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# NOTETAKER CHECKLIST FORM

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**Speaker's Name:** Bhargav Bhatt

**Talk Title:** Prismatic cohomology and applications

**Date:** 3 / 25 / 19 **Time:** 11 : 00 [✓ **am**] / **pm (circle one)**

**Please summarize the lecture in 5 or fewer sentences:**

Prismatic cohomology allows for new approaches to results in crystalline, étale, and de Rham cohomology. This talk defines the theory and gives some concrete applications to those particular cases

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# PRISMATIC COHOMOLOGY AND APPLICATIONS

BHARGAV BHATT

The goal of this walk, unlike previous talks, is to be get to applications and perhaps miss a bit of detail on the way. Fix $p$ a prime, as everything is $p$-local.

**Theorem 1** (Bhatt-Morrow-Scholze)**.** Let $C/\mathbb{Q}_p$ be a complete and algebraically closed field, and let $\mathcal{O}_C$ be the valuation ring. Let $X/\mathcal{O}_C$ be a proper, smooth (formal) scheme. Then $\dim H^i_{\text{ét}}(X_C; \mathbb{F}_p) \leq \dim H^i_{\text{dR}}(X_k)$, where $X_k$ denotes the special fibre.

In the process of proving this, they made a new cohomology theory that specialised down to both sides, specifically a deformation of de Rham cohomology. How can we broaden the context in which such a thing is possible?

Goal: present this context site-theoretically, giving a uniform framework for finding deformations of our favourite cohomology theories.

## I. Prisms

**Definition 2.** A *prism* is a pair $(A, I)$ that:

(1) $A$ is a commutative ring with a (derived) lift of Frobenius $\phi$, along with higher homotopical data telling us why $\phi$ lifts Frobenius.
(2) $I \subset A$ is an ideal defining a Cartier divisor, and $A$ is $p$-adically and $I$-adically complete.
(3) $p \in (I, \phi(I))$. That is, any local generator $d$ of $I$ satisfies $\phi(d) = d^p + p \cdot u$ for some unit $u \in A^\times$.

The idea is that we are working in mixed characteristic, and $\phi$ doesn't change $I$ in characteristic $p$ but it does everywhere else.

**Example 3.** (1) Crystalline prisms. Let $A$ be any $p$-complete, $p$-torsion free $\delta$-ring. Let $I = pA$ and ensure $\phi(p) = p$. Note that $p = p^p + p(1 - p^{p-1})$, and $1 - p^{p-1} \in A$ is a unit since $A$ is $p$-complete, so we satisfy (3) above.

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Notes by Ian Coley.

Specifically, we could let $A = \mathbb{Z}_p$ and $\phi = \mathrm{id}$. *Note:* We never had an actual definition for $\delta$-ring in this talk, but in this case it stands for "$I$ is a Cartier divisor".

(2) Breuil-Kisin prisms. $A = \mathbb{Z}_p[[u]]$, $\phi(u) = u^p$, and $I = (u - p)A$.

(3) q-dR (quantum de Rham) prisms. $A = \mathbb{Z}_p[[q-1]]$, $\phi(q) = q^p$. This is the same ring as above, but has a different structure as a $\delta$-ring. $I = [p]_q A$, where $[n]_q = \frac{q^n-1}{q-1}$ for any $n \in \mathbb{N}$.

In each of the above examples, the de Rham cohomology of $A/I$ was known to be liftable to $A$.

(4) Perfect prisms. Call $(A, I)$ *perfect* if $\phi\colon A \xrightarrow{\cong} A$. Observe: any prism has a perfection given by $\operatorname{colim}(A \xrightarrow{\phi} A \xrightarrow{\phi} \cdots)$, which may need to be $(p, I)$-adically completed afterwards. Thus we might as well think of all our prisms as pefect.

**Proposition 4.** The map $(A, I) \mapsto A/I$ gives an equivalence

$$\{\text{perfect prisms}\} \xrightarrow{\sim} \{\text{perfectoid rings}\}$$

### II. Prismatic cohomology

Fix a prism $(A, I)$ and $X$ a $p$-adic (formal) scheme over $A/I$.

**Definition 5.** The *prismatic site of* $X$ denoted $(X/A)_\Delta$ has objects prisms $(B, J)$ with $(A, I) \to (B, J)$ a map of prisms (so a map of commutative rings compatible with $\phi$ and sending $I$ to $J$) anlong with a map $\operatorname{Spf}(B/J) \to X$. Take for the topology the étale topology on $\operatorname{Spf}(B)$ with respect to the $p$-adic completion (not the $J$-adic one).

If $X$ is affine, the topology won't turn out to matter, so might as well use the indiscrete one.

We have two natural presheaves $\mathcal{O}_\Delta$ and $\overline{\mathcal{O}}_\Delta$ which send $(B, J)$ to $B$ or $B/J$ respectively. In the topologies allowed above, these are both sheaves.

**Lemma 6** (Rigidity Lemma)**.** If $(A, I) \to (B, J)$ is a map of prisms, then $I \otimes_A B \cong J$.

As a corollary, $\overline{\mathcal{O}}_\Delta \cong \mathcal{O}_\Delta \otimes^{\mathbb{L}}_A A/I$. Further note that $\overline{\mathcal{O}}_\Delta$ is a sheaf of $\mathcal{O}(X)$-algebras.

**Theorem 7** (Hodge-Tate comparison)**.** Assume $X = \mathrm{Spf}(R)$ is affine and smooth. Let $\Delta_{R/A} = \mathbb{R}\Gamma((X/A)_\Delta; \mathcal{O}_\Delta)$, where we note that $\phi$ acts on the site $(X/A)_\Delta$, giving an action on the global sections also. Let

$$\overline{\Delta}_{R/A} = \Delta_{R/A} \otimes^{\mathbb{L}}_A A/I \cong \mathbb{R}\Gamma((X/A)_\Delta; \overline{\mathcal{O}}_\Delta)$$

upon which $\phi$ also acts. Then there is a canonical isomorphism $H^n(\overline{\Delta}_{R/A}) \overset{\cong}{\to} \Omega^n_{R/A}\{n\}$, where $\{n\}$ denotes $\otimes_{A/I} I^n/I^{n+1}$. In particular, $\overline{\Delta}_{R/A}$ is a perfect complex.

**Corollary 8** (de Rham and crystalline comparison)**.** If $I = pA$, there exists a canonical isomorphism (up to Frobenius twist)

$$\phi_A^* \Delta_{R/A} \overset{\cong}{\to} \mathbb{R}\Gamma_{\mathrm{crys}}(R/A)$$

In general, there exists an isomorphism

$$(\phi_A^* \Delta_{R/A}) \otimes^{\mathbb{L}}_A A/I \overset{\cong}{\to} \Omega^*_{R/(A/I)}$$

**Lemma 9** (Key Lemma)**.** If $A$ is a p-torsionfree $p$-adically complete ring with a lift of Frobenius (aka $\delta$-ring), and $x \in A$ such that $\phi(x) \in pA$ if and only if $x^p \in pA$, then $\frac{x^n}{n!} \in A$ for all $n \in \mathbb{N}$, i.e. all divided powers of $x$ are in $A$.

Now what if $X$ is singular? *Derive everything*, because the site gets nasty.

### III. Derived prismatic cohomology

Fix a prism $(A, I)$, and assume all $A$-modules are $(p, I)$-complete.

**Definition 10.** Consider the subcategory

$\{$formally smooth $A/I$-algebras$\} \subset \{$$p$-complete simplicial commutative $A/I$-algebras$\}$

Then from the smaller category we had a functor $\Delta_{-/A}$ to $D_{\mathrm{complete}}(A)$. We denote still by $\Delta_{-/A}$ the left Kan extension of that functor along the inclusion of the subcategory, thus giving us

$$\Delta_{-/A} \colon \{p\text{-complete simplicial commutative } A/I\text{-algebras}\} \to D_{\mathrm{comp}}(A)$$

This is how we define the complex for singular schemes. Concretely, if $R$ is any $A/I$-algebra, then simplicially resolve it by smooth ones $P_\bullet \to R$ and take the colimit $\Delta_{P_\bullet/A} = \mathrm{Tot}(\Delta_{P_\bullet/A}) =: \Delta_{R/A}$

**Theorem 11** (Derived Hodge-Tate comparison)**.** For all $p$-complete $A/I$-algebras $R$, there exists a functorial increasing exhaustive filtration ("conjugate filtration") on $\overline{\Delta}_{R/A}$ (analogously defined) such that the associated graded is $\mathrm{gr}_i \cong \Lambda^i L_{R/(A/I)}[-i]\{-i\}$, where $L_{R/(A/I)}$ is the cotangent complex and the shift and twist is as before.

**Example 12.** Say $R = (A/I)/(f)$, $f$ a nonzerodivisor on $A/I$. Then the HT comparison theorem plus work implies that $\Delta_{R/A}$ is concentrated in degree zero, so it's just a ring! In fact, if we write $I = dA$, it is concretely

$$\Delta_{R/A} = A\left\{\frac{\widetilde{f}}{d}\right\}^{\wedge}$$

as a $\delta$-ring so that the lift of Frobenius survives the adjoining of these new elements (and $\widetilde{f}$ is a lift of $f$).

**Remark 13.** For $f = p^n$, this is useful in computing algebraic K-theory via THH.

### IV. Perfection in mixed characteristic

Fix a perfect prism $(A, I)$. Observe the following: if $R$ is a smooth $\mathbb{F}_p$-algebra, then

$$\begin{aligned} R_{\text{perf}} :&= \operatorname{colim}(R \xrightarrow{\phi} R \xrightarrow{\phi} \cdots) \\ &= \operatorname{colim}(\Omega^*_{R/\mathbb{F}_p} \xrightarrow{\phi} \Omega^*_{R/\mathbb{F}_p} \xrightarrow{\phi} \cdots) \\ &= \Big[\operatorname{colim} \mathbb{R}\Gamma_{\text{crys}}(R/\mathbb{Z}_p) \xrightarrow{\phi} \mathbb{R}\Gamma_{\text{crys}}(R/\mathbb{Z}_p) \xrightarrow{\phi} \cdots)\Big]/p \end{aligned}$$

That last construction is still doable in mixed characteristic.

**Definition 14.** For any $A/I$-algebra $R$, let $R_{\text{pfd}} = \Big[\operatorname{colim}(\Delta_{R/A} \xrightarrow{\phi} \cdots)\Big]^{\wedge}/I$.

A priori this is a derived object in $D(R)$ i.e. an $E_\infty$-ring if you are so inclined.

**Remark 15.** If $I = pA$, $R_{\text{pfd}} = R_{\text{perf}}$ is a ring, but this fails in general. However, $R_{\text{pfd}} \in D(R)^{\geq 0}$ is connective (a nontrivial statement).

**Example 16.** Where $R_{\text{pfd}}$ is concentrated in degree zero:

1. If $A/I \to R$ is surjective, $R_{\text{pfd}}$ is a quotient of $R$.
2. If $A/I \to R$ is finite (or integral), then $R_{\text{pfd}}$ is in degree zero.

**Remark 17.** One final useful isomorphism: if $I = dA$,

$$\left[\Delta_{R/A}\left[\frac{1}{d}\right]/p^n\right]^{\phi=1} \xrightarrow{\cong} \mathbb{R}\Gamma_{\text{'et}}\left(R\left[\frac{1}{p}\right], \mathbb{Z}/p^n\right)$$