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Name: Ian Coley **Email/Phone:** msri@iancoley.org

Speaker's Name: Constantin Teleman

Talk Title: Characters of categorical representations: theory and applications

Date: 3 / 25 / 19 **Time:** 9 : 30 [✓ am] / pm (circle one)

Please summarize the lecture in 5 or fewer sentences:
Categorical representations generalize topological representations on vector spaces. This talk discusses what categories we want to study representations on and some concrete applications to symplectic geometry, quantum cohomology, and physics-inspired results

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CHARACTERS OF CATEGORICAL REPRESENTATIONS: THEORY AND APPLICATIONS

CONSTANTIN TELEMAN

For this talk, let G be a compact (usually connected) Lie group. The agenda: a survey of what we currently know, with details left out to make sure we talk about as much as possible.

Why study categorical representations? First, categories are for 2D quantum field theories are what vector spaces are for 1D QFTs. Second, there are many examples of categorical representations that come from symplectic geometry.

If X is a compact symplectic manifold with a Hamiltonian G-action, then G acts on $\mathcal{F}(X)$ the Fukaya category.

How do we begin studying representations? Start with the character theory. In fact, the character theory in this categorical settings seems to have “cleaner” output than input, e.g. the input is a derived category but the output is underived.

We put the following hypothesis on the categories on which G will act: they will be smooth, proper, Calabi-Yau, and their deformation theory is unobstructed . This is an unlikely but ideal situation which will still arise in practice.

Note: categorical representation theory is *not* an analogue of representation theory of group actions on a vector space, but rather an analogue of *topological* actions on a vector space. That is, a flat vector bundle over BG gives a local system of $\pi_0 BG$-actions. If we take *complexes* of vector bundles, we then get information about our group past π_0.

The topological classification of such vector bundles is the cohomology group $H^1(BG; \mathrm{GL}(V)_{\mathrm{disc}})$ giving the coefficients the discrete topology. If G is connected, we don't see anything here ordinarily.

Notes by Ian Coley.

However, if we replace V the vector space by a chain complex of vector spaces, then we can describe the above as the set of differential graded algebra maps $C_*(G) \to \mathrm{End}(V)$, and that's the description that we are going to categorify.

At just the level of vector spaces, we have a total answer about topological representations of V but it's a but nasty to compute. We also have the total story once we lift to categories, but the answer is (surprisingly) nicer.

Example 1. Let $G = T$ a torus, acting on **Vect** the category of complex vector spaces. Write $T = \mathfrak{t}/\pi_1$. Then if we look at the topological actions of T, the contractible parts of T must act trivializably, so $\mathfrak{t}$ acts trivializably and π_1 acts trivially. An automorphism of the trivial action of π_1 on **Vect** correponds to a map $\pi_1 \to \mathbb{C}^\times$, i.e. a point in the complexified dual torus $T^\vee_\mathbb{C}$.

So $T^\vee_\mathbb{C} = H^2(BT; \mathbb{C}^\times_{\text{disc}})$. But something is wrong here: if we let $G = SO(2)$ or $SU(n)$, this group vanishes. Yet there is a nontrivial representation $SU(n)$ acting on $\mathbb{P}^{n-1}$ and $\mathcal{F}(\mathbb{P}^{n-1}) \cong \mathbf{Vect}^{\oplus n}$.

Where's the error? Well, we actually have to collapse the cohomological grading from $\mathbb{Z}$ to $\mathbb{Z}/2$. Here is the scheme to fix this: let β be an indeterminate of degree 2. Then consider $H^2(BG; \mathbb{C}((\beta))^\times)$, which is our stand-in for $H^{\text{even}}(BG; \mathbb{C}^\times)$. This has the form of a Brauer group $H^2(-; \mathcal{O}^\times)$, where here we are using $\mathcal{O} = H^*\mathbb{C}((\beta))$ but there is also a story for $\mathcal{O} = K^{\text{top}}$.

At any rate, $\mathbb{C}((\beta))^\times = \mathrm{GL}_1(\mathbb{C}((\beta)))$ so we can identify $H^0(BG; \mathbb{C}((\beta))\times)$ as maps

$$\mathfrak{g}/G = \mathfrak{t}/W \to \mathbb{C}^\times.$$

If we transgress over S^2, we get a class in $H^0(\Omega^2 BG/_{\mathrm{Ad}}G; \mathbb{C}((\beta))^\times)$. Assuming that $G = T$ a torus for the moment, this group is $H^0(BT \times \pi_1; \mathbb{C}((\beta))^\times)$ of maps linear in π_1, which in tern allows us to identify it as $H^0(BT; H^0(\pi_1; \mathbb{C}((\beta))^\times))$, where this coefficient system is our dual torus $T^\vee_\mathbb{C}$ with the grading collapsed. So ultimately, we have a map $\mathfrak{t} \to T^\vee_\mathbb{C}$ again which is the exponentiated differential of a function, giving rise to a Lagrangian $L \subset \mathfrak{t}_\mathbb{C} \times T^\vee_\mathbb{C} = T^*T^\vee_\mathbb{C}$ the cotangent bundle over the dual torus, which is degree 1 and finite over $\mathfrak{t}_\mathbb{C}$. This gives us a geometric way to think about a particular "Brauer class".

Well, what if G is nonabelian? We can guess that the Lagrangian should now live in $(T^*T^\vee_\mathbb{C})/W$. This has a symplectic resolution, which we can think about in one of the following 5 ways:

Theorem 2 (Bezrukavnikov, Mirković, Finkelberg)**.** The symplectic resolution of the above, which we call $C_3(G)$, has the following properties:

1. Its functions are Weyl-invariants in $\mathbb{C}[T^*T^\vee_\mathbb{C}][e^{\alpha^\vee}-1/\alpha]$, where α are the roots
2. It is isomorphic to $T^*_{\mathrm{reg}}G^\vee_\mathbb{C}//_{\mathrm{Ad}}G^\vee_\mathbb{C}$
3. It is also isomorphic to $N_\chi\backslash\backslash T^*G^\vee_\mathbb{C}//_{-\chi}N$, where χ is a regular nilpotent character
4. It is also isomorphic to $\operatorname{Spec} H^G_*(\Omega G)$, equipped with a Pontryagin product.
5. It is algebraic symplectic with a symplectic form of degree 2, which provides an E_3-structure on its functions.

Theorem 3. Let $\mathcal{C}$ be a "nice" (e.g. satisfying all the assumptions made above) super-dg category with a topological G-action.

- $HH^*_G(\mathcal{C})$ is an E_2-algebra and an algebra over $C_3(G)$ compatible with the E_3-structure.
- There is a sheaf of categories over $\operatorname{Spec} HH^*_G(\mathcal{C})$ with E_3-action compatible with that of $C_3(G)$.
- $\operatorname{Spec} HH^*_G(\mathcal{C})$ has Lagrangian support which is finite over the base $\mathfrak{t}_\mathbb{C}$.

Example 4. If $\mathcal{C}$ is nice with a trivial G-action, then for $\psi\colon \mathfrak{t} \to HH^{\mathrm{even}}(\mathcal{C})$, the element $e^\psi \in H^{\mathrm{even}}(BT; \mathrm{GL}_1(HH^{\mathrm{even}}))$ gives an interesting action on $\mathcal{C}$. Moreover, a nontrivial flat structure on $HH^*(\mathcal{C})$ gives a deformation of the trivial G-action. As a special case, if $\mathcal{C} \cong \bigoplus \mathbf{Vect}$ is semisimple, then such a deformation gives a curving for $\bigoplus \mathbf{Vect}$ over $\mathfrak{t}$.

Model representations analogous to finite dimension representations of G:

1. The invariant category computation, $\mathcal{C}^G$: $\mathcal{C} \otimes \mathrm{Coh}(\mathfrak{t})$ with deformation class ψ. As a subexample, if $\mathcal{C} = \mathbf{Vect}$, then the invariant category is $MF(\mathfrak{t};\psi)$ the matrix factorisation category.
2. The "space of states" $HH^*(\mathcal{C}^G)$. G acts on $\mathcal{C}$, which means that every $g \in G$ has a corresponding element $HH^*(g\cdot;\mathcal{C})$ on which the stabilizer of g acts. Because the action is topological, this gives rise to a derived local system over G, equivariant under the action of conjugation, hence $HH^*(\mathcal{C}^G) = H^G_*(G;\mathcal{HH}^*)$ with a Pontryagin product, where $\mathcal{HH}^*$ is the local system above.
3. Quantum GTT Conjecture. If X is Fano, and G acts freely on $X^{\mathrm{semistable}}$, then $HH^*(\mathcal{C}^G) = QH^*_G(X//_G)$ (where $\mathcal{C}$ is the Fukaya category of X). The proof that they are the same multiplicatively goes back to Wehrheim-Woodward, coming from the quantum Kirwan map. That they are the same additively comes from a Witten deformation of Floer homotopy. These proofs seem to be totally separate.

(4) Batyrev presentation of $QH^*(\mathbb{C}^N/!/\text{torus})$ in the Fano case. We have

$$QH^*(\mathbb{C}^N//\text{torus}) = QH^*_T(\mathbb{C}^N)/(\text{Seidel monodromies } = 1) = H^*(T; H^*(BT))$$

where this last coefficient system needs to be twisted.

(5) Mirrors of flag varieties. We have from a theorem above that

$$T^*_{\text{reg}}G^\vee_\mathbb{C}//_{\text{Ad}}G^\vee_\mathbb{C} \cong N_\chi\backslash\backslash\, T^*G^\vee_\mathbb{C}\,//_{-\chi}N$$

The lefthand side is foliated by G-equivariant mirrors of flag varieties, giving a vertical Lagrangian foliation on the righthand side, which recovers T-equivariant Rietsch mirrors. There is also follow-up work on generalized flags.

(6) Coloumb branches for nonzero representations.. Nakajima and Brauerman-Nakajima-Finkelberg defined a version of $C_3(G; V)$ for V a G-representation. Fact: we can actually derive their definition from just $C_3(G; 0)$ and a rational Lagrangian section, or equivalently a mirror of V as a symplectic G-fold.