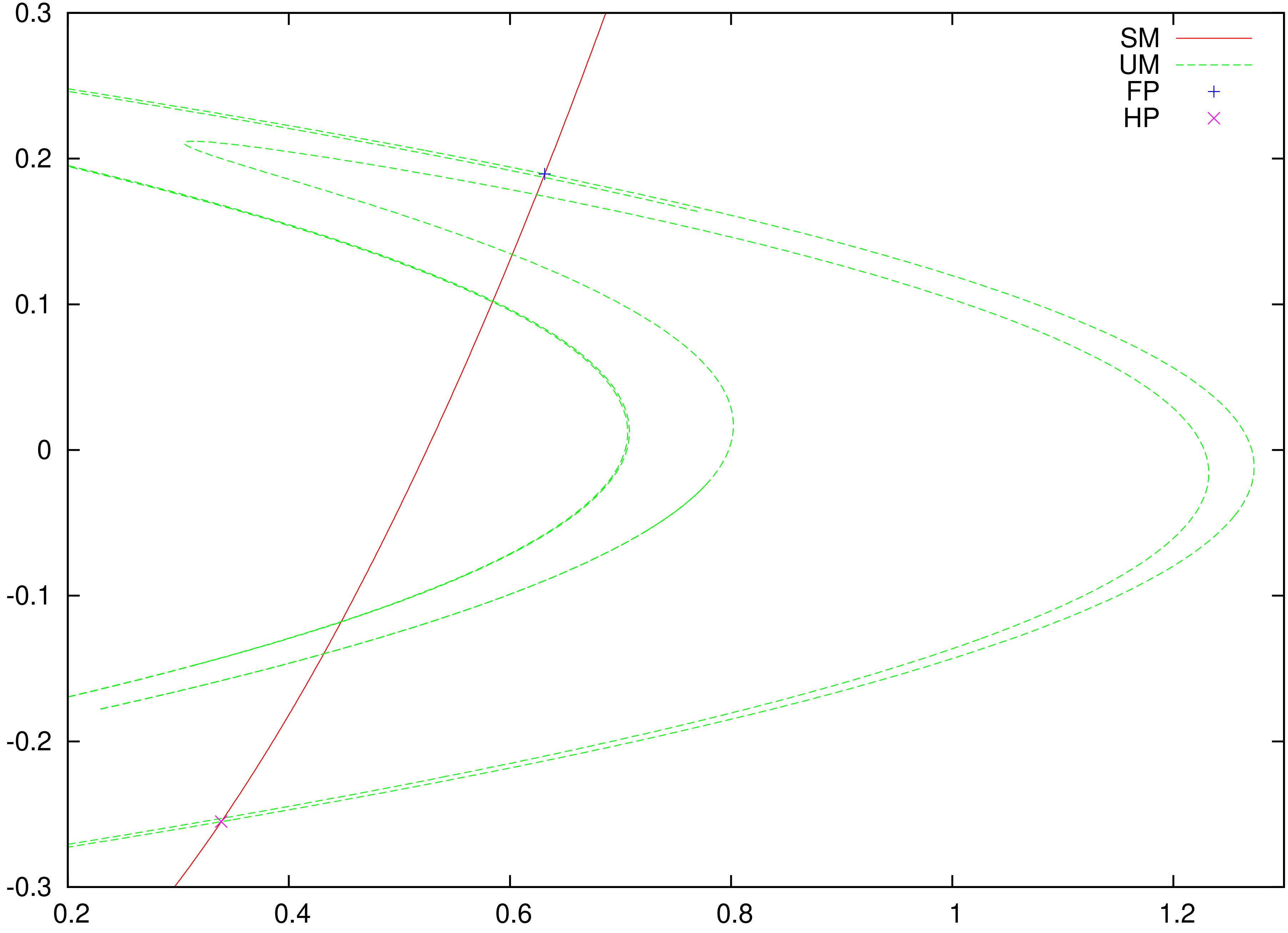
- Contains a piece of the unstable manifold
- The unstable manifold leaves through the "narrow sides"
- The unstable manifold does not leave through the "long sides"

By considering the inverse map, we can analogously obtain rigorous enclosures of the stable manifolds.

Unstable Manifold of a Henon map (a=1.4, b=0.3) represented by 450 pieces of TMs




SM
UM
FP
HP
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2



SM
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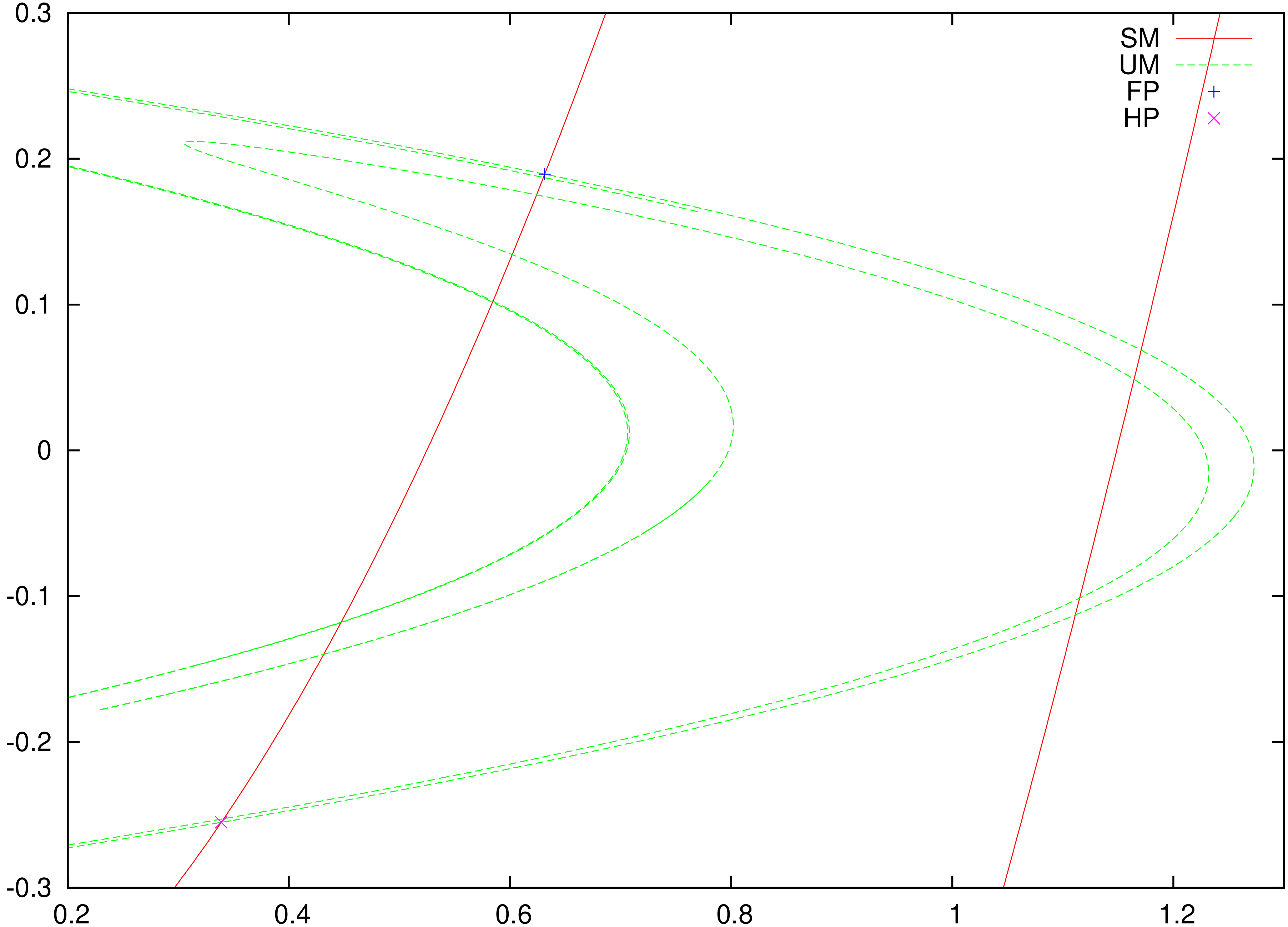

SM
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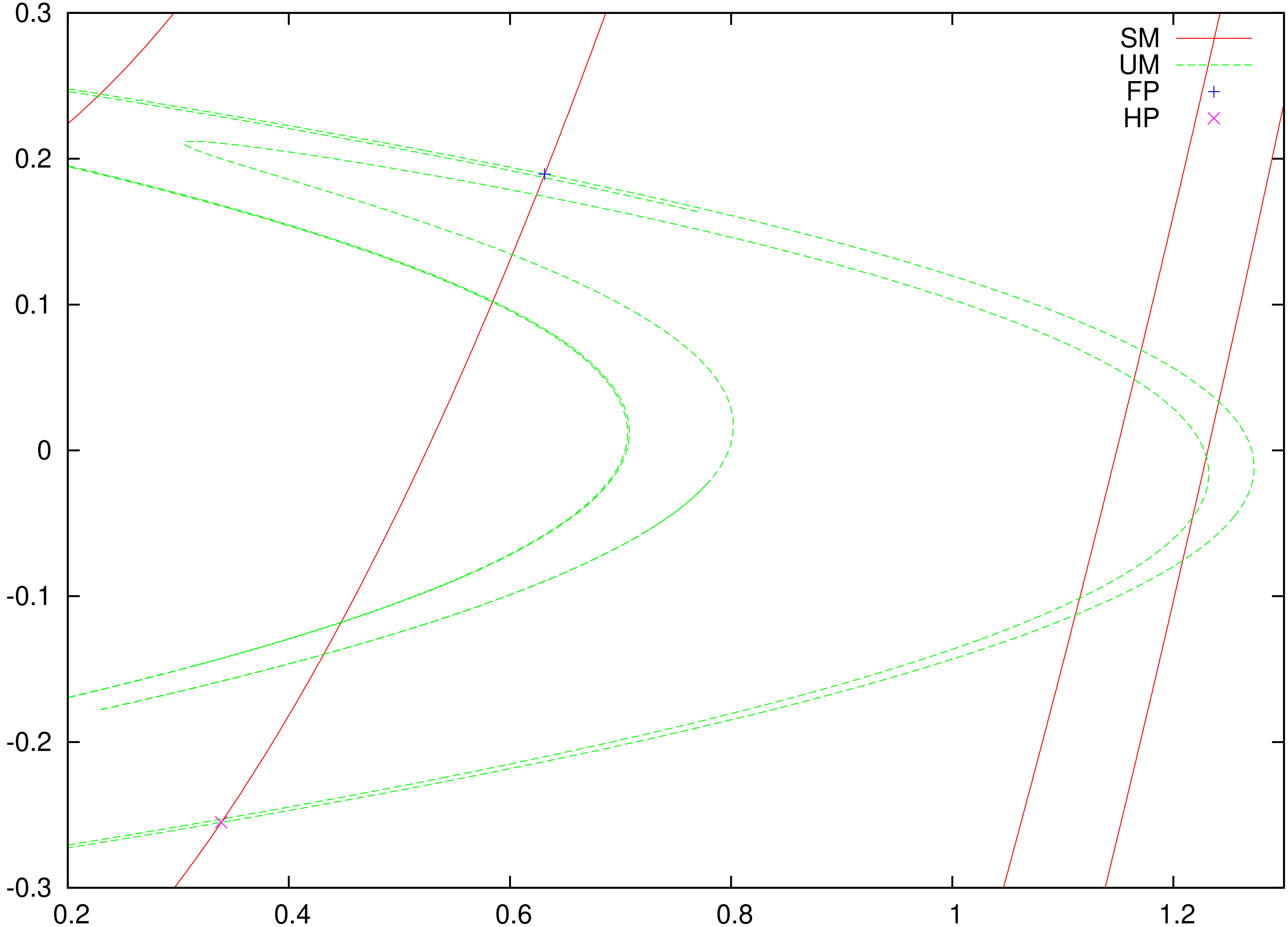
1.2


SM
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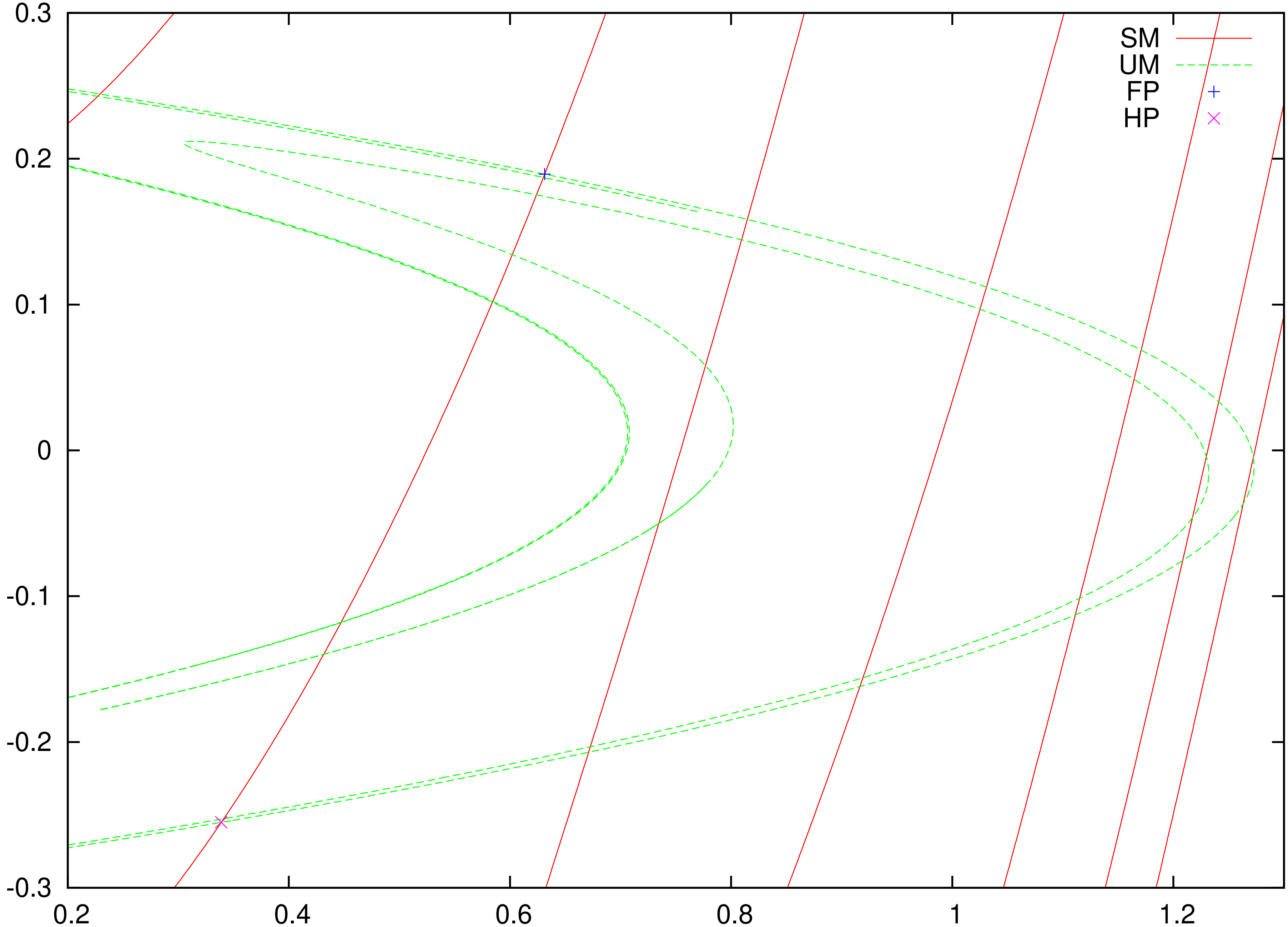

0.3
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0.2
0.4
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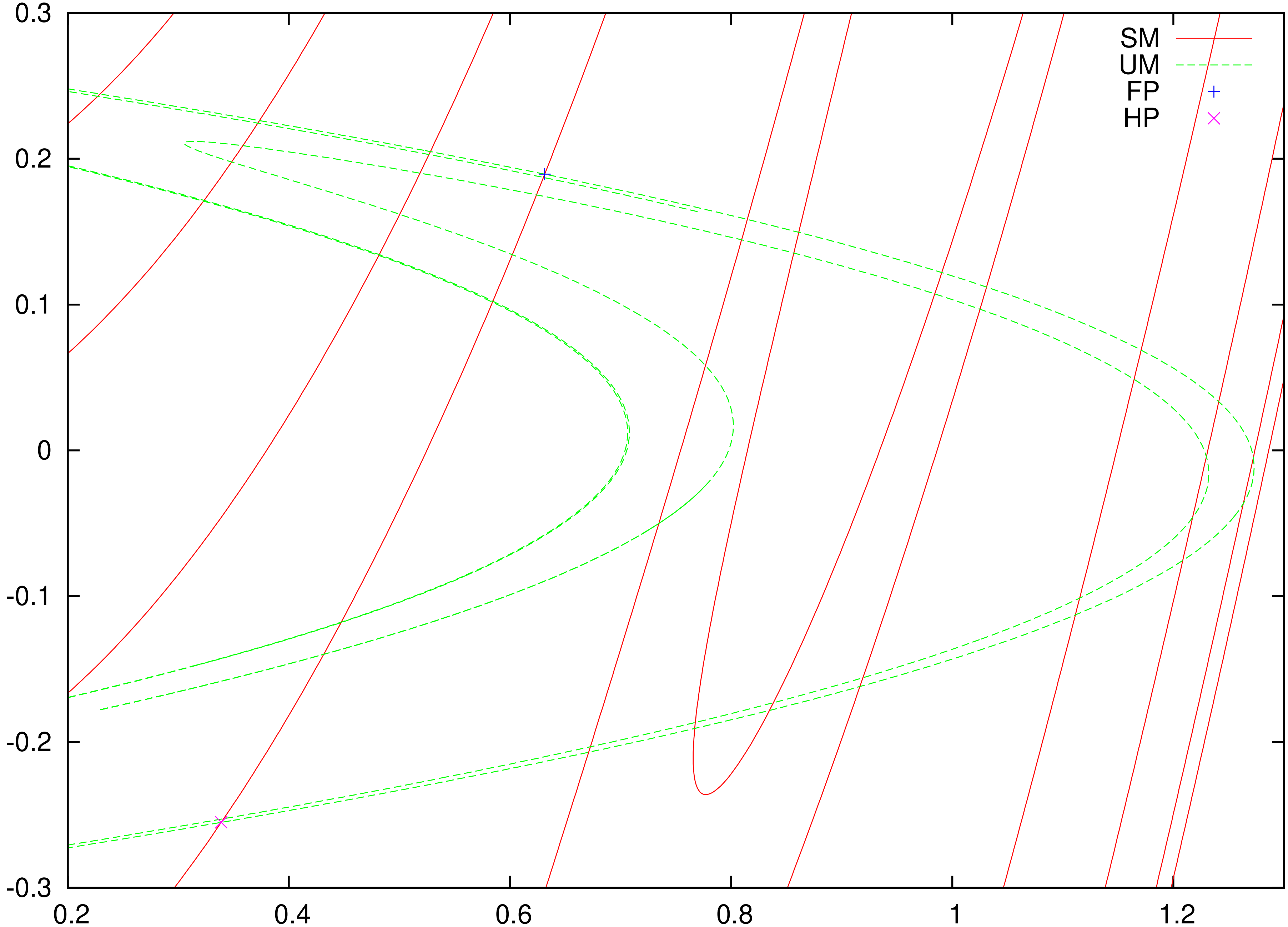
1.2
SM
UM
FP
HP



SM
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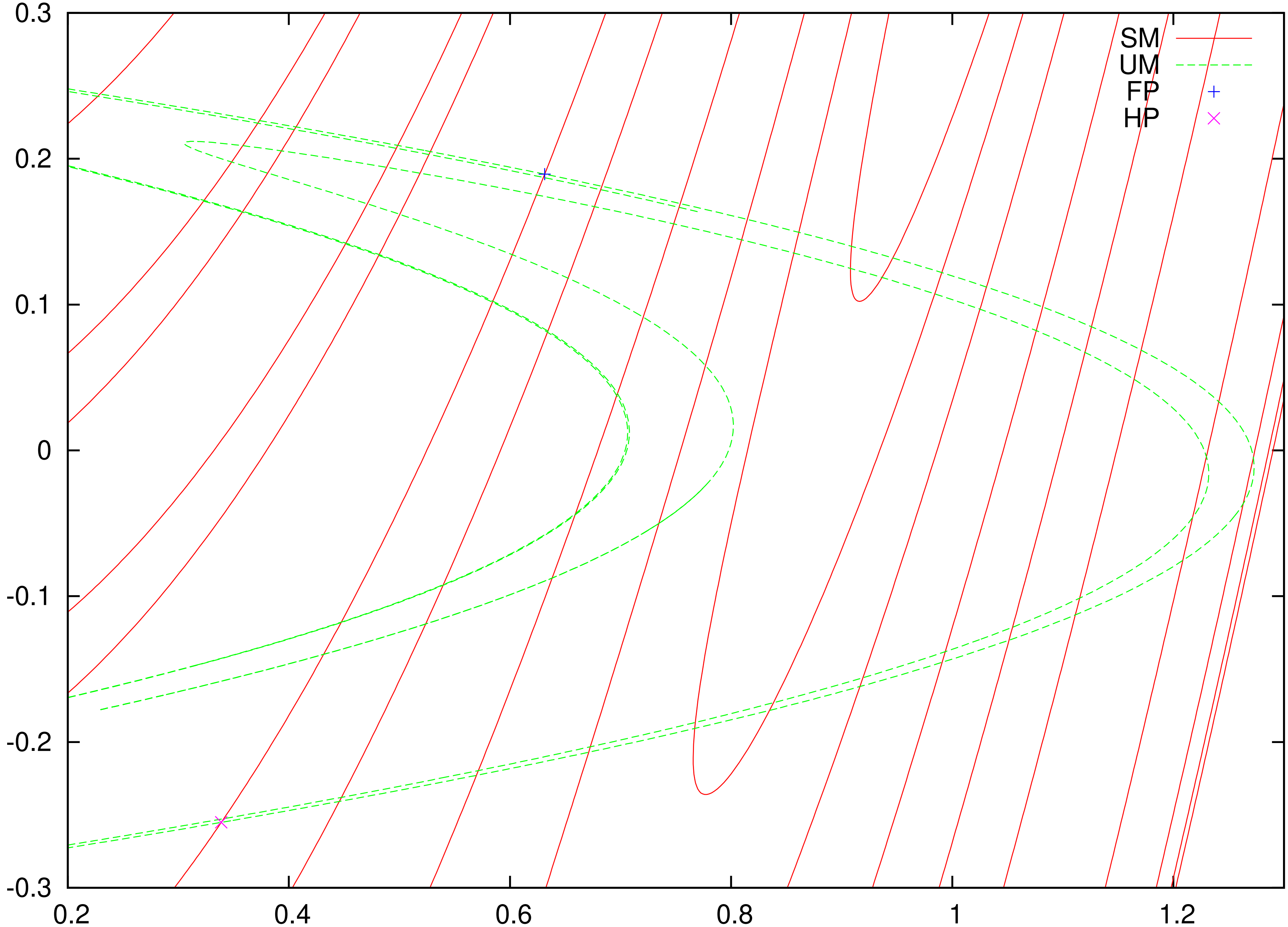

SM
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HP
0.3
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0.4
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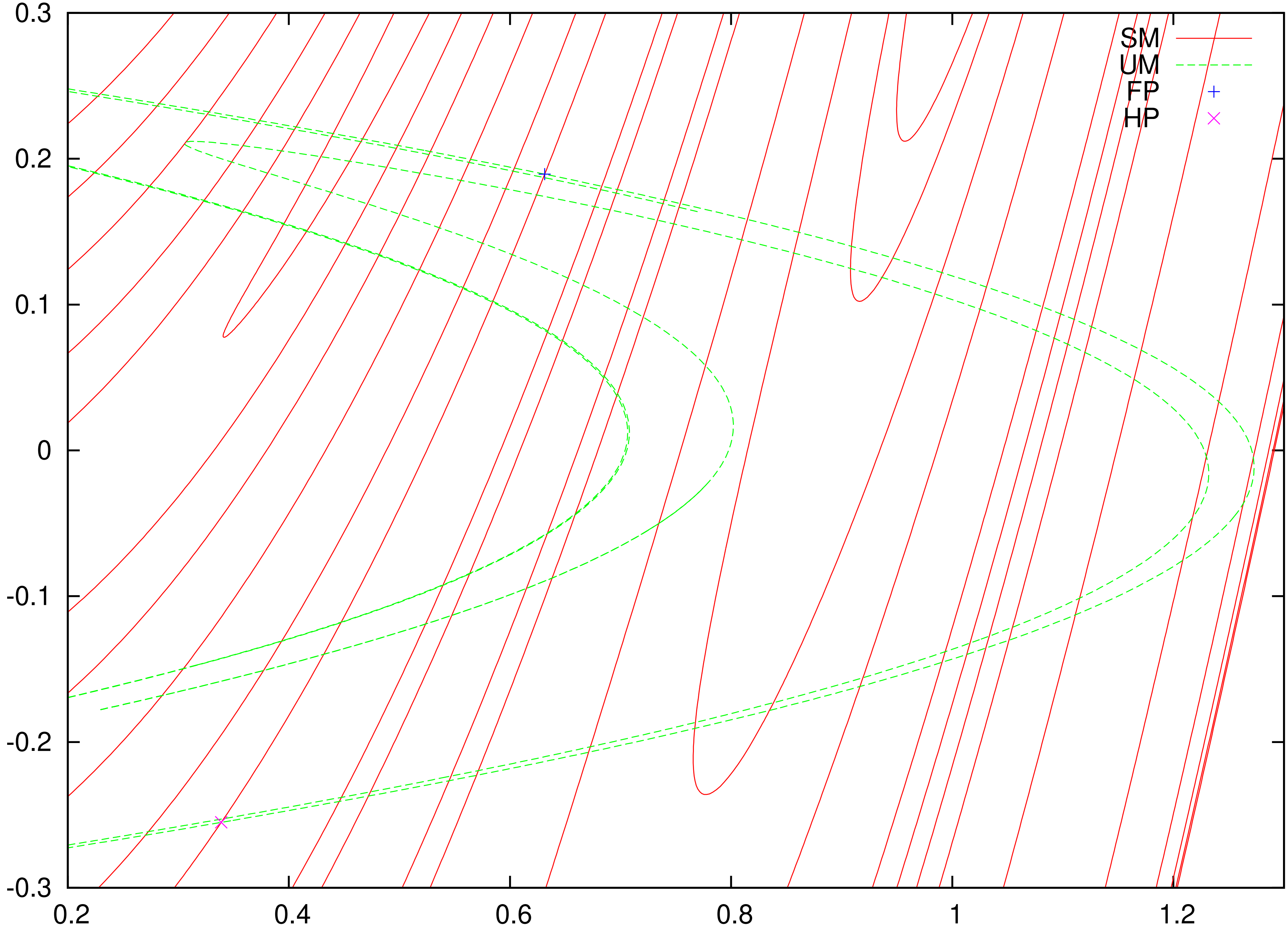
1.2



SM
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HP
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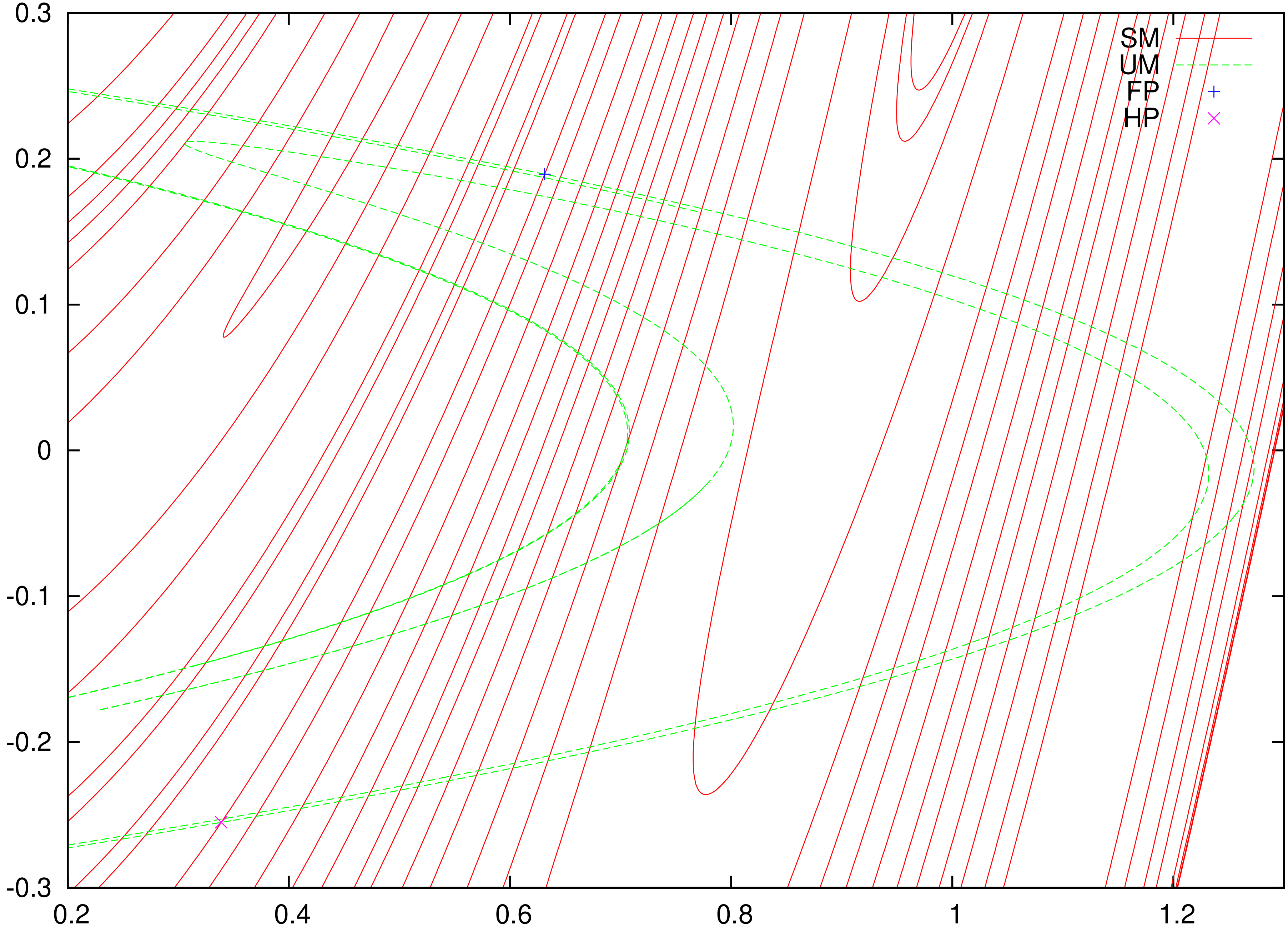
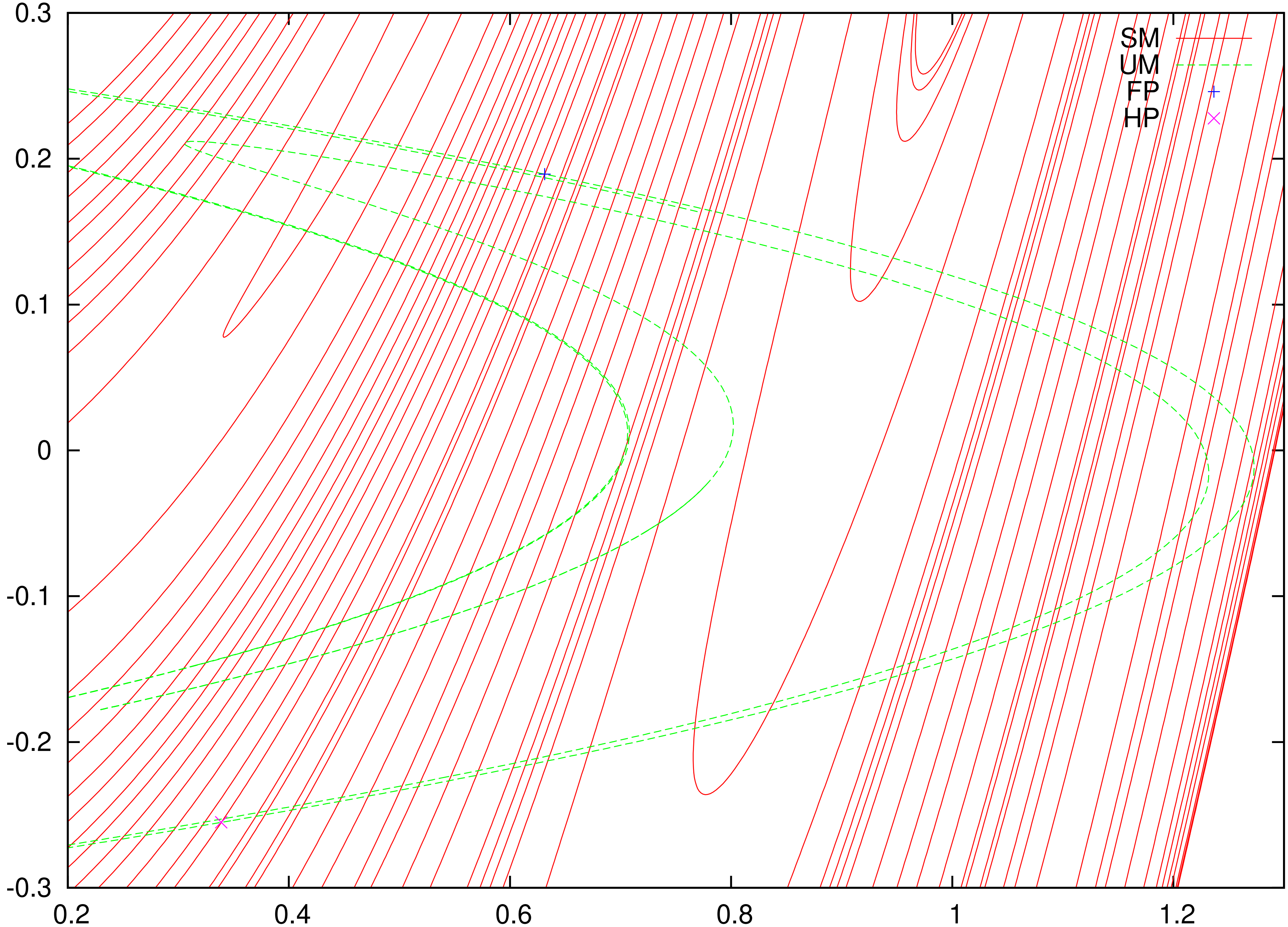


SM
UM
FP
HP
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2



0.3
0.2
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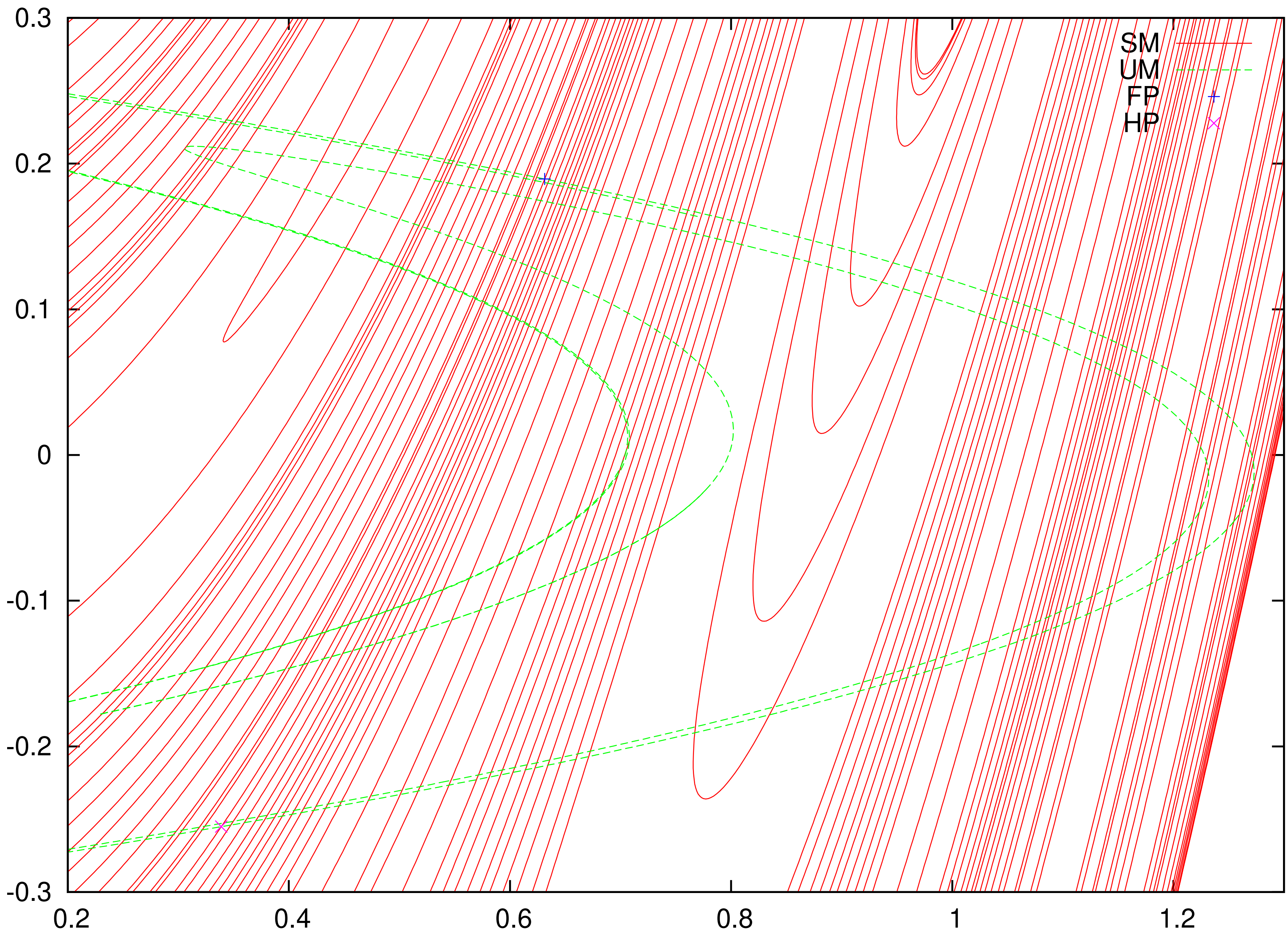
-0.3
0.2
0.4
0.6
0.8
1
1.2
SM
UM
FP
HP


SM
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FP
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0.1
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-0.1
-0.2
-0.3
0.2
0.4
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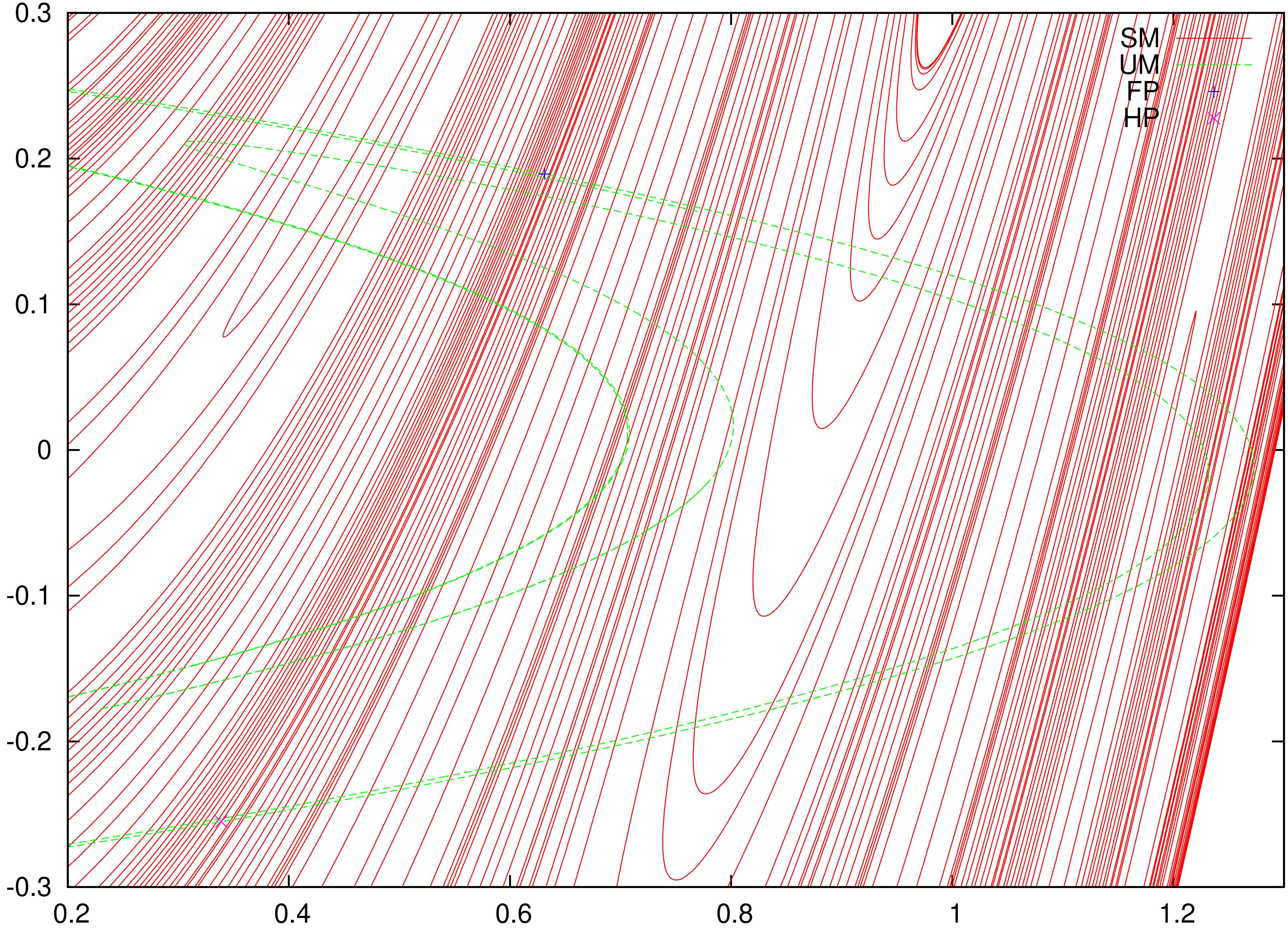
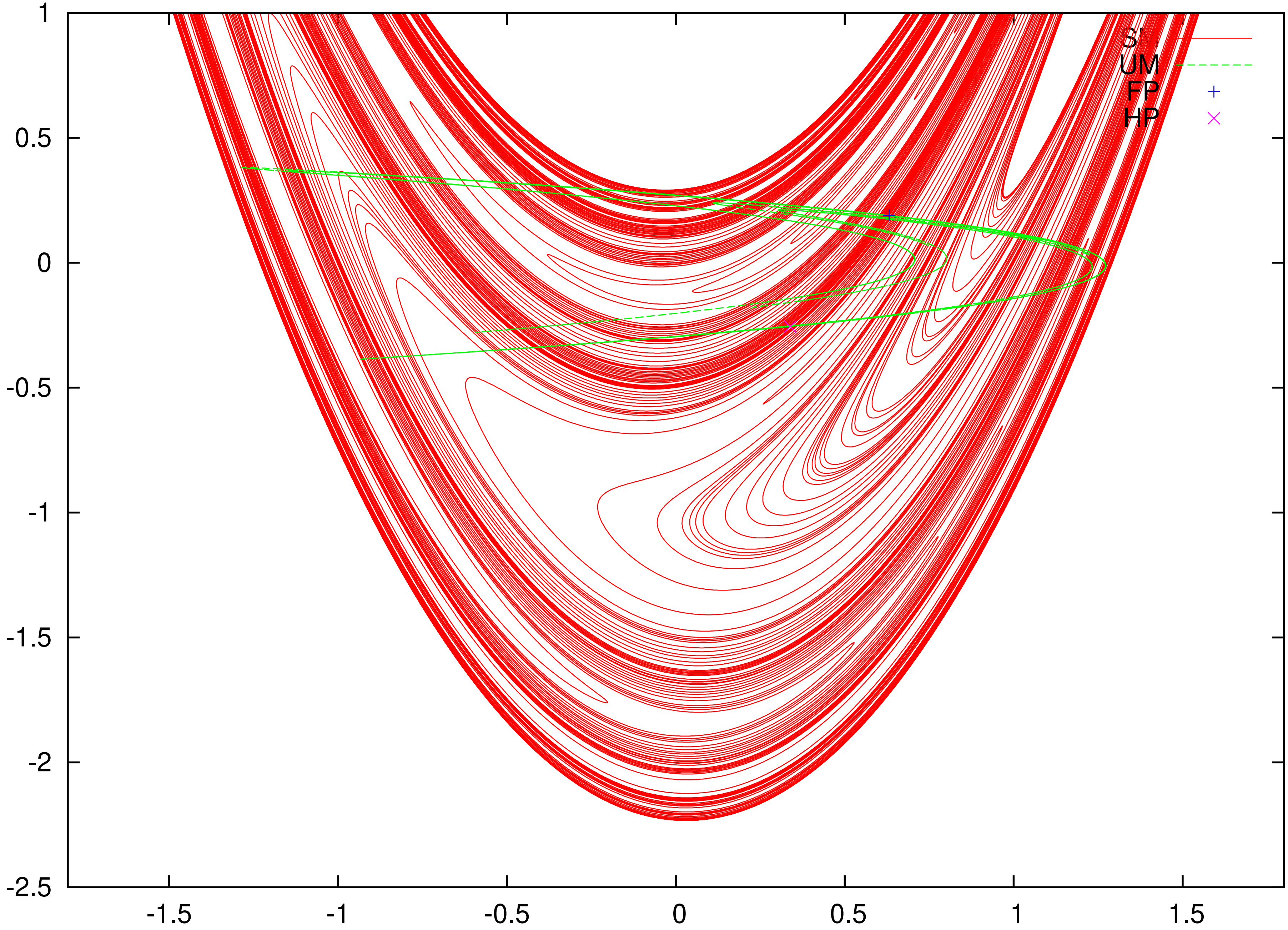
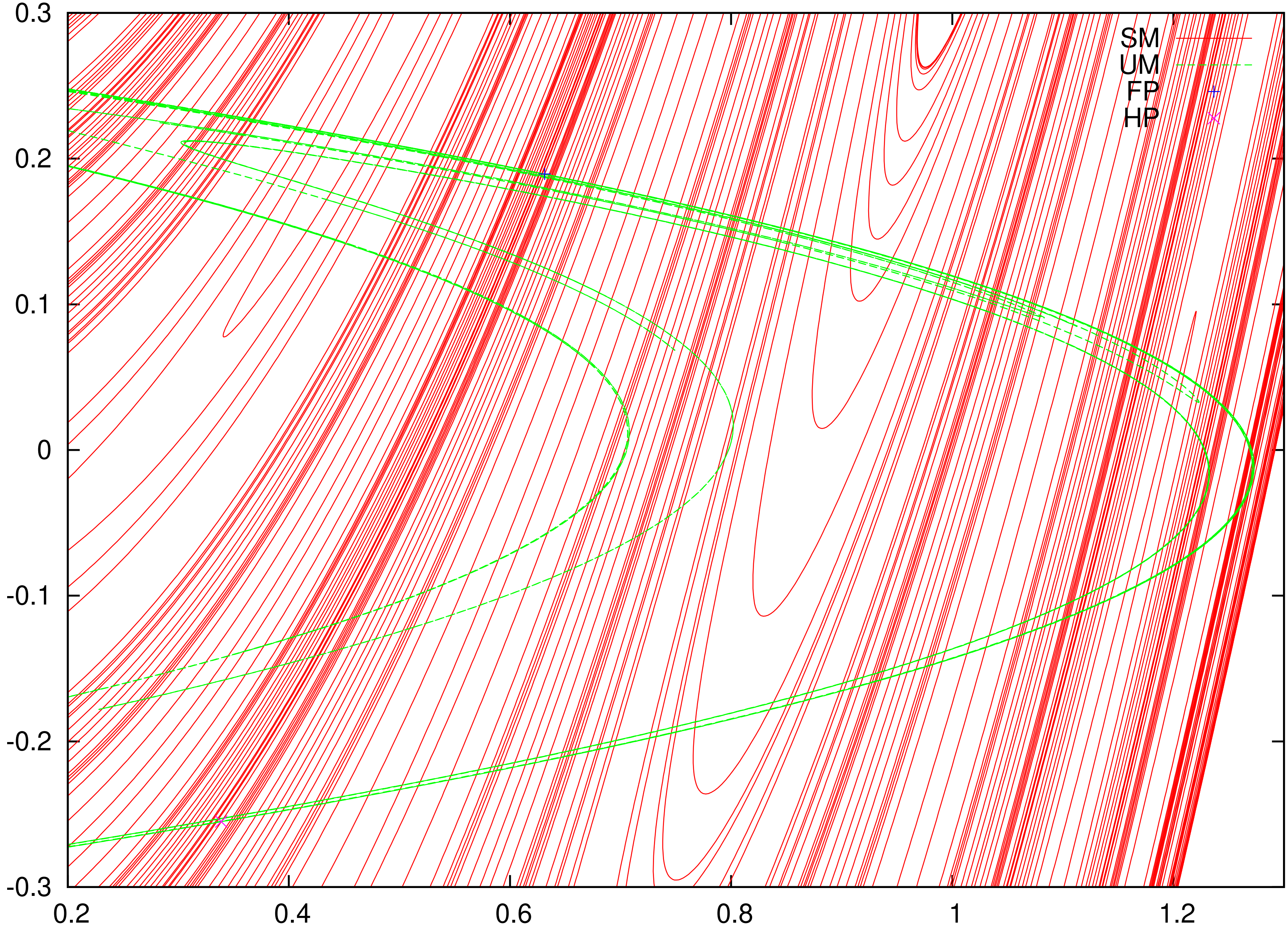
0.8
1
1.2



SM
UM
FP
HP
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.2
0.4
0.6
0.8
1
1.2



SM
UM
FP
HP
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-1.5
-1
-0.5
0
0.5
1
1.5



SM
UM
FP
HP
0.3
0.2
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0
-0.1
-0.2
-0.3
0.2
0.4
0.6
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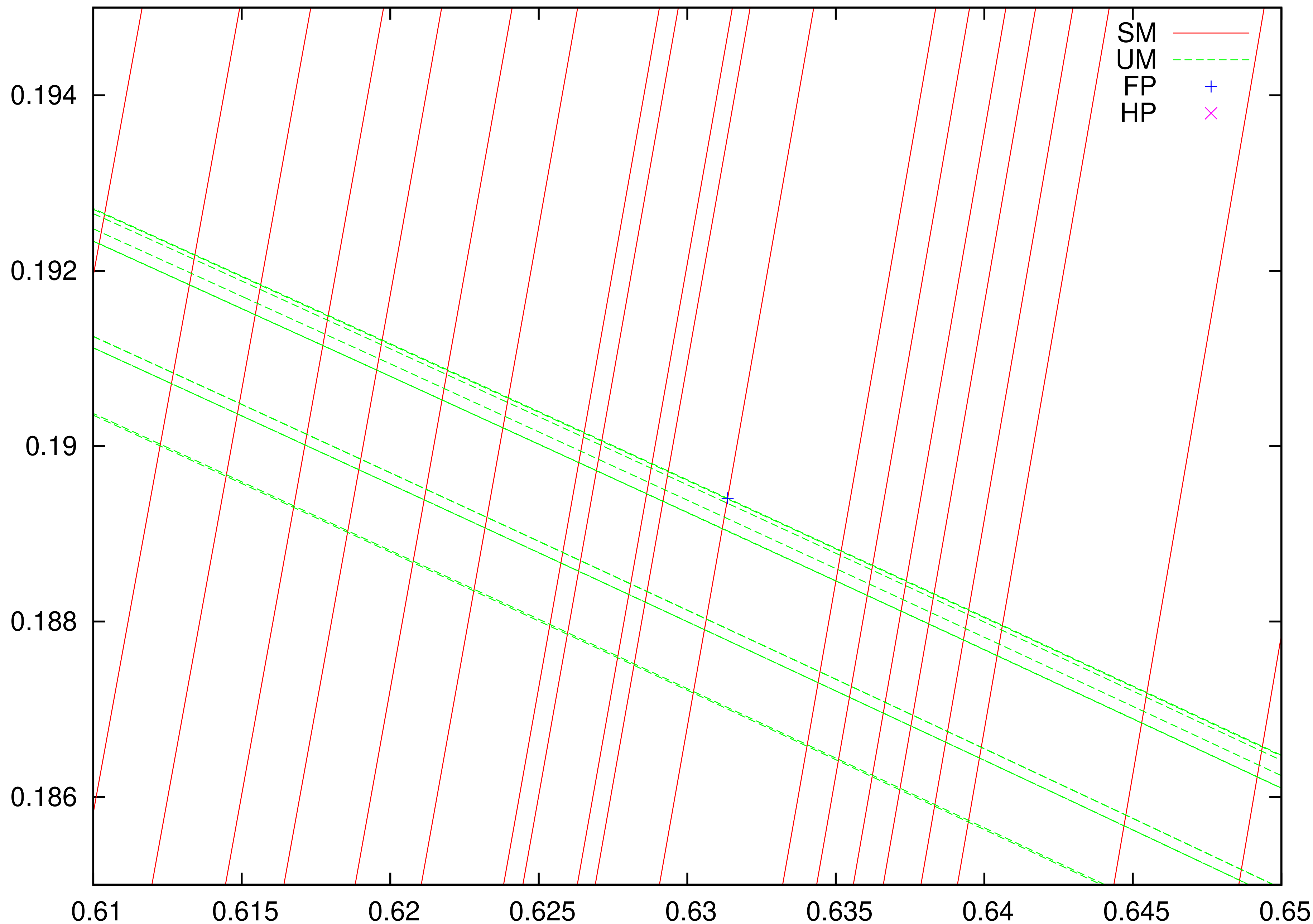
1
1.2


SM
UM
FP
HP
0.194
0.192
0.19
0.188
0.186
0.61
0.615
0.62
0.625
0.63
0.635
0.64
0.645
0.65

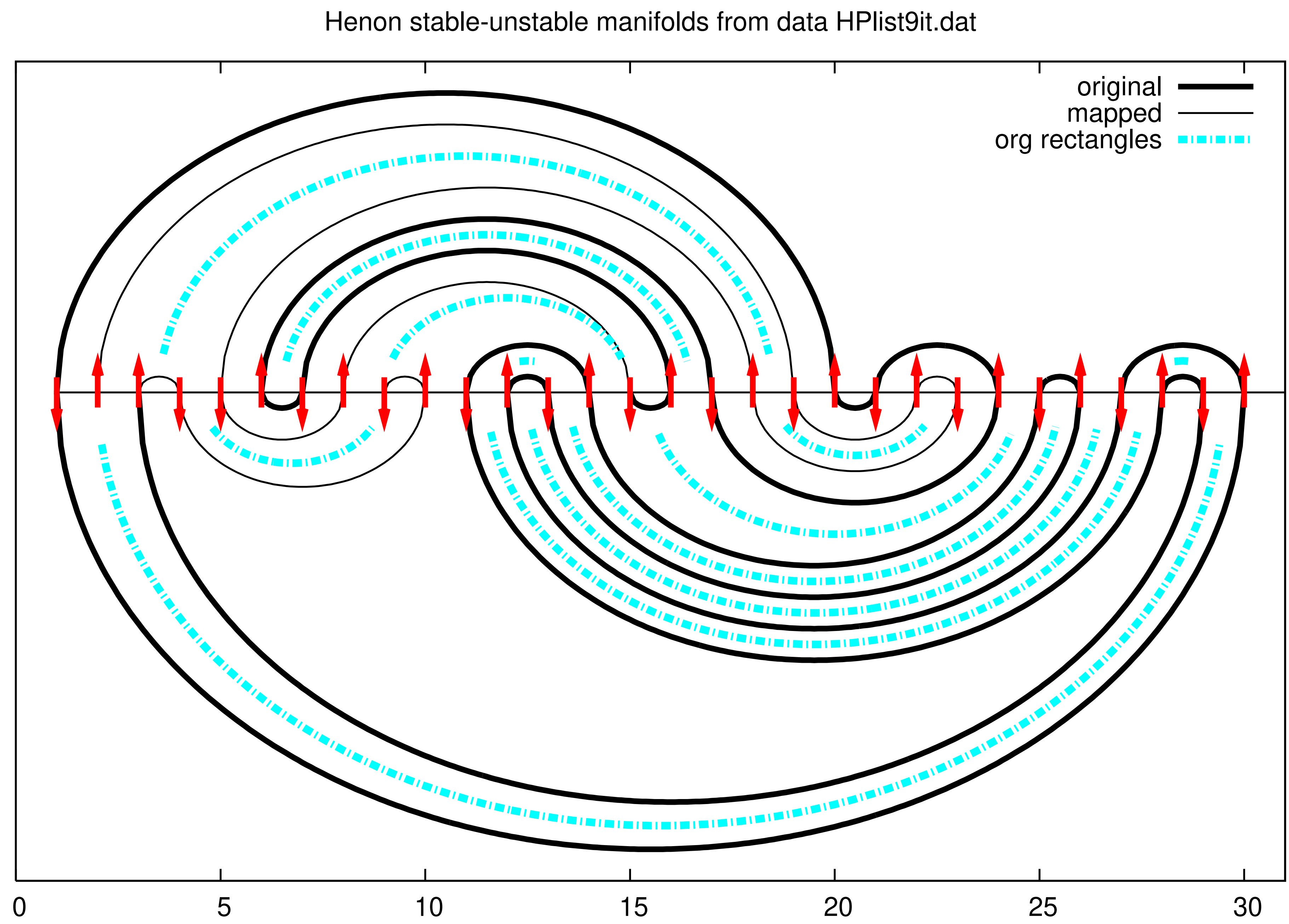
Rigorous Computational Symbolic Dynamics

Using Taylor model based flow integrators and normal form methods, can set up even very complicated symbolic dynamics. Let two initial pieces of stable and unstable manifold be given.

1. Rigorously enclose **ALL** homoclinic points of using the rigorous global optimizer COSY-GO.
2. Determine rigorous **parent-child relationships** of these homoclinic points.

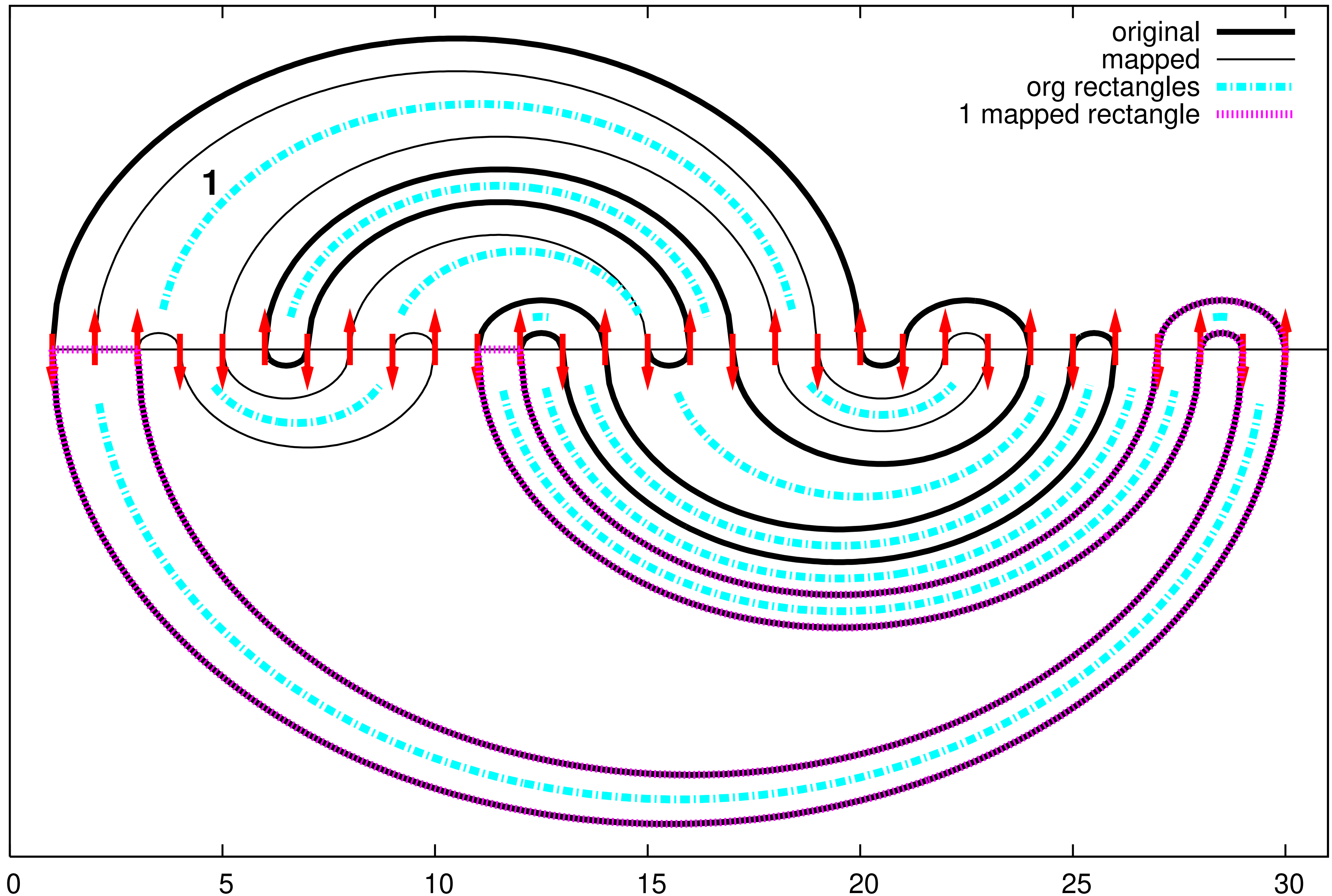
This allows the rigorous determination the mapping properties of curvilinear rectangles, which can be described by the so-called incidence matrix. The largest eigenvalue of it is a lower bound of the topological entropy.

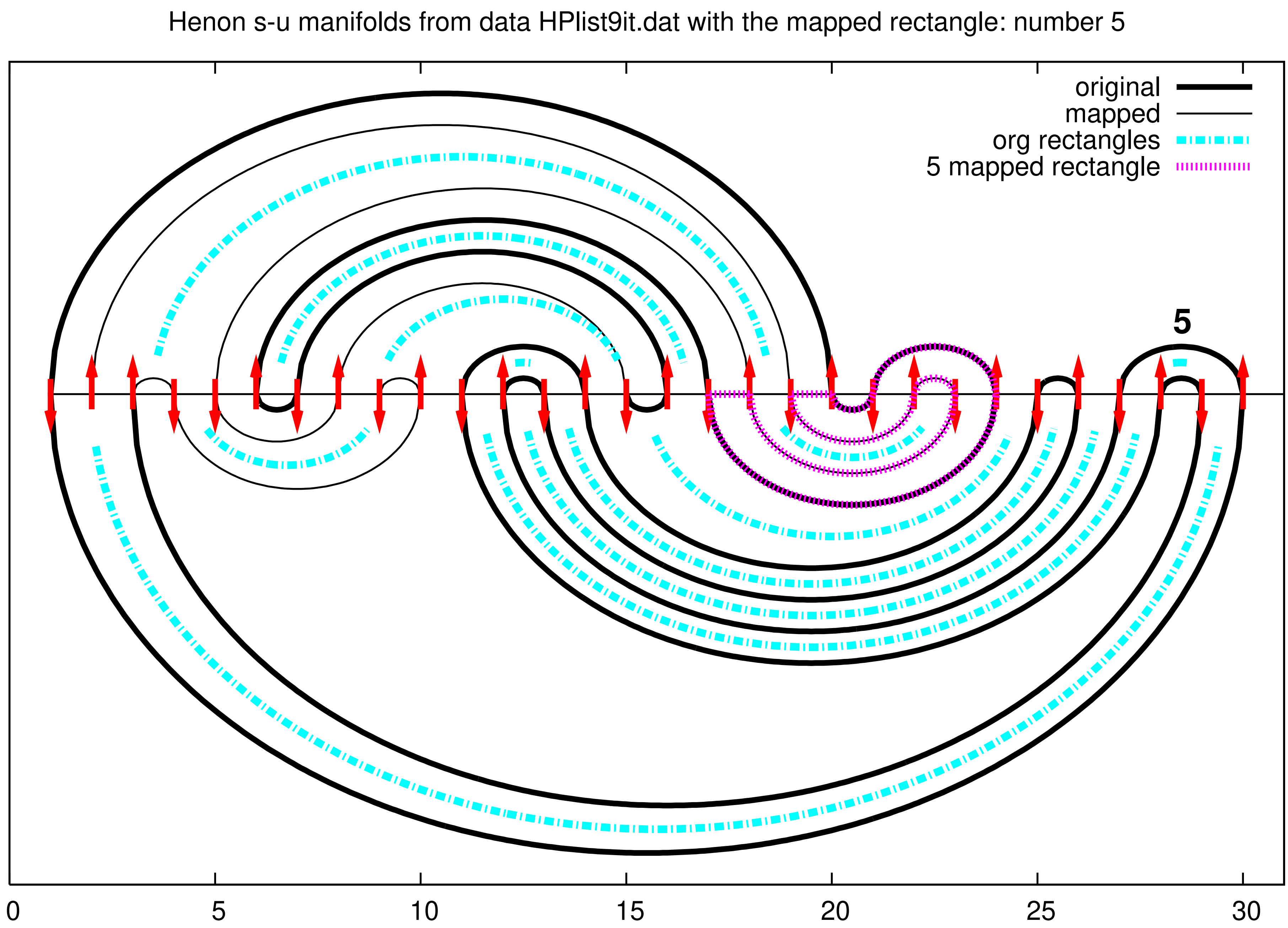
Note: probably the first such attempt at a rigorous dynamics was done by Piotr Zgliczynski for the Henon map, proving that it follows a horseshoe dynamics, with

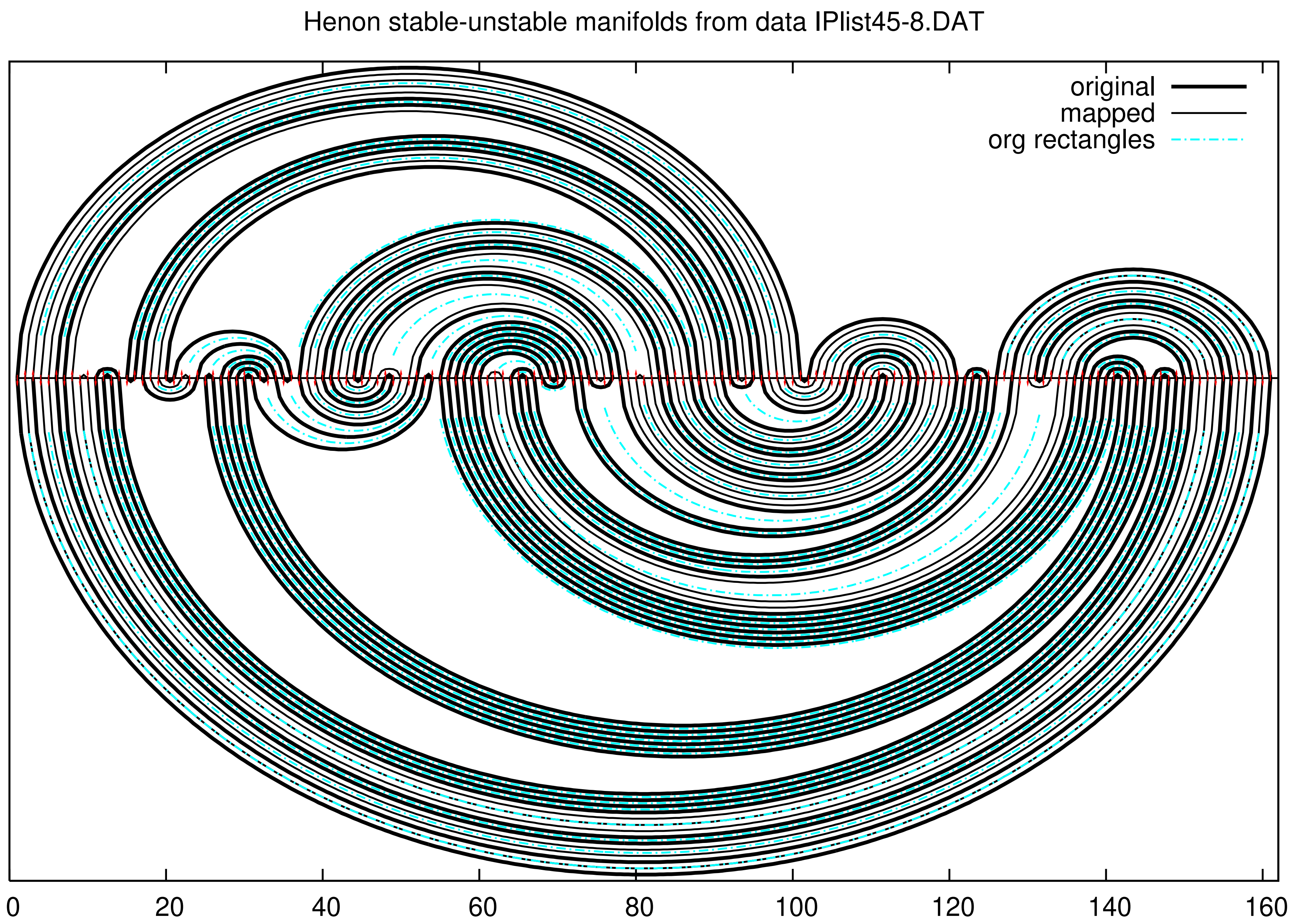


Henon stable-unstable manifolds from data HPlist9it.dat
original
mapped
org rectangles
0
5
10
15
20
25
30

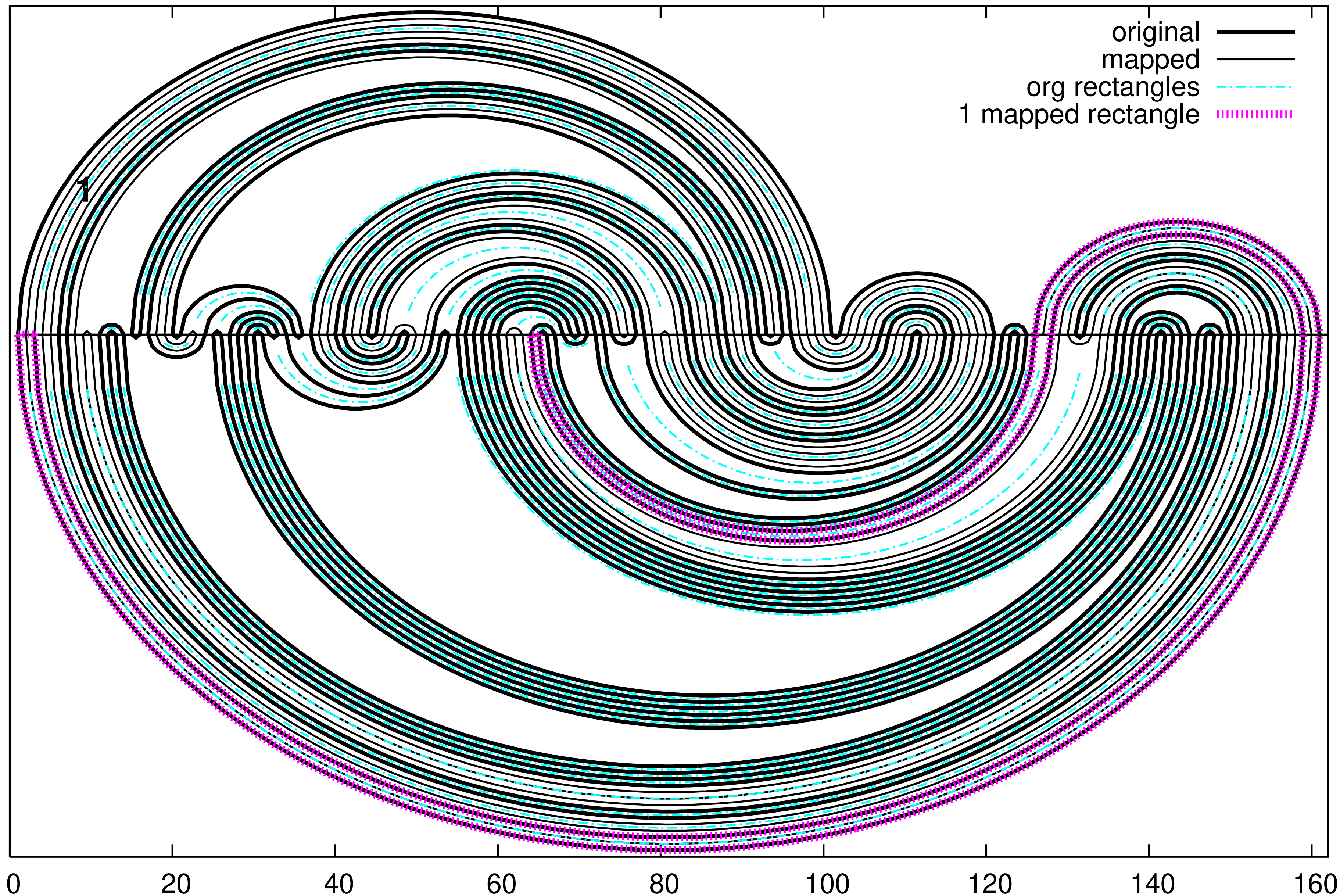
Henon s-u manifolds from data HPlist9it.dat with the mapped rectangle: number 1



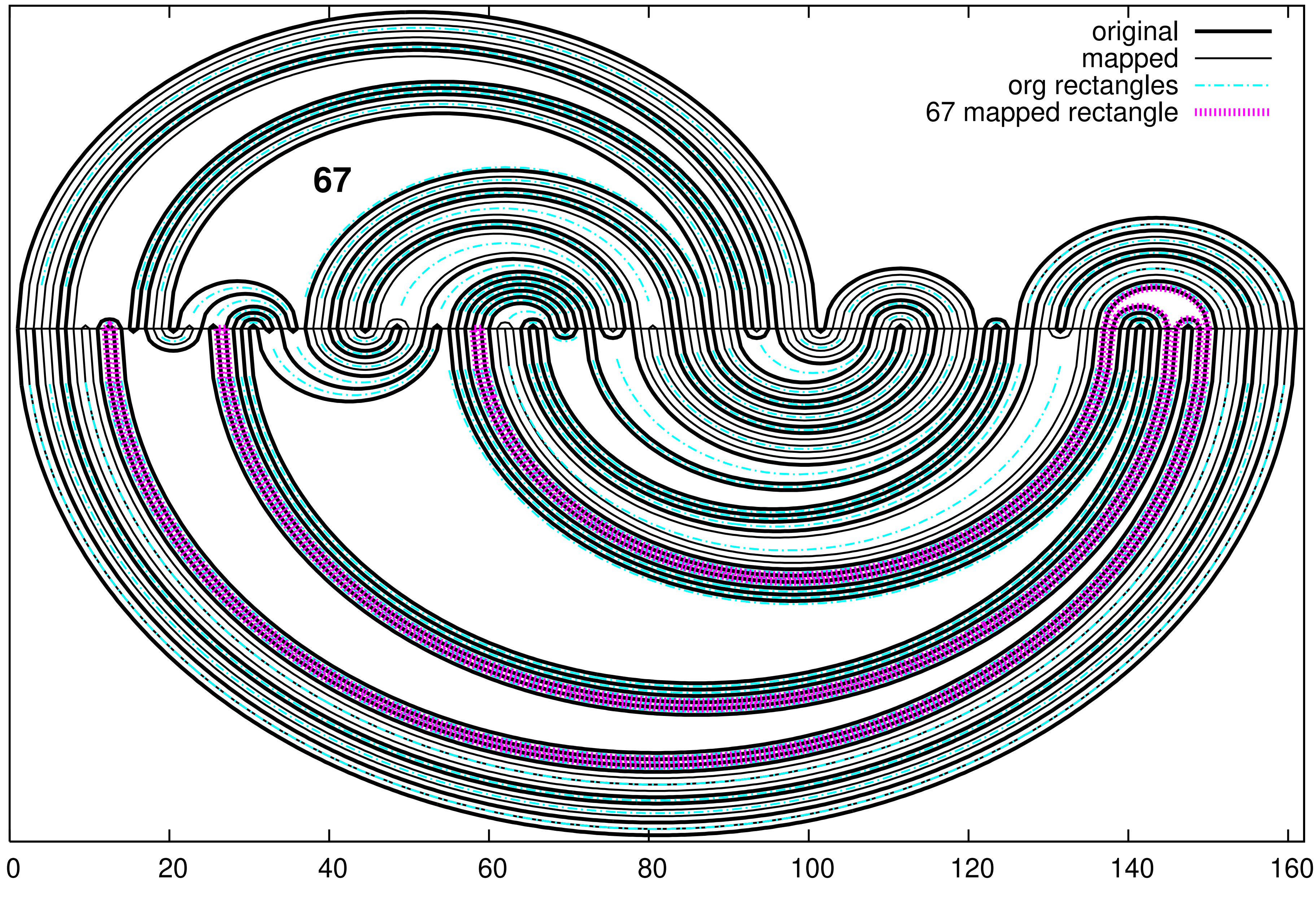

Henon s-u manifolds from data HPlist9it.dat with the mapped rectangle: number 5
original
mapped
org rectangles
5 mapped rectangle
5
0
5
10
15
20
25
30


Henon stable-unstable manifolds from data IPlist45-8.DAT
original
mapped
org rectangles
0
20
40
60
80
100
120
140
160

Henon s-u manifolds from data IPlist45-8.DAT with the mapped rectangle: number 1



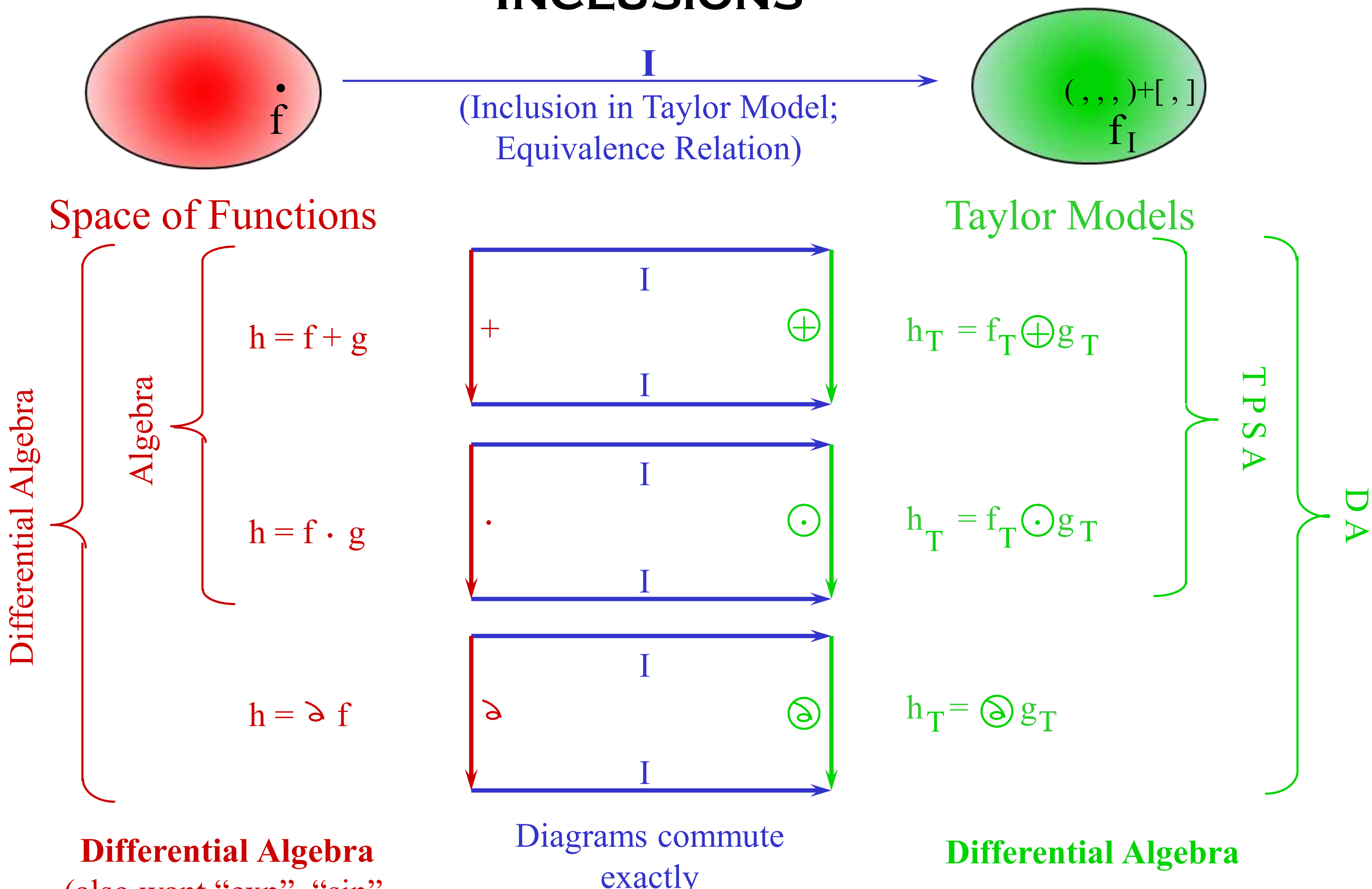
Henon s-u manifolds from data IPlist45-8.DAT with the mapped rectangle: number 67



Entropy Estimates from Trellis Untangling

1. 161 HP's, Pure Rectangles, 66 Symbols, 94 Crossings: 0.4131
2. 161 HP's, Rect +Hexagons, 77 Symbols, 110 Crossings: 0.4309
3. 267 HP's, Pure Rectangles, 119 Symbols, 185 Crossings: 0.4131
4. 267 HP's, Rect +Hexagons, 130 Symbols, 205 Crossings: 0.4402
5. 437 HP's, Pure Rectangles, 218 Symbols, 346 Crossings: 0.4282
6. 437 HP's, Rect +Hexagons, 229 Symbols, 366 Crossings: 0.4499
7. 707 HP's, Pure Rectangles, 381 Symbols, 603 Crossings: 0.4417
8. 707 HP's, Rect +Hexagons, 392 Symbols, 621 Crossings: 0.4536

Function Algebra Inclusions



Differential Algebra
(also want "exp", "sin" etc: Banach DA)

Differential Algebra

T: Extracts information considered relevant

The Use of Schauder's Theorem

Re-write differential equation as integral equation

$$\vec{r}(t) = \vec{r}_0 + \int_{t_0}^{t} \vec{F}(\vec{r}(t'), t')\, dt'.$$

Now introduce the operator

$$A : \vec{C}^0[t_0, t_1] \to \vec{C}^0[t_0, t_1]$$

on space of continuous functions via

$$A\left(\vec{f}\right)(t) = \vec{r}_0 + \int_{t_0}^{t} \vec{F}\left(\vec{f}(t'), t'\right)\, dt'.$$

Then the solution of ODE is transformed to a fixed-point problem on space of continuous functions

$$\vec{r} = A(\vec{r}).$$

Theorem (Schauder): *Let A be a continuous operator on the Banach Space X. Let $M \subset X$ be compact and convex, and let $A(M) \subset M$. Then A has a fixed point in M, i.e. there is an $\vec{r} \in M$ such that $A(\vec{r}) = \vec{r}$.*

The Polynomial of the Self-Including Set

Attempt sets M^* of the form

$$M^* = M_{\vec{P}^* + \vec{I}^*} \text{ where}$$
$$\vec{P}^* = \mathcal{M}_n(\vec{r}_0, t),$$

the n-th order Taylor expansion of the flow of the ODE. It is to be expected that $\vec{I}^*$ can be chosen smaller and smaller as order n of $\vec{P}^*$ increases.

This requires knowledge of nth order flow $\mathcal{M}_n(\vec{r}_0, t)$, including time dependence. It can be obtained by iterating in polynomial arithmetic, or Taylor models without treatment of a remainder. To this end, one chooses an initial function $\mathcal{M}_n^{(0)}(\vec{r}, t) = \mathcal{I}$, where $\mathcal{I}$ is the identity function, and then iteratively determines

$$\mathcal{M}_n^{(k+1)} =_n A(\mathcal{M}_n^{(k)}).$$

This process converges to the exact result $\mathcal{M}_n$ in exactly n steps.

The Remainder of the Self-Including Set

Now try to find $\vec{I}^*$ such that

$$A(\mathcal{M}_n + \vec{I}^*) \subset \mathcal{M}_n + \vec{I}^*,$$

the Schauder inclusion requirement. Suitable choice for $\vec{I}^*$ requires experimenting, but is greatly simplified by the observation

$$\vec{I}^* \supset \vec{I}^{(0)} = A(\mathcal{M}_n(\vec{r}, t) + [\vec{0}, \vec{0}]) - \mathcal{M}_n(\vec{r}, t).$$

Evaluating the right hand side in RDA yields a lower bound for $\vec{I}^*$, and a benchmark for the size to be expected. Now iteratively try

$$\vec{I}^{(k)} = 2^k \cdot \vec{I}^{(0)},$$

until computational inclusion is found, i.e.

$$A(\mathcal{M}_n(\vec{r}, t) + \vec{I}^{(k)}) \subset \mathcal{M}_n(\vec{r}, t) + \vec{I}^{(k)}.$$

Preconditioning the Flow

It can be viewed as a coordinate transformation.

Definition (Preconditioning the Flow) Let $(P + I)$ be a Taylor model. We say that $(P_l + I_l), (P_r + I_r)$ is a factorization of $(P + I)$ if $B(P_r + I_r) \in [-1, 1]$ and

$$(P + I) \in (P_l + I_l) \circ (P_r + I_r) \text{ for all } x \in D$$

where D is the domain of the Taylor model $(P_r + I_r)$.

Proposition Let $(P_{l,n} + I_{l,n}) \circ (P_{r.n} + I_{r,n})$ be a factored Taylor model that encloses the flow of the ODE at time t_n. Let $(P^*_{l,n+1}, I^*_{l,n+1})$ be the result of integrating $(P_{l,n} + I_{l,n})$ from t_n to t_{n+1}. Then

$$(P^*_{l,n+1}, I^*_{l,n+1}) \circ (P_{r.n} + I_{r,n})$$

is a factorization of the flow at time t_{n+1}.

Example Preconditionings: QR, Blunted, Curvilinear.

Curvilinear Preconditioning

Definition (Curvilinear Preconditioning) Let $x^{(m)} = f(x, x', ...x^{(m-1)}, t)$ be an m-th order ODE in n variables. Let $x_r(t)$ be a solution of the ODE and $x'_r(t)$, ..., $x_r^{(k)}(t)$ its first k time derivatives. Let $\vec{e}_1, ..., \vec{e}_l$ be the l unit vectors not in the span of $x'_r(t)$, ..., $x_r^{(k)}(t)$, sorted by distance from the span. Then we call the Gram-Schmidt orthonormalization of the set ($x'_r(t)$, ..., $x_r^{(k)}(t), \vec{e}_1, ..., \vec{e}_l$) the curvilinear basis of depth k.

Example (Solar System and Particle Accelerators)
In this case, $n = 3$, and one usually chooses $k = 2$.
(1) The first basis vector points in the direction of reference orbit motion.
(2) The second is perpendicular to it and points approximately to the sun or the center of the accelerator.
(3) The third is chosen perpendicular to the plane of the previous two.

The Duffing Equation

The equation describes a damped and driven oscillator.
Exhibits sensitive dependence on initial conditions and chaoticity.

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega t)$$

Example: Study

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - \delta y - x^3 + \gamma \cos(t) \end{aligned}$$

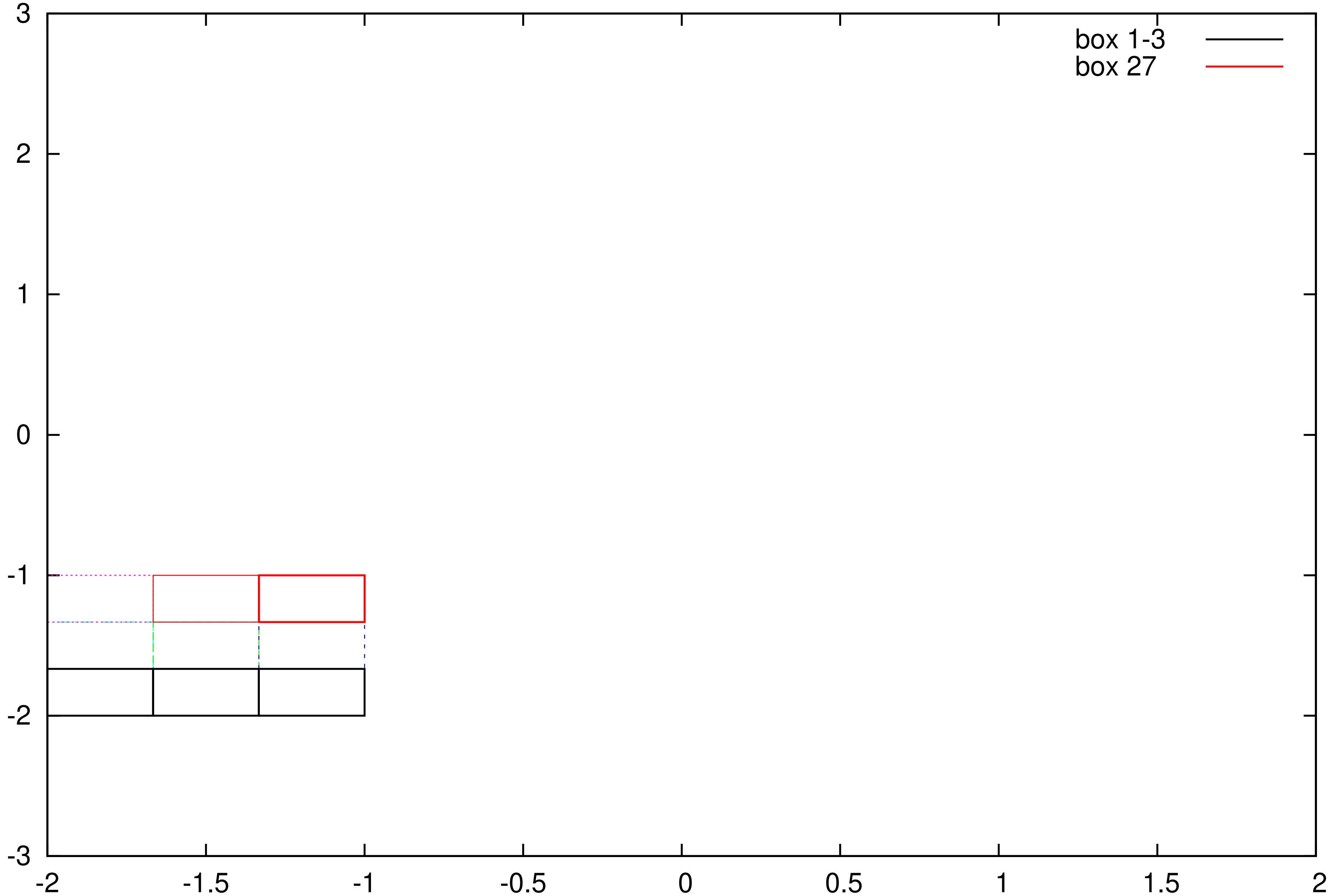
with

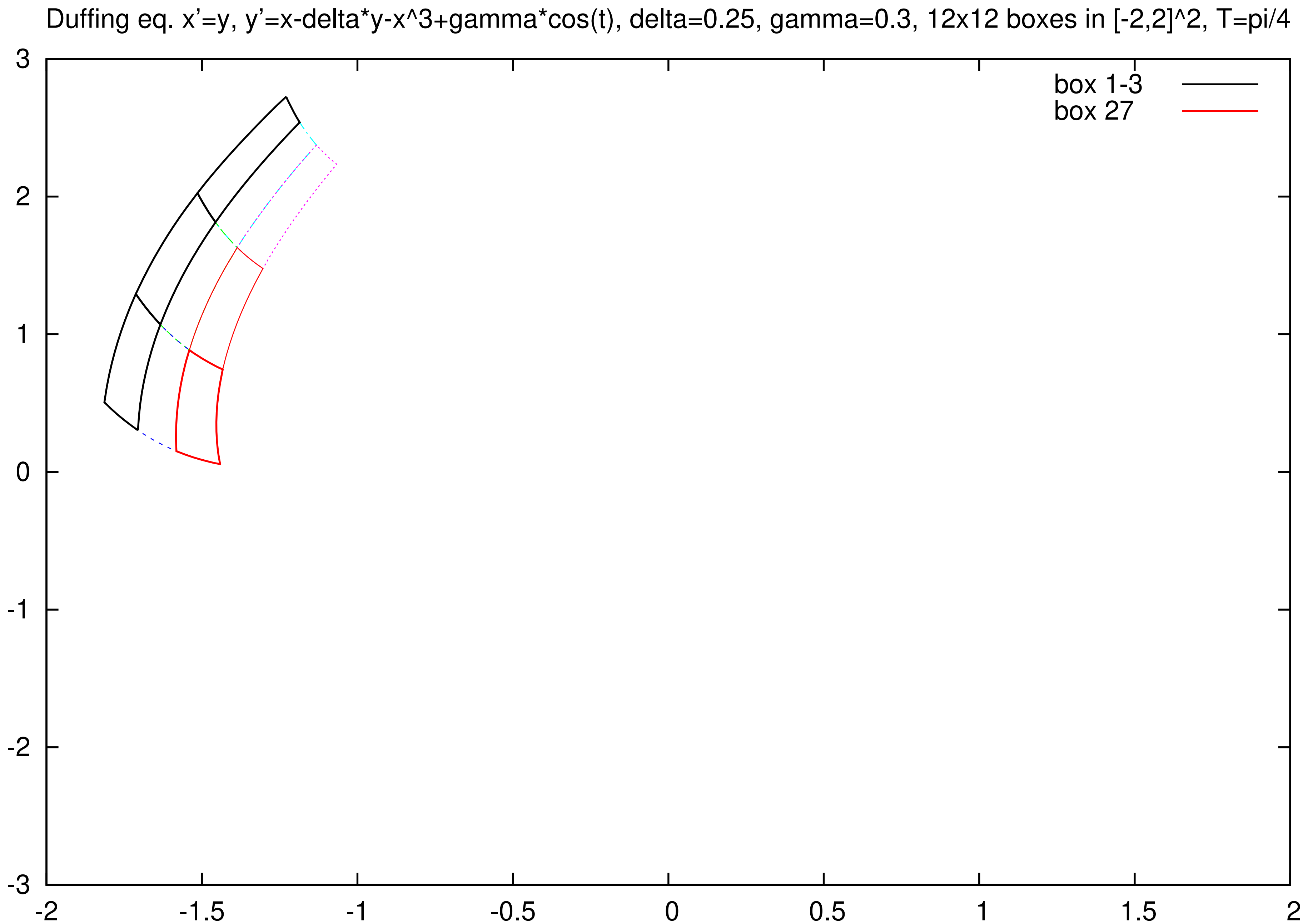
$$\delta = 0.25, \quad \gamma = 0.3,$$

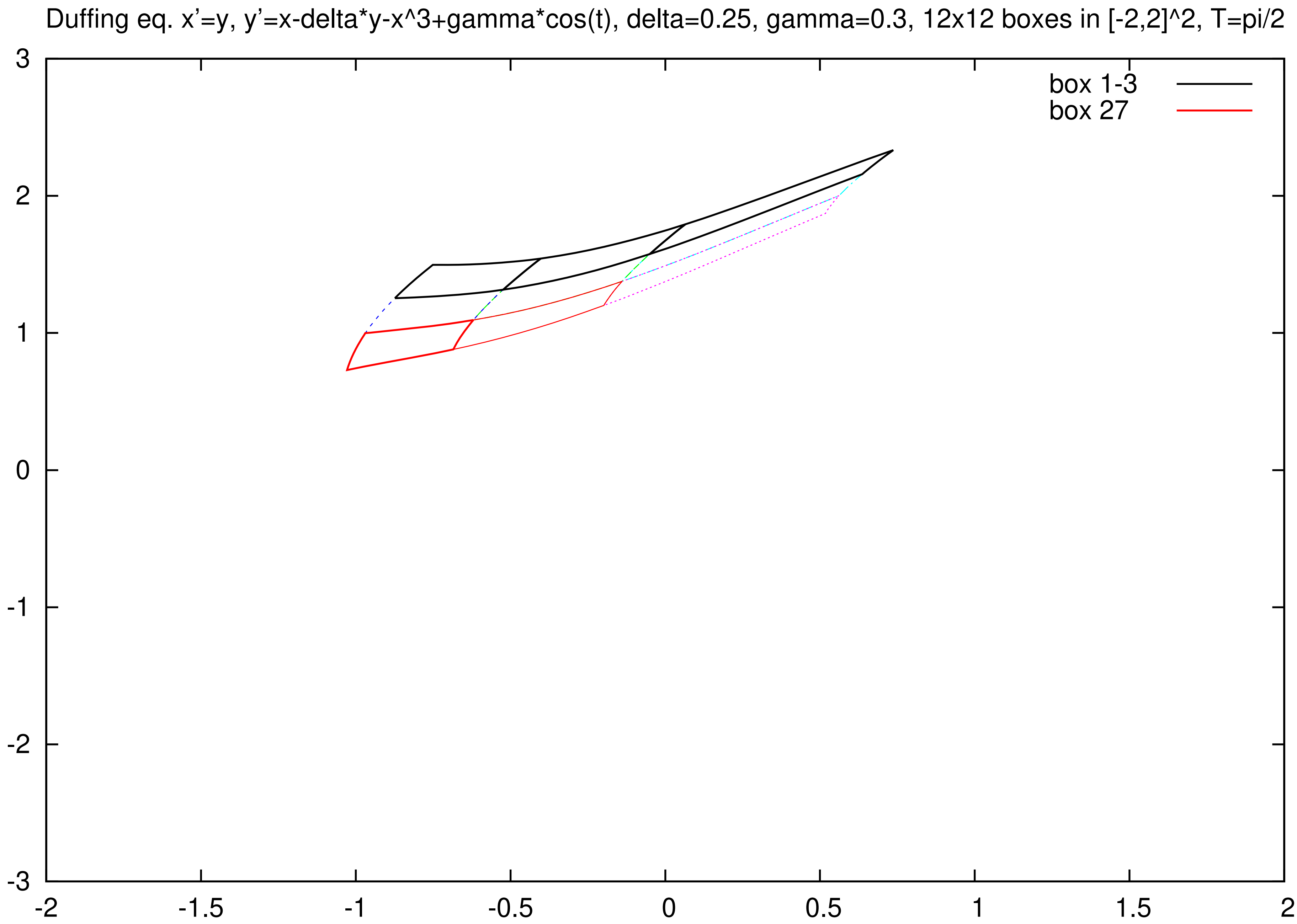
for

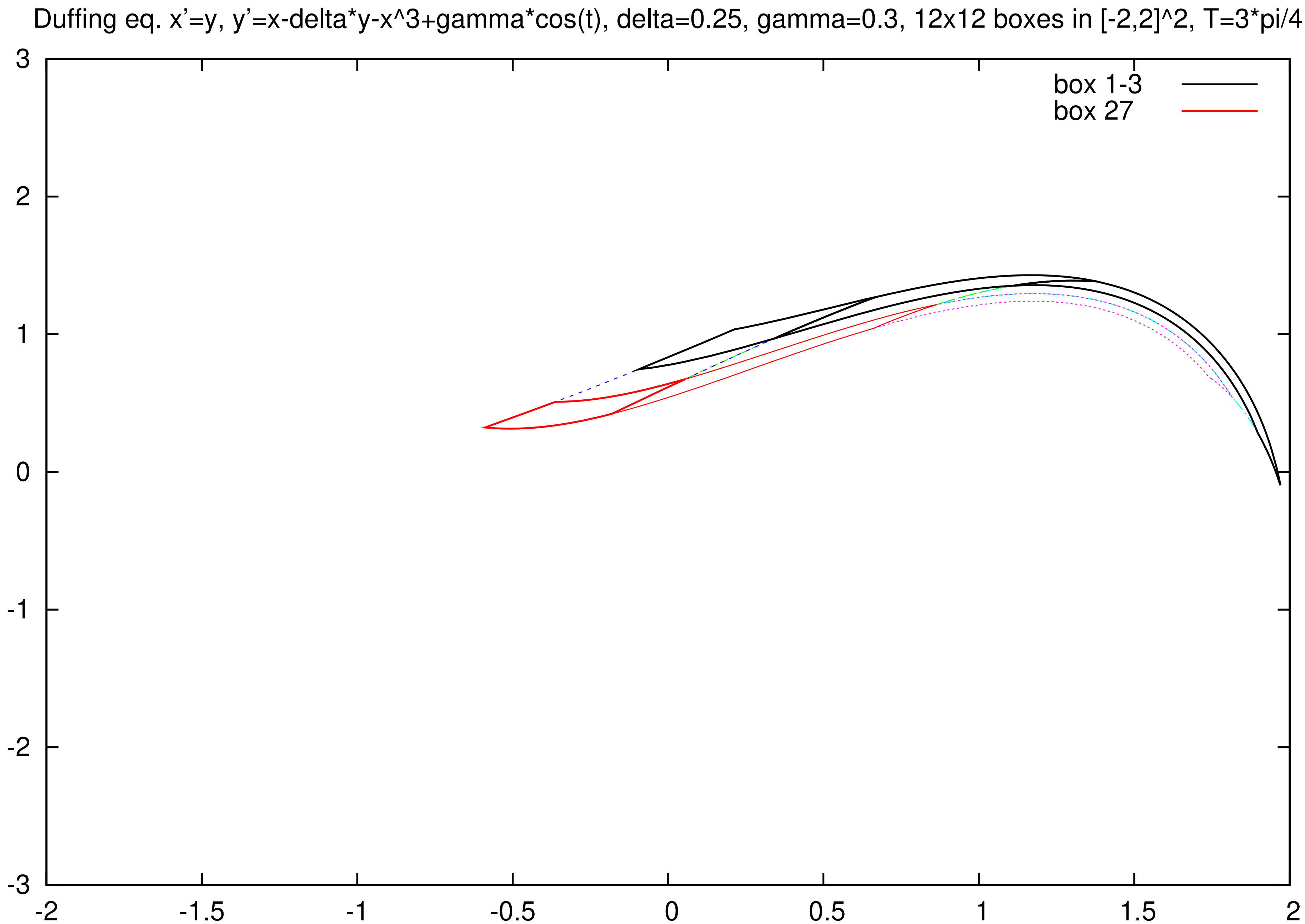
$$t \in [0, \pi], \quad (x, y)_{IC} \in [-2, 2] \times [-2, 2].$$

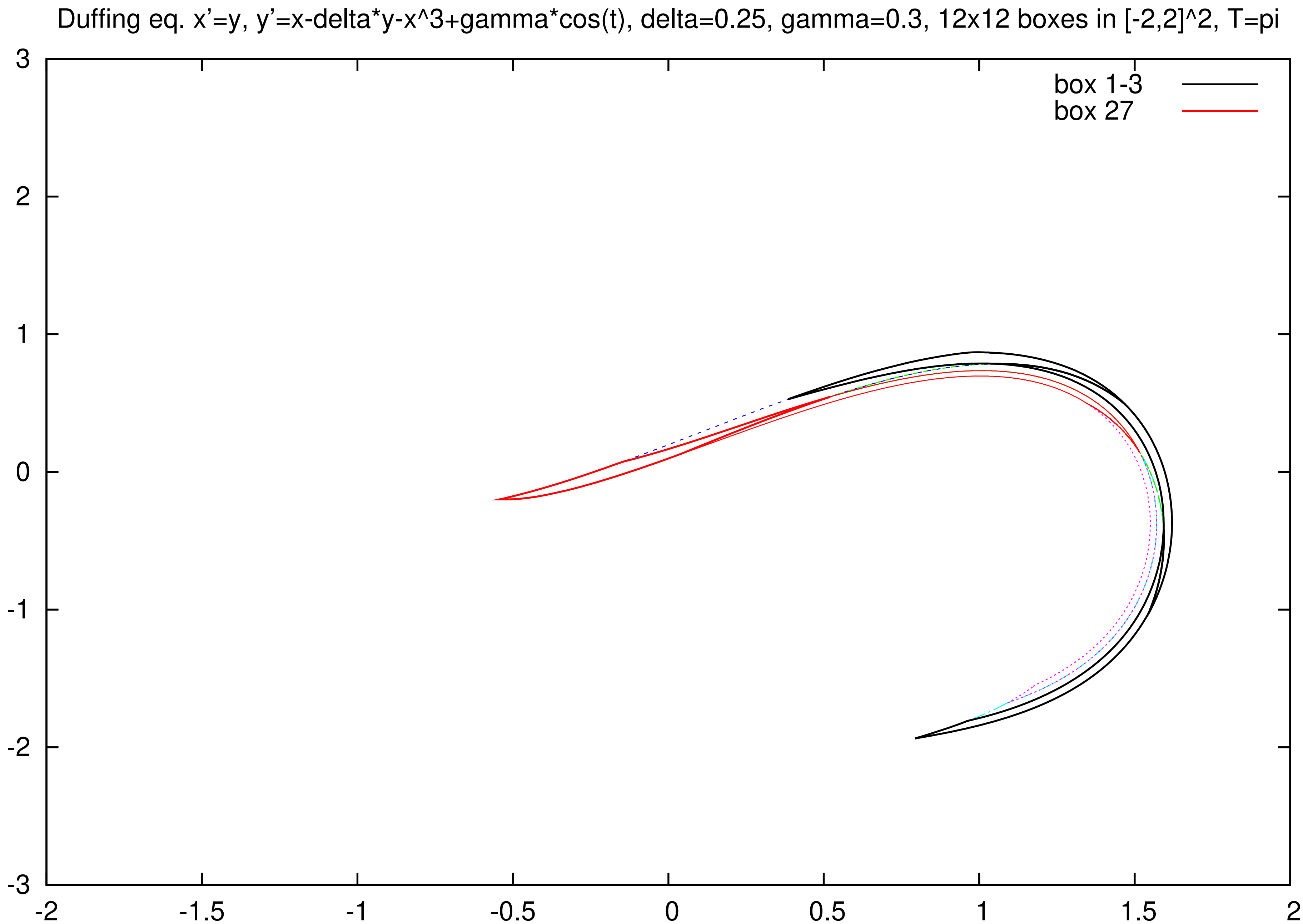
Duffing eq. x'=y, y'=x-delta*y-x^3+gamma*cos(t), delta=0.25, gamma=0.3, 12x12 boxes in [-2,2]^2, T=0 (IC)



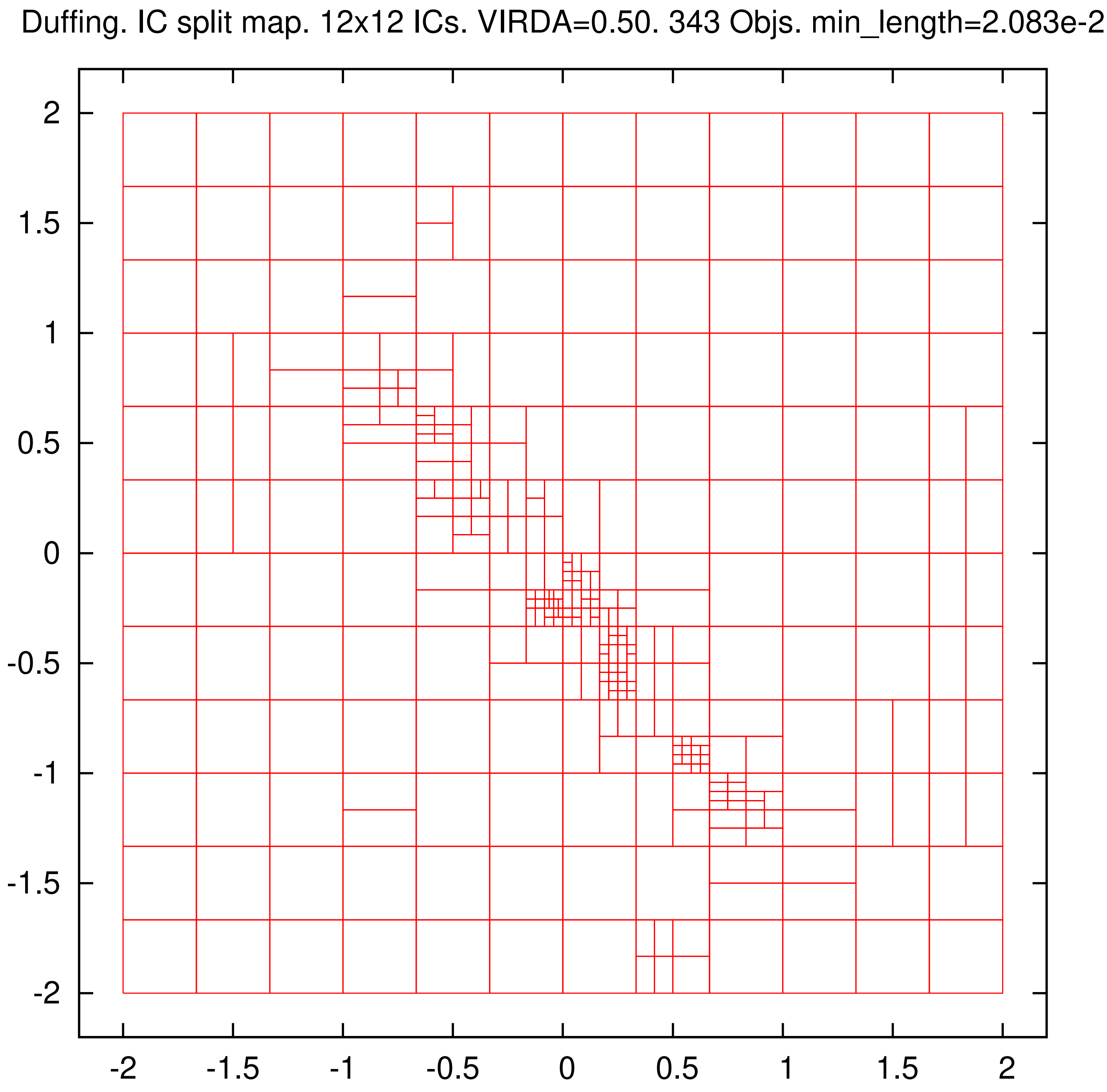

Duffing eq. x'=y, y'=x-delta*y-x^3+gamma*cos(t), delta=0.25, gamma=0.3, 12x12 boxes in [-2,2]^2, T=pi/4
box 1-3
box 27
3
2
1
0
-1
-2
-3
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2


Duffing eq. x'=y, y'=x-delta*y-x^3+gamma*cos(t), delta=0.25, gamma=0.3, 12x12 boxes in [-2,2]^2, T=pi/2
box 1-3
box 27
3
2
1
0
-1
-2
-3
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

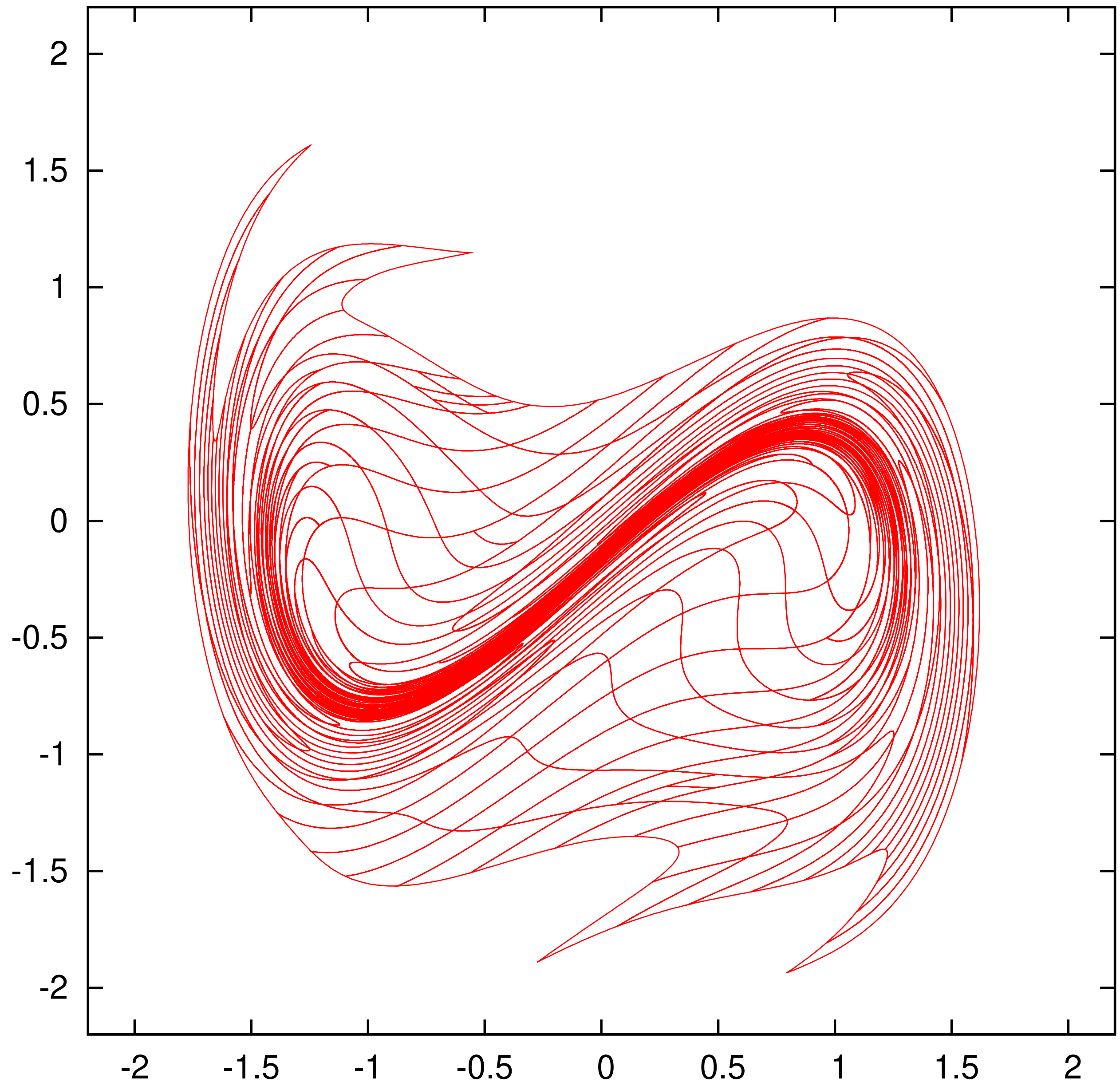

Duffing eq. x'=y, y'=x-delta*y-x^3+gamma*cos(t), delta=0.25, gamma=0.3, 12x12 boxes in [-2,2]^2, T=3*pi/4
box 1-3
box 27
3
2
1
0
-1
-2
-3
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2



Duffing eq. x'=y, y'=x-delta*y-x^3+gamma*cos(t), delta=0.25, gamma=0.3, 12x12 boxes in [-2,2]^2, T=pi
box 1-3
box 27
3
2
1
0
-1
-2
-3
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2


Duffing. IC split map. 12x12 ICs. VIRDA=0.50. 343 Objs. min_length=2.083e-2
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Duffing. Time 0 to pi. 12x12 ICs. VIRDA=0.50. 343 Objs

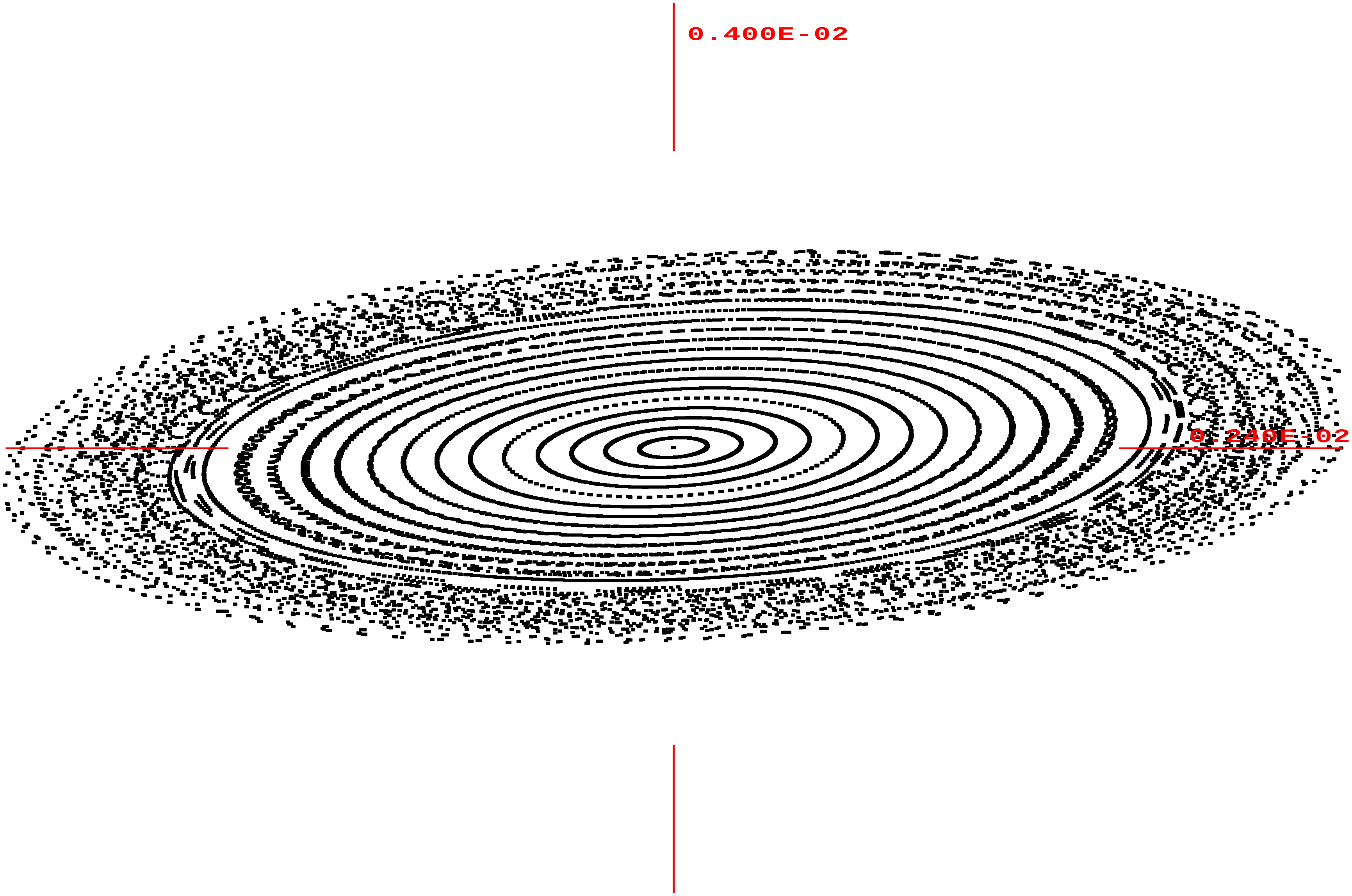


Allows graph theoretical treatment (Morse decomposition, Conley index etc)

CPU Time:

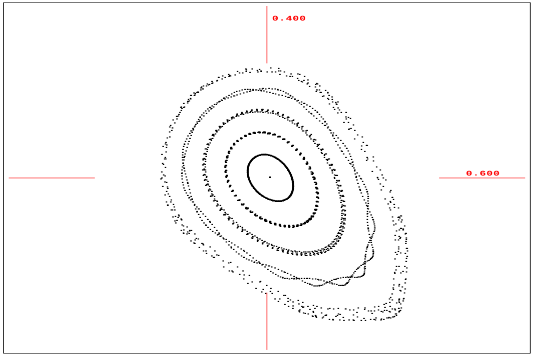
~ 20 min (1E-5 accuracy)

~ 100 min (1E-10 accuracy)

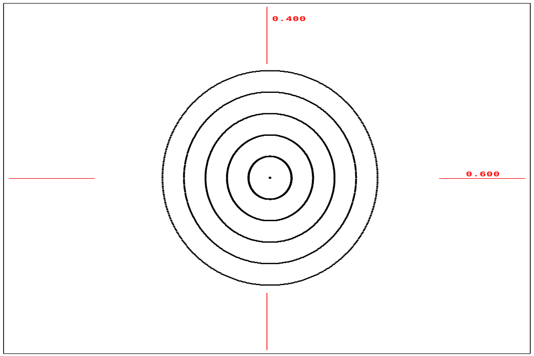


Tracking y-py Phase Space Moition of the Tevatron

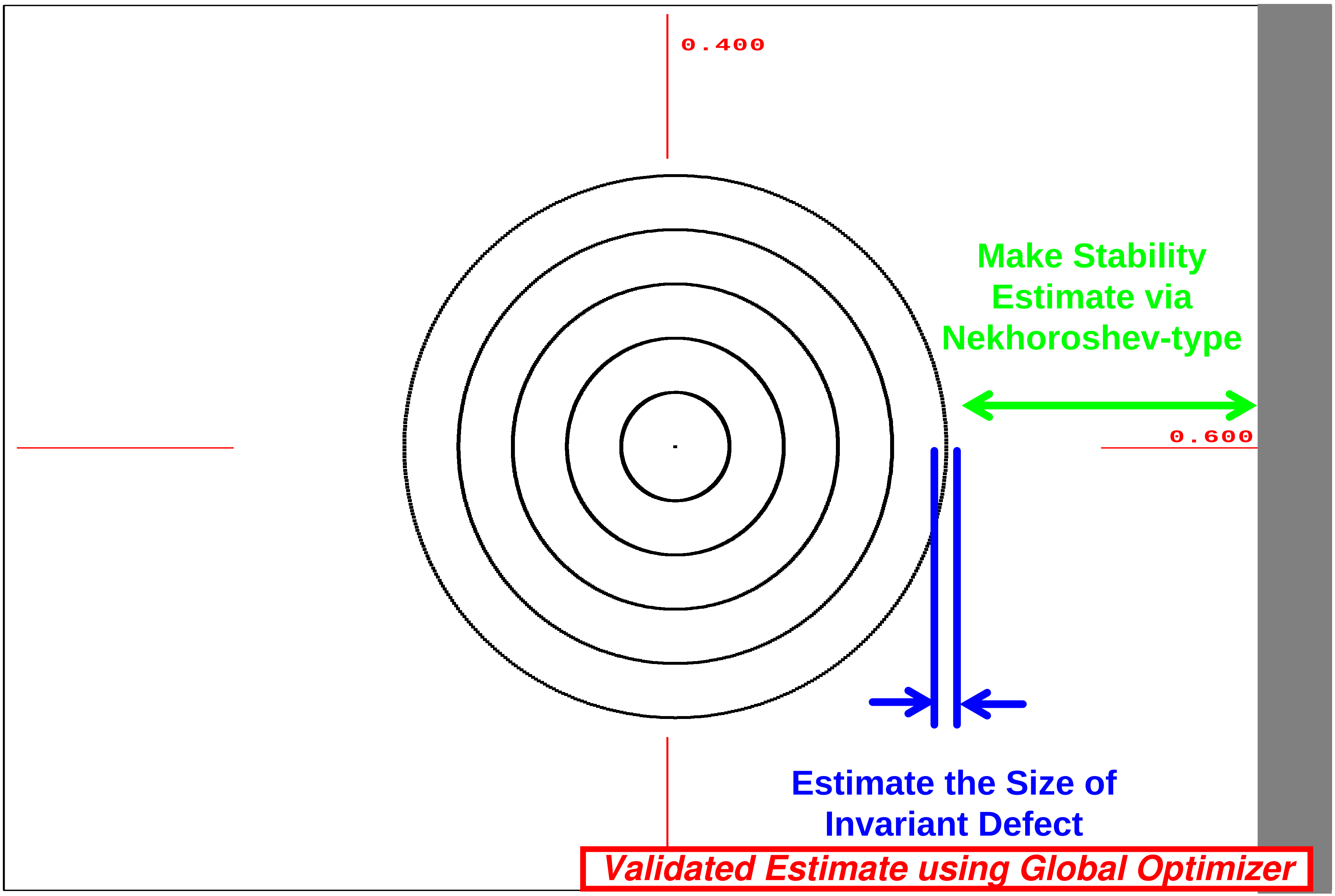
Example of Phase Space Motion



Tracking Phase Space Motion of 5 Particles in Regular Coordinates



Tracking Phase Space Motion of 5 Particles in Normal Form Coordinates



Tracking Phase Space Motion of 5 Particles in Normal Form Coordinates

The Normal Form Defect Function

- **Extreme cancellation**; one of the reasons TM methods were invented
- Six-dimensional problem from dynamical systems theory
- Describes invariance defects of a particle accelerator
- Essentially composition of three tenth order polynomials
- The function vanishes identically to order ten
- Study for $a\cdot\ (1,1,1,1,1,1)$ for $a = .1$ and $a = .2$
- Interesting **Speed observation**: on same machine,
  * one CF in INTLAB takes 45 minutes
  * one TM of order 7 takes 10 seconds

$$f_4(x_1,..,x_6) = \sum_{i=1}^{3}\left(\sqrt{y_{2i-1}^2 + y_{2i}^2} - \sqrt{x_{2i-1}^2 + x_{2i}^2}\right)^2$$

$$\text{where } \vec{y} = \vec{P}_1\left(\vec{P}_2\left(\vec{P}_3(\vec{x})\right)\right)$$

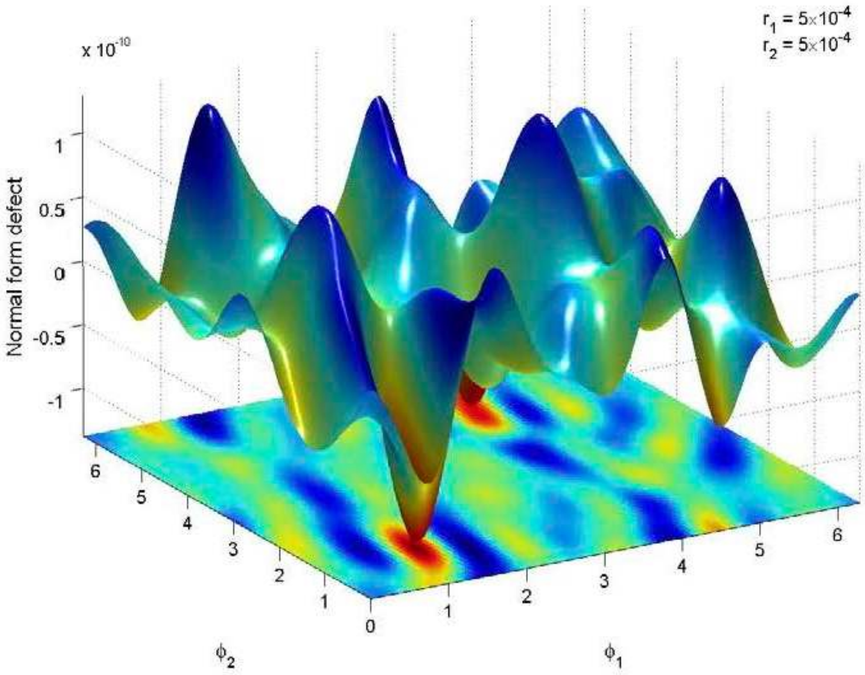
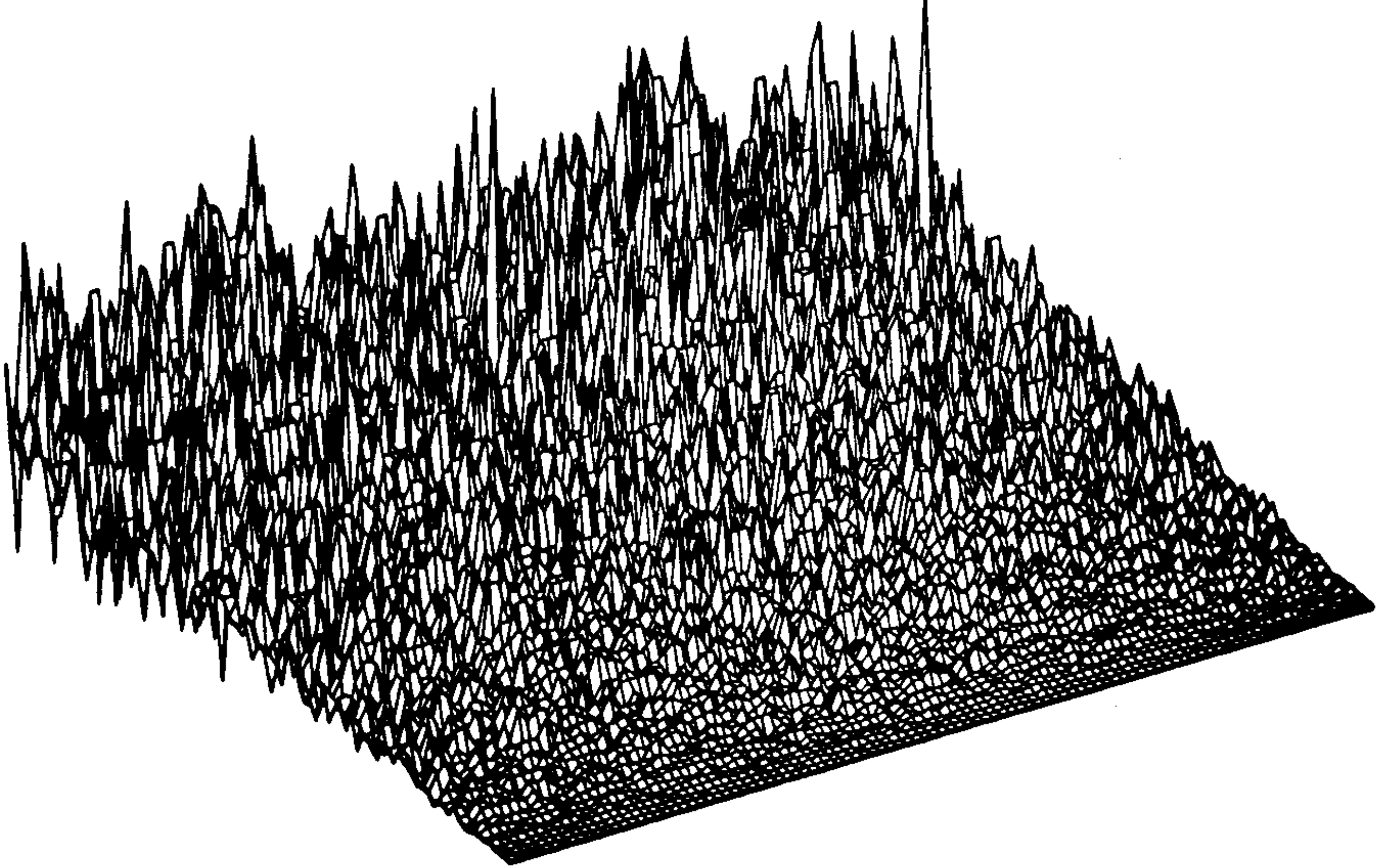


Fig. 9. Projection of the normal form defect function. Dependence on two angle variables for the fixed radii $r_1 = r_2 = 5 \cdot 10^{-4}$



The TM based Global Optimizer, COSY-GO

has utilized various algorothms based on Taylor models.

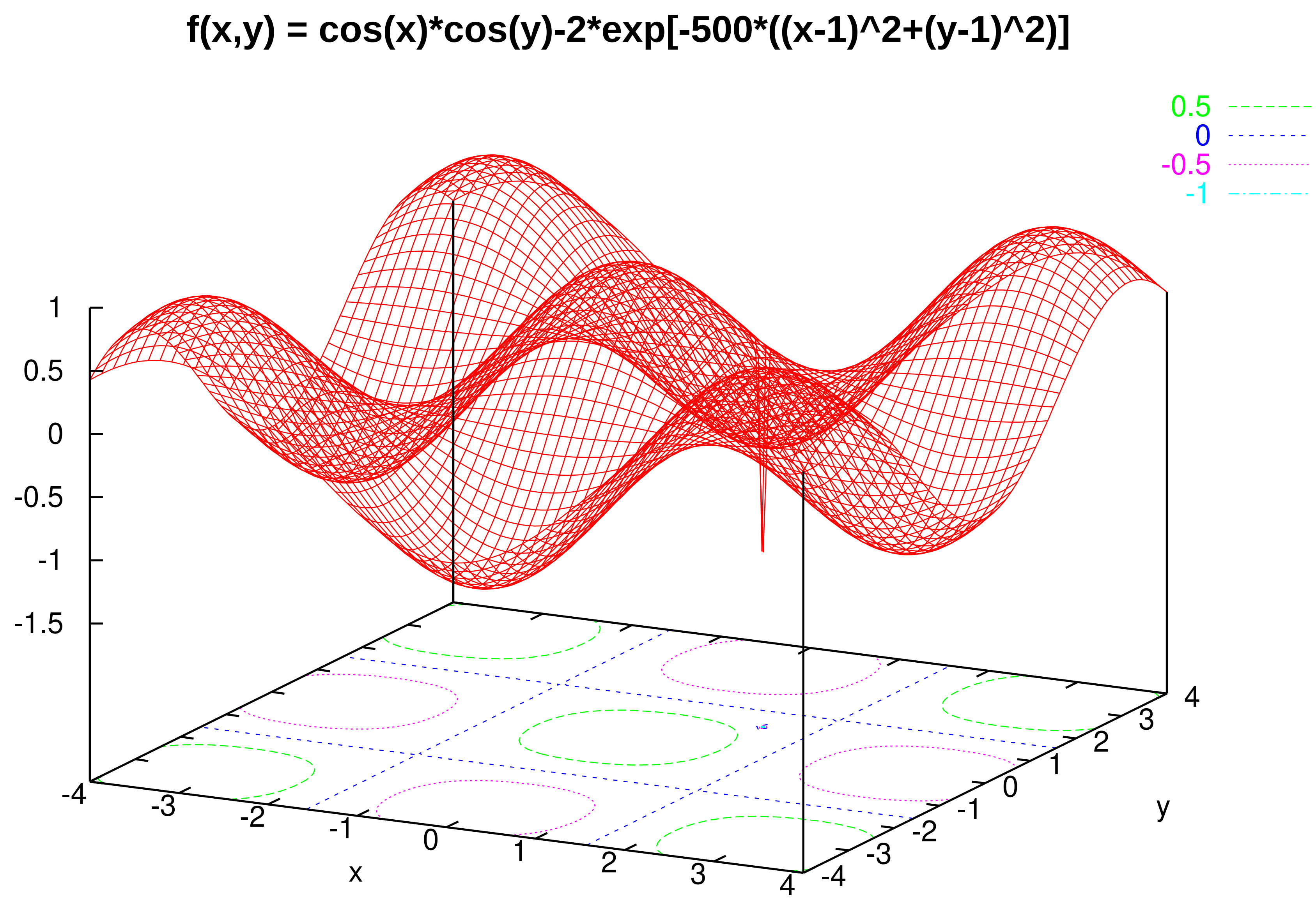
- LDB (Linear Dominated Bounding) bounding and domain reduction
- QFB (Quadratic Fast Bounding) bounding and domain reduction for positive definate cases (Quadratic pruning)
- Various cutoff value update schemes

And, we have completed

- Adjustment to pallarel environments with low inter-processoer communication rate
- Restart capability
- Continuation of computations while the underlying arithmetic fails
- COSY INFINITY Version 9.0 has been released

And, what we are doing further...

- High-order derivative based box rejection and the domain reduction
- Supporting high multiple precision computations for TMs

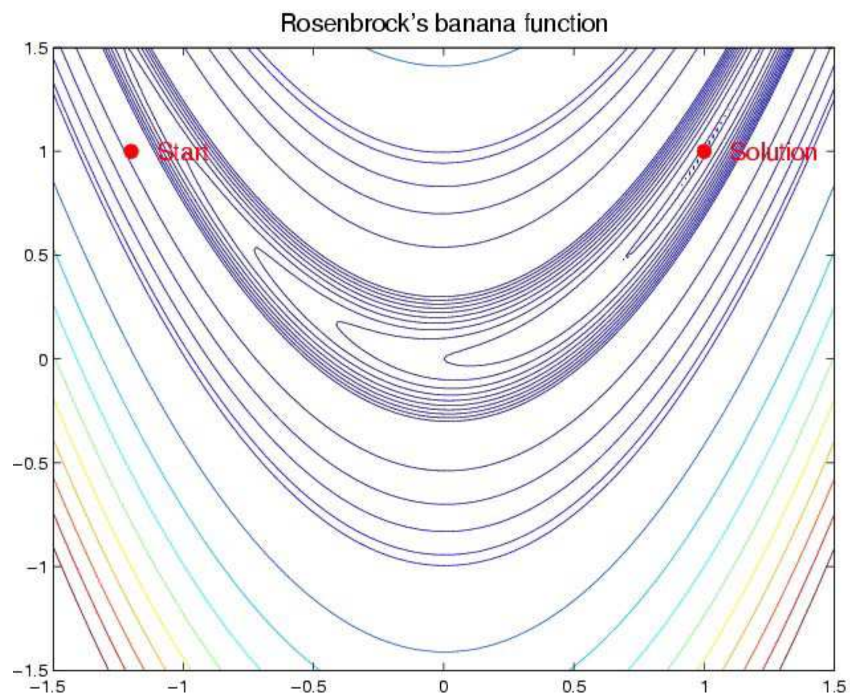


f(x,y) = cos(x)*cos(y)-2*exp[-500*((x-1)^2+(y-1)^2)]
0.5
0
-0.5
-1
1
0.5
0
-0.5
-1
-1.5
-4
-3
-2
-1
0
1
2
3
4
x
-4
-3
-2
-1
0
1
2
3
4
y

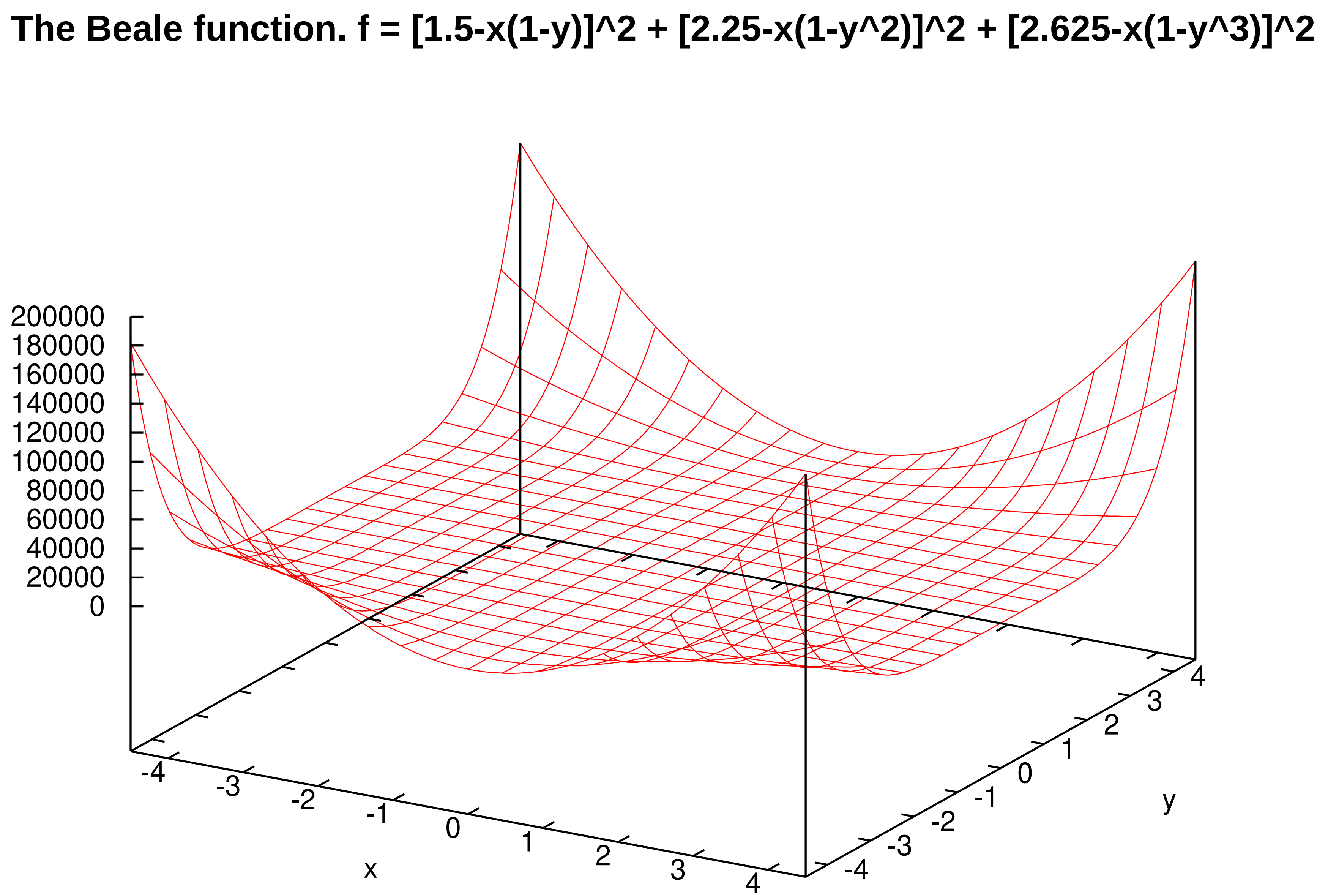
Rosenbrock's "Banana" Function

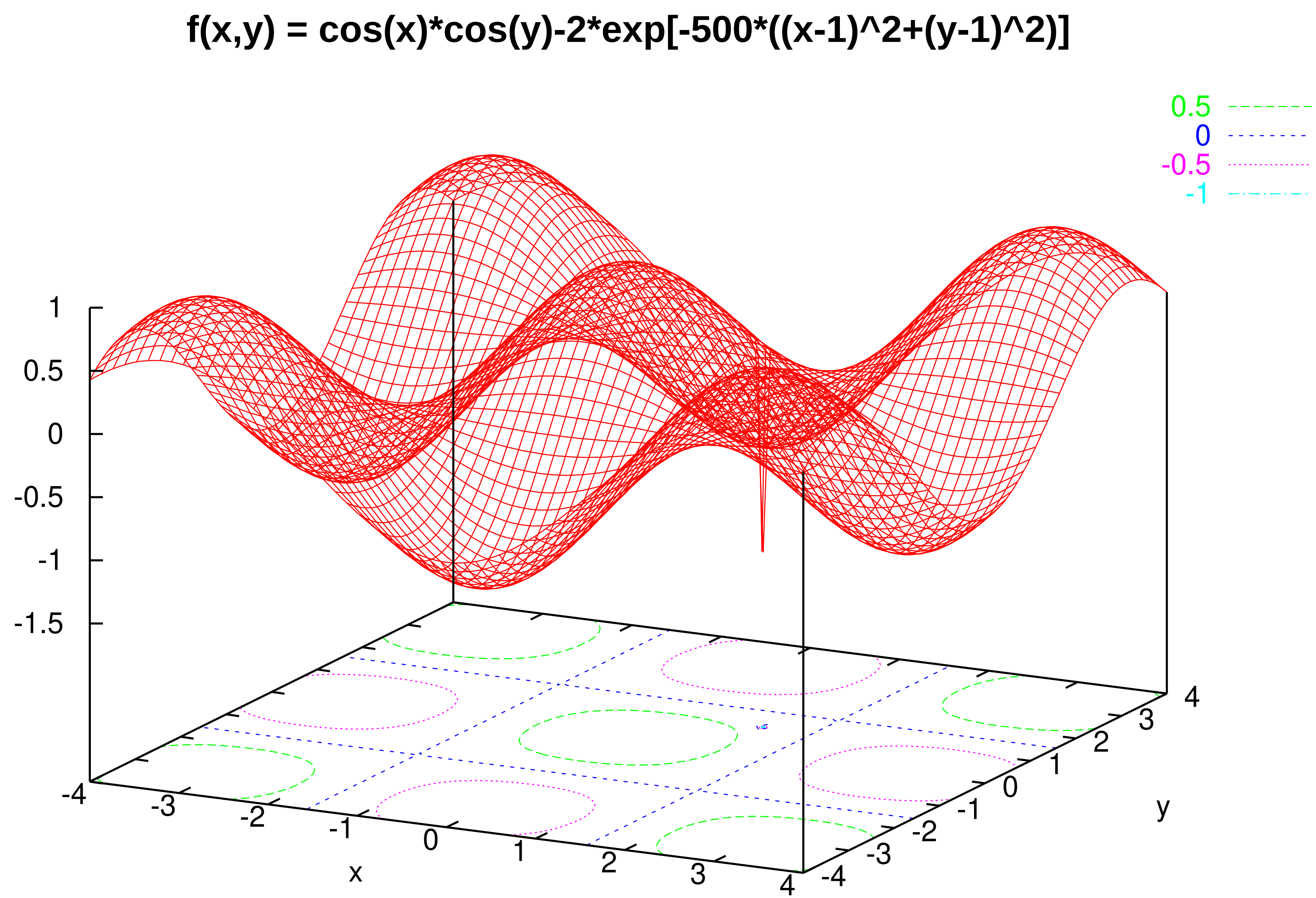
$$f(x, y) = 100 \cdot (y - x^2)^2 + (1 - x)^2$$

Study on $[-1.5, 1.5] \times [-1.5, 1.5]$.
Assumes min 0 at $(1, 1)$, but it is very difficult for gradient methods.



Picture from http://www.math.wm.edu/~buckaroo/classes/csci638/homework/project2.html


The Beale function. f = [1.5-x(1-y)]^2 + [2.25-x(1-y^2)]^2 + [2.625-x(1-y^3)]^2
200000
180000
160000
140000
120000
100000
80000
60000
40000
20000
0
-4
-3
-2
-1
0
1
2
3
4
x
-4
-3
-2
-1
0
1
2
3
4
y


f(x,y) = cos(x)*cos(y)-2*exp[-500*((x-1)^2+(y-1)^2)]
0.5
0
-0.5
-1
1
0.5
0
-0.5
-1
-1.5
-4
-3
-2
-1
0
1
2
3
4
x
-4
-3
-2
-1
0
1
2
3
4
y

Example

An extremely simplified model of some QCD optimization problems, scaled down to 2D for the illustration. Find the minimum of the function $f(x, y)$:

$$f(x, y) = \cos x \cos y - 2 \exp\left[-500 \cdot \left((x-1)^2 + (y-1)^2\right)\right]$$

- A spike on top of the periodic cos behavior.

$$\min = -1.70817675216073^{26}_{39} \quad \text{at } x = y = 1.0002272^{92}_{85}.$$

- Use various local optimization algorithms

  Simulated Annealing, Simplex, LMDIF (available as "**FIT**" in COSY)

Algorithm	Without Constraint		Constraint $\|x\|, \|y\| \le 4$	
	Steps	min	Steps	min
Simulated Annealing	1000	~ -1	1000	10^{-2} accurate
Simplex	130	~ -1	130	~ -1
LMDIF	27	~ 0	57	~ -1

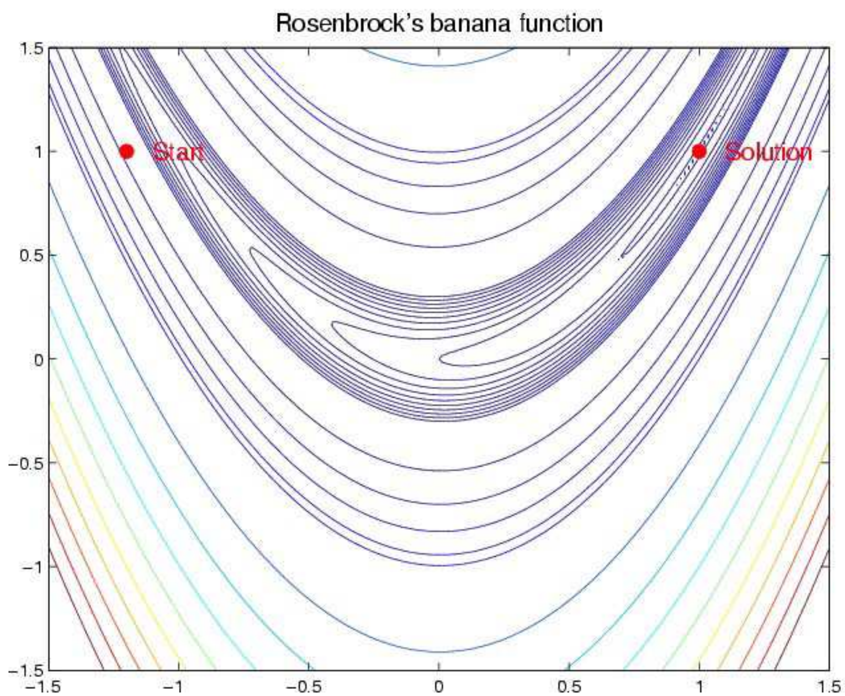
- Use COSY-GO (**verified** global optimizer)

  In the search domain $[-4, 4] \times [-4, 4]$, the minimum is found with 10^{-14} accuracy in 129 steps. The minimizer is localized in the volume $5 \cdot 10^{-17}$.

Rosenbrock's "Banana" Function

$$f(x, y) = 100 \cdot (y - x^2)^2 + (1 - x)^2$$

Study on $[-1.5, 1.5] \times [-1.5, 1.5]$.
Assumes min 0 at $(1, 1)$, but it is very difficult for gradient methods.

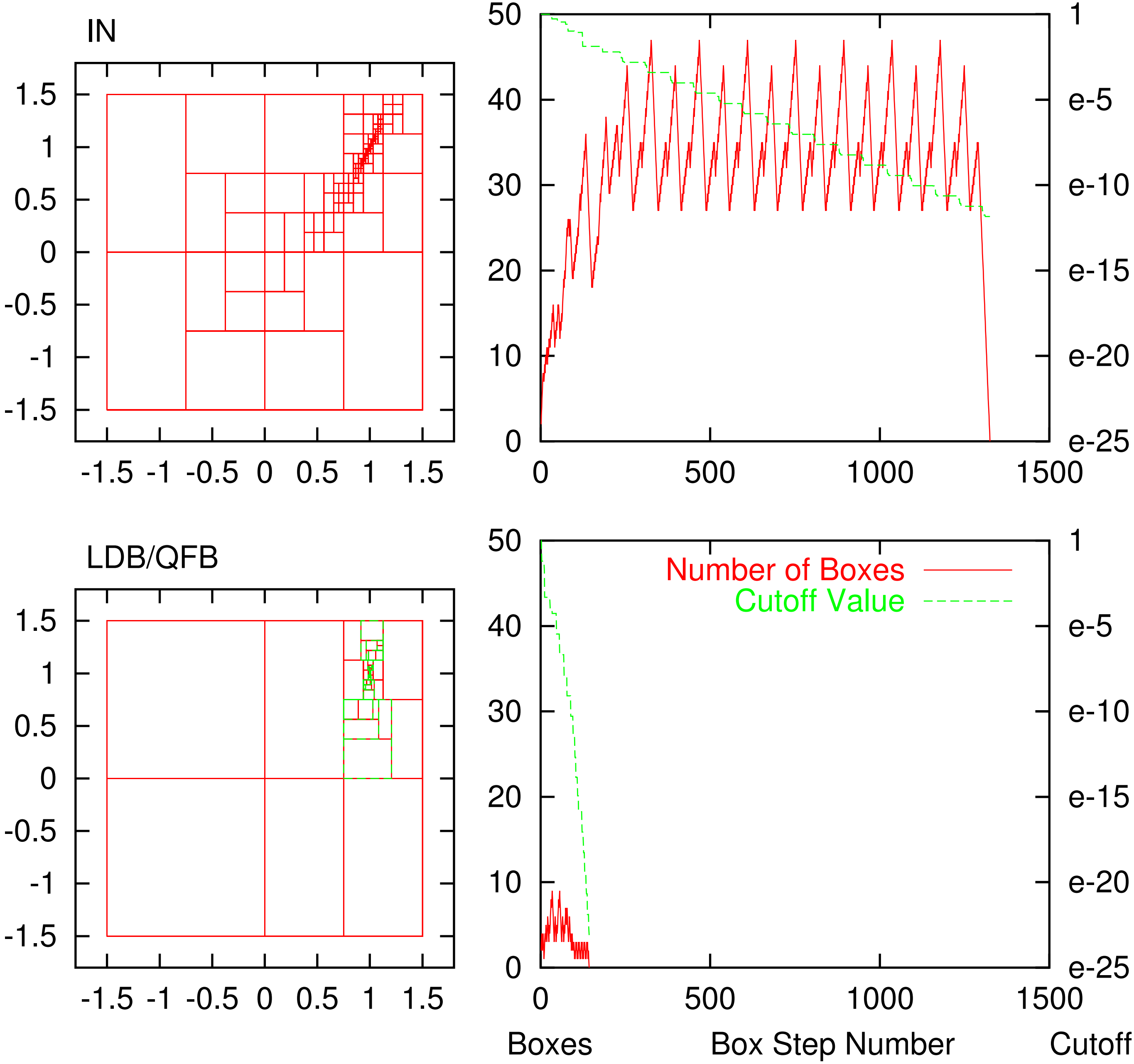


Picture from http://www.math.wm.edu/~buckaroo/classes/csci638/homework/project2.html

Nonvalidated Results of Rosenbrock's Function

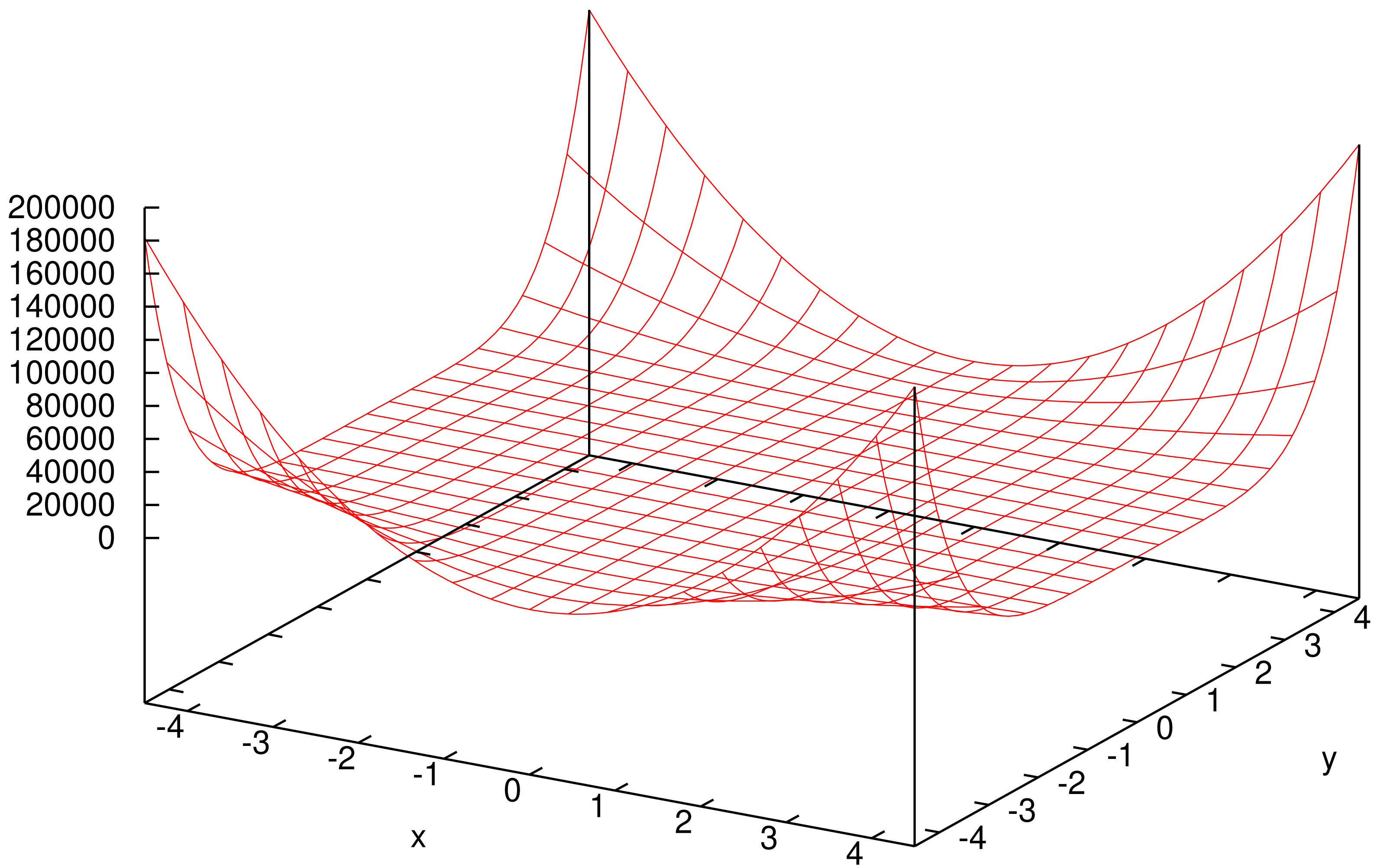
Using COSY's true and tested default optimizers:

```
Starting point (X,Y)> (-1.2,1.0)

Optimizer #> 1 : Simplex
Number of steps> 251
F> 0.1711168421282399E-16
(X,Y)> (1.000000004115731 ,1.000000008272989 )

Optimizer #> 2 : LMDIF
Number of steps> 70424
F> 0.9124815296170133E-10
(X,Y)> (0.9999904485839599,0.9999809108990141)

Optimizer #> 3 : Simulated Annealing
Number of steps> 100003
F> 0.5106520406572324E-05
(X,Y)> (0.9977499955081044,0.9954840773134492)
```



IN
LDB/QFB
Number of Boxes
Cutoff Value
Boxes
Box Step Number
Cutoff

The Beale function. f = [1.5-x(1-y)]^2 + [2.25-x(1-y^2)]^2 + [2.625-x(1-y^3)]^2



Beale's 2D and 4D Function

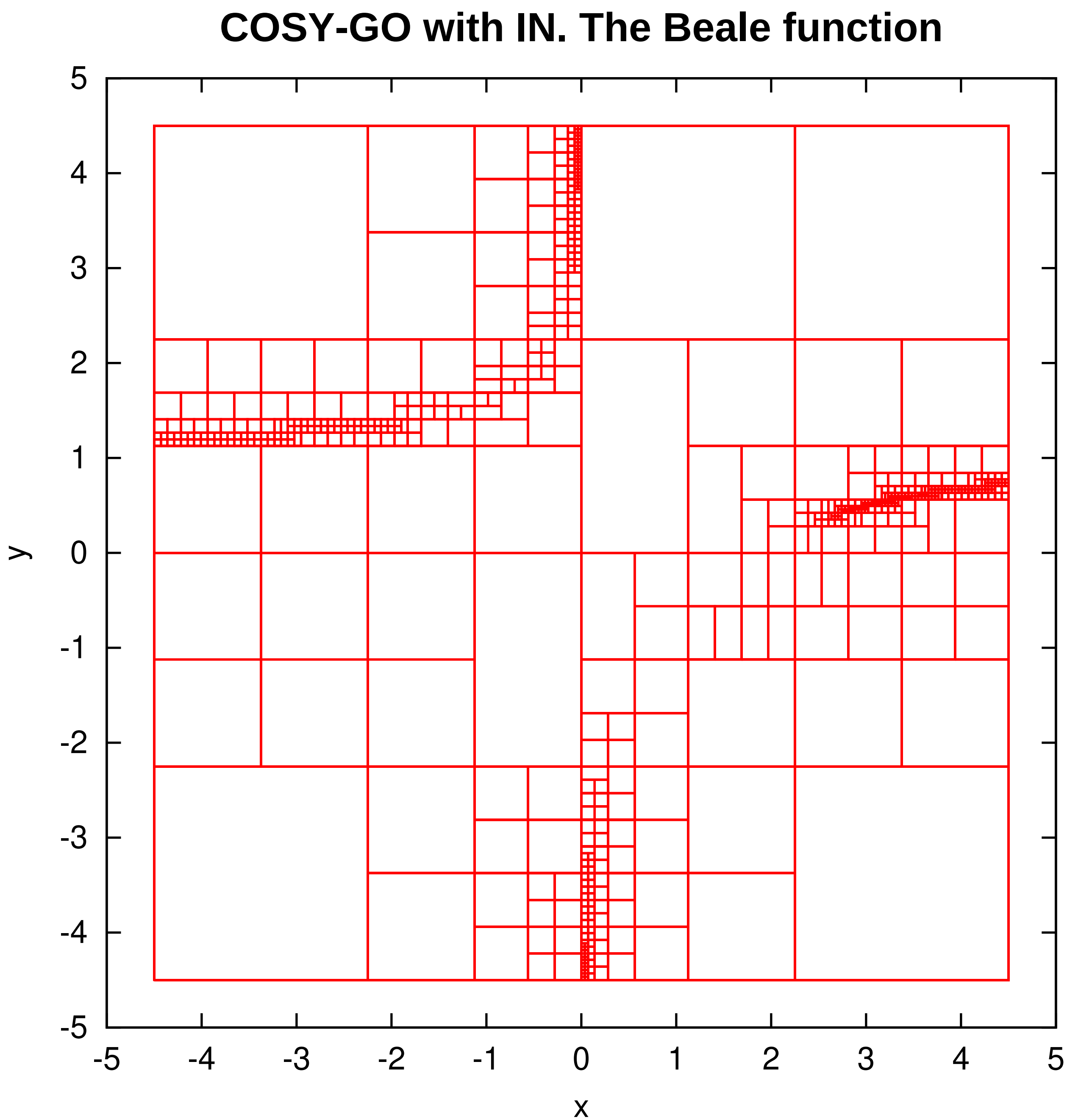
$$f(x_1, x_2) = (1.5 - x_1(1 - x_2))^2 + \left(2.25 - x_1(1 - x_2^2)\right)^2 + \left(2.625 - x_1(1 - x_2^3)\right)^2$$

Domain $[-4.5, 4.5]^2$. Minimum value 0 at $(3, 0.5)$.

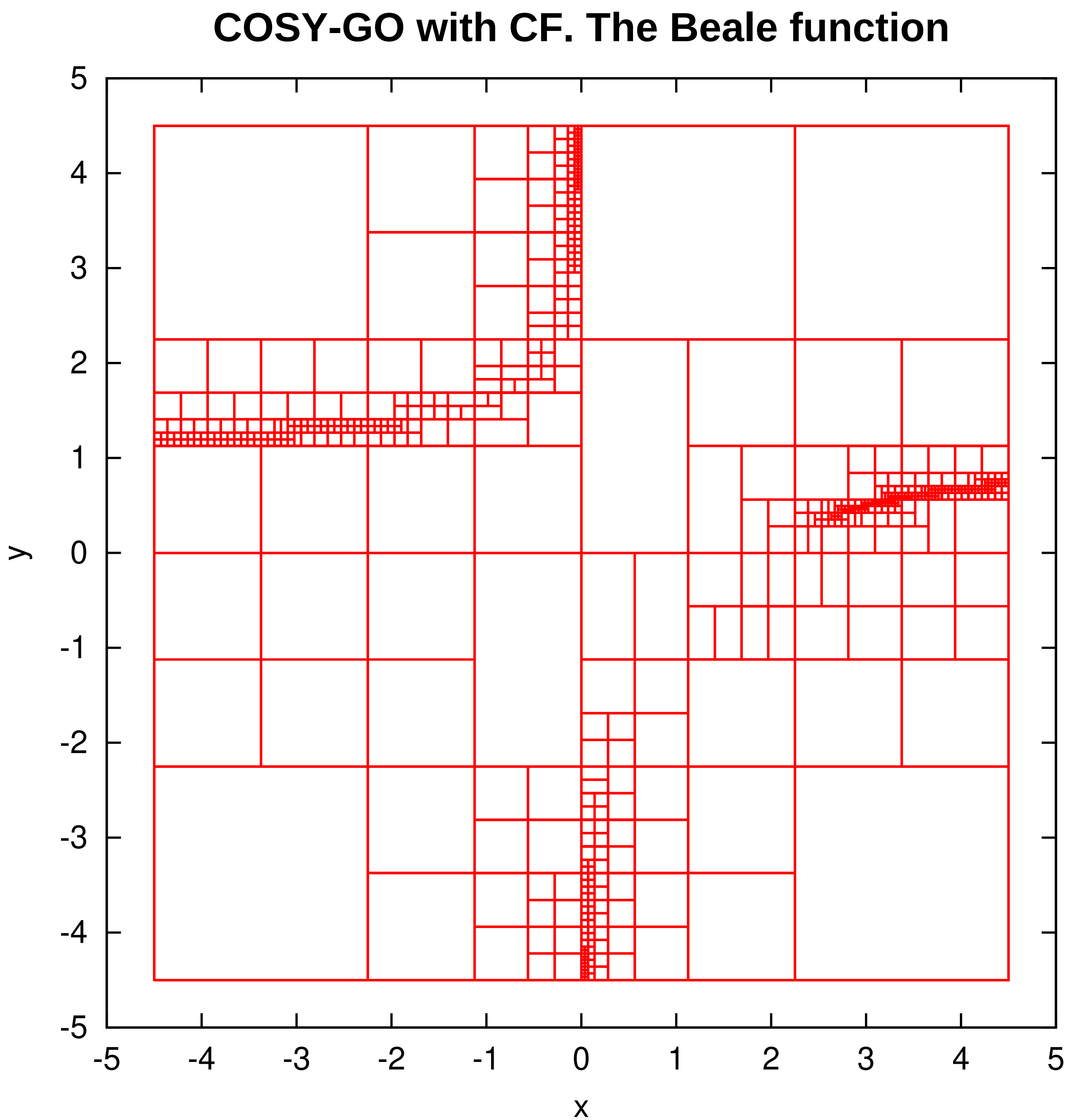
Little dependency, but tricky very shallow behavior.
Generalization to 4D:

$$\begin{aligned} f(x_1, x_2, x_3, x_4) = {} & (1.5 - x_1(1 - x_2))^2 + \left(2.25 - x_1(1 - x_2^2)\right)^2 + \left(2.625 - x_1(1 - x_2^3)\right)^2 \\ & + (1 + x_3(1 - x_4))^2 + \left(3 + x_3(1 - x_4^2)\right)^2 + \left(7 + x_3(1 - x_4^3)\right)^2 \\ & + (3 + x_1(1 - x_4))^2 + \left(9 + x_1(1 - x_4^2)\right)^2 + \left(21 + x_1(1 - x_4^3)\right)^2 \\ & + (0.5 - x_3(1 - x_2))^2 + \left(0.75 - x_3(1 - x_2^2)\right)^2 + \left(0.875 - x_3(1 - x_2^3)\right)^2 \end{aligned}$$

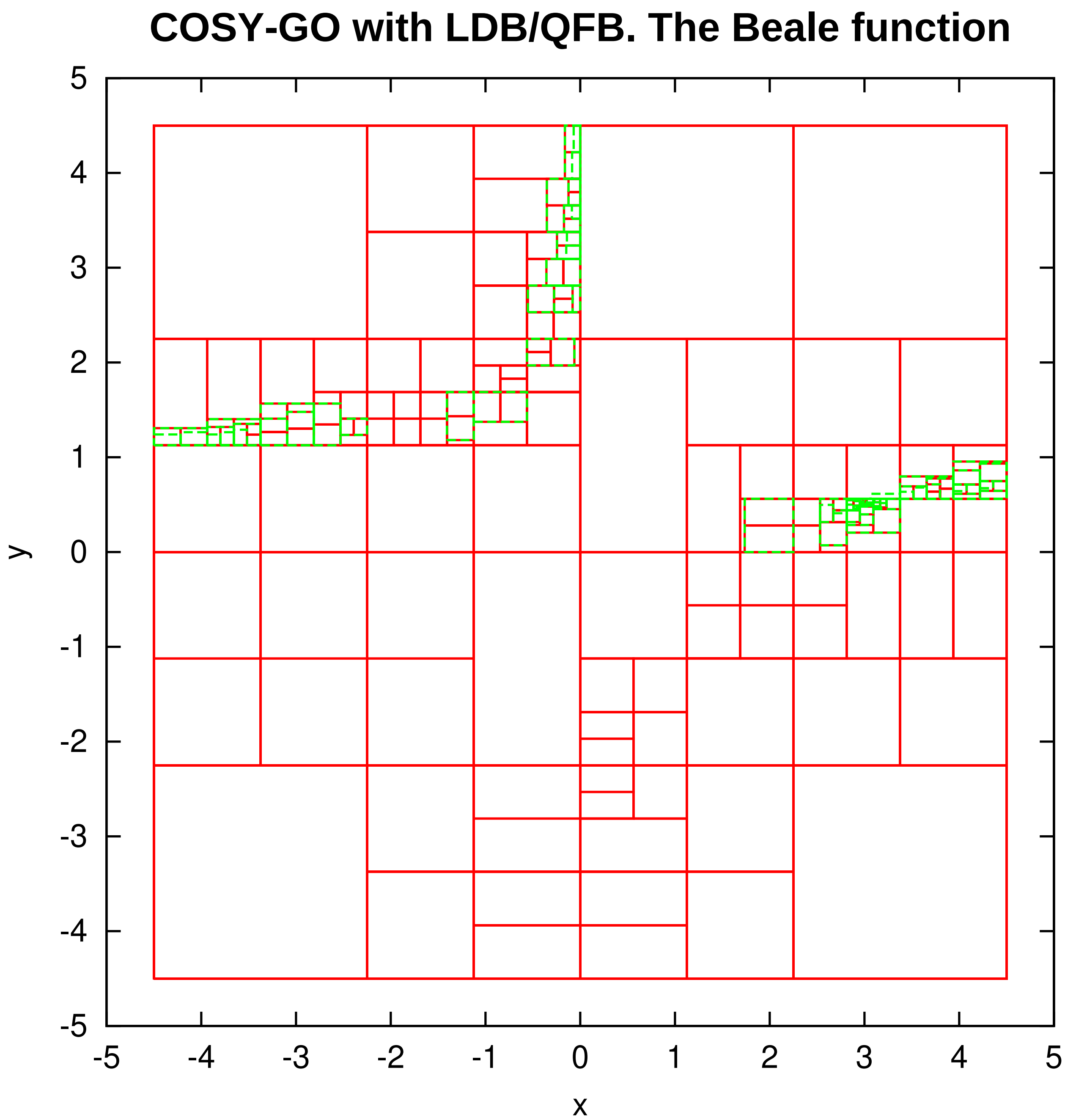
Domain $[0, 4]^4$. Minimum value 0 at $(3, 0.5, 1, 2)$



COSY-GO with IN. The Beale function
5
4
3
2
1
0
-1
-2
-3
-4
-5
y
-5
-4
-3
-2
-1
0
1
2
3
4
5
x



COSY-GO with CF. The Beale function
5
4
3
2
1
0
-1
-2
-3
-4
-5
y
-5
-4
-3
-2
-1
0
1
2
3
4
5
x


COSY-GO with LDB/QFB. The Beale function
5
4
3
2
1
0
-1
-2
-3
-4
-5
y
-5
-4
-3
-2
-1
0
1
2
3
4
5
x

The Normal Form Defect Function

- **Extreme cancellation**; one of the reasons TM methods were invented
- Six-dimensional problem from dynamical systems theory
- Describes invariance defects of a particle accelerator
- Essentially composition of three tenth order polynomials
- The function vanishes identically to order ten
- Study for $a\cdot\ (1,1,1,1,1,1)$ for $a = .1$ and $a = .2$
- Interesting **Speed observation**: on same machine,
  * one CF in INTLAB takes 45 minutes
  * one TM of order 7 takes 10 seconds

$$f_4(x_1,..,x_6) = \sum_{i=1}^{3}\left(\sqrt{y_{2i-1}^2 + y_{2i}^2} - \sqrt{x_{2i-1}^2 + x_{2i}^2}\right)^2$$

$$\text{where } \vec{y} = \vec{P}_1\left(\vec{P}_2\left(\vec{P}_3(\vec{x})\right)\right)$$

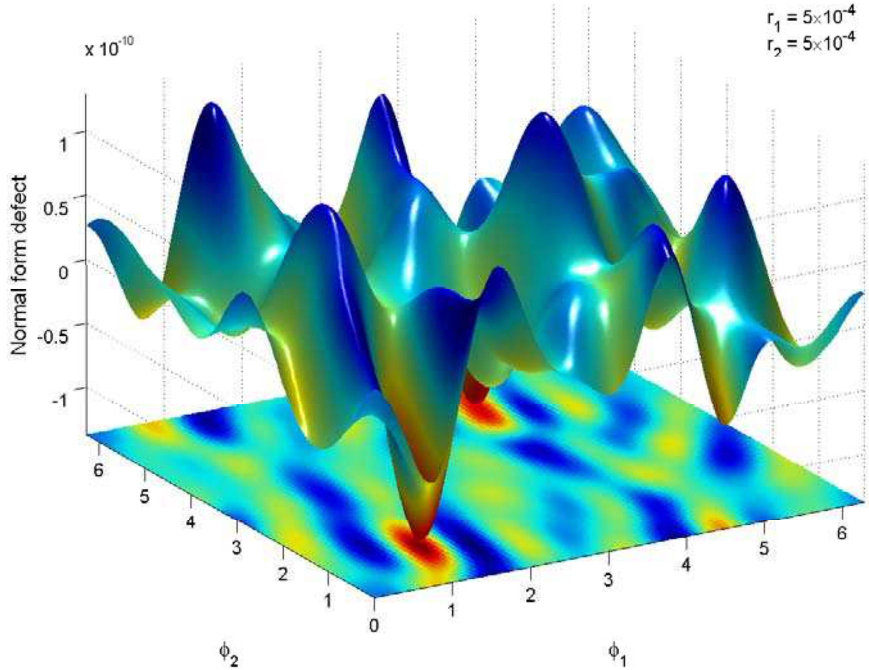


Fig. 9. Projection of the normal form defect function. Dependence on two angle variables for the fixed radii $r_1 = r_2 = 5 \cdot 10^{-4}$

Region	Boxes studied	CPU-time	Bound	Transversal Iterations
$[0.2, 0.4] \cdot 10^{-4}$	82,930	30,603 sec	$0.859 \cdot 10^{-13}$	$2.3283 \cdot 10^{8}$
$[0.4, 0.6] \cdot 10^{-4}$	82,626	30,603 sec	$0.587 \cdot 10^{-12}$	$3.4072 \cdot 10^{7}$
$[0.6, 0.9] \cdot 10^{-4}$	64,131	14,441 sec	$0.616 \cdot 10^{-11}$	$4.8701 \cdot 10^{6}$
$[0.9, 1.2] \cdot 10^{-4}$	73,701	13,501 sec	$0.372 \cdot 10^{-10}$	$8.0645 \cdot 10^{5}$
$[1.2, 1.5] \cdot 10^{-4}$	106,929	24,304 sec	$0.144 \cdot 10^{-9}$	$2.0833 \cdot 10^{5}$
$[1.5, 1.8] \cdot 10^{-4}$	111,391	26,103 sec	$0.314 \cdot 10^{-9}$	$0.95541 \cdot 10^{5}$

Table 8

Global bounds obtained for six radial regions in normal form space for the Tevatron. Also computed are the guaranteed minimum transversal iterations.

Rigorous High-Order Methods in the Description of Large Particle Accelerators

Martin Berz   Notes by Jeffrey Heninger

Particle Accelerators

much of the early work was done at Berkeley by Lawrence

van der Graaf – high voltage
step up a tube so the particles traveling ~~between~~ from sphere go into chamber
double energy obtained – start at zero voltage and end – change sign of beam so it accelerates both times
linear accelerator – repeat this

cyclotron
particle traveling in circle due to magnetic field
goes ~~between~~ between two metal "D"s – held at different voltages
voltage of "D"s oscillates with cyclotron frequency so the particles gain energy each ~~[illegible]~~ time around
limitations – to get more energy, you need either stronger magnetic field or larger radii.
once the radius becomes large enough, they leave the cyclotron.

instead of making solid "D"s, only confine along the path of the beam
adjust the magnetic field strengths as the particles gain energy
~~[illegible]~~ synchrotrons
dipole magnets to bend beam, combination of quadrupole fields to focusing

motion in a large synchrotron
large size, high speed ($\simeq c$), long storage time, many magnets → 10^{14} contacts with fields

a closed orbit (center) in the beam
perturbation theory has been used for decades.
how can we do better?

function algebras → Taylor polynomials
↓ $+, \cdot, \partial$ ↓ $\oplus, \odot, \textcircled{\partial}$
differential algebras → differential algebra that works on a computer

11^{th} order – takes ~2000 terms

How do you design the accelerator so it is stable?

p2

Rigorous Methods

Set Intervals – commuting diagram for algebra.
as you map intervals, they can grow → can't guarantee the interval stays on attractor

take the object that encloses the attractor
 do a Taylor model
 map this forward – it lies within itself
 this guarantees that there is an attractor

ex. Hénon Map

enclosure of manifold
 high order Taylor approx. of manifold near fixed point
 take a strip around it & show that its iterate is narrower
 → rigorous enclosures
as you go along the manifold, if it becomes too nonlinear, split it into pieces &
 do a Taylor model for each

allows you to find all homoclinic intersections up to a given order number
untangle stable & unstable manifolds
 stable = straight line, unstable loops around it
 → rigorous symbolic dynamics
 → bounds for topological entropy

make this into a rigorous integrator
re-write differential equation as an integral
the Taylor model just sees this as an operator.

Schauder's Thm

to get an N^{th} order integrator, iterate this N times starting with identity

try to find an interval that is mapped into itself by this operator
this guarantees that the solution is in this interval.

ex. Duffing Equation

if things get too large, split the box in two
so each one can be rigorously pushed forward with known error
 up to the final time

Phase Space Motion for Particle Accelerators

convert to normal form coordinates (to high order)
→ approximate invariant circles
Nekhoroshev - type estimates for ~~error~~ ~~or~~ worst error after a certain number of turns
ignores a lot of subtle details, but is rigorous

normal form defect function - how good of an invariant is it?
extreme cancelation problem - substract two similar numbers
Taylor methods ~~much~~ are much better to deal with this

size of radius / normal form defect ≈ ~~bound for~~ bound for number of iterations it's stable for

tries the method out for various pathological examples

shows that the accelerator is stable for at least $10^7 - 10^8$ turns