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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Ian Coley **Email/Phone:** msri@iancoley.org

Speaker's Name: Sarah Scherotzke

Talk Title: The Chern character and categorification

Date: 3 / 29 / 19 **Time:** 9 : 30 [✓ **am**] / **pm (circle one)**

Please summarize the lecture in 5 or fewer sentences:

Categorifying the Chern character gives us a connection with noncommutative motives, as the classical Chern character relates to K-theory

An application to the Grothendieck-Riemann-Roch theorem is proven and some consequences are sketched.

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THE CHERN CHARACTER AND CATEGORIFICATION

SARAH SCHEROTZKE

Joint with M. Hoyois, N. Sibilla, and P. Safronov. HSS = Hoyois-Sibilla-Scherotzke, HSSS = everyone.

Agenda:

(1) Classical Chern character
(2) Categorified Chern character
(3) Grothendieck-Riemann-Roch and applications

Example 1. Instances of the classical Chern character:

(1) Algebraic topology: M a compact manifold, then
$$\mathrm{ch}\colon \{\mathbb{C}\text{-vector bundles on } M\} \to H^*_{\mathrm{dR}}(M)$$
(2) Representation theory: G a finite group,
$$\mathrm{ch}\colon \{\text{f.d. reps over a field } k\} \to HH_*(k[G])$$
which sends V to the class function $g \in G \mapsto \mathrm{tr}(V \xrightarrow{\cdot g} V)$.
(3) Algebraic geometry, X a scheme or a stack.
$$\mathrm{ch}\colon \mathrm{Perf}(X) \to HH_*(X)$$
Note that $3 \implies 2$ for the case $X = [\bullet/G]$ the quotient stack.

What's in common? The lefthand sides are stable (= triangulated or abelian) symmetric monoidal categories $\mathcal{C}$. The righthand sides are Hochschild homology ring of that category. So in general, $\mathrm{ch}\colon \mathrm{Ob}\,\mathcal{C} \to HH_*(\mathcal{C})$ satisfying:

(P1) ch is additive, i.e. it splits short exact sequences or triangles. Specifically,
$$(0 \to M \to N \to L \to 0) \implies \mathrm{ch}(N) = \mathrm{ch}(L) + \mathrm{ch}(M)$$
This means that ch factors through $K_0(\mathcal{C})$.
(P2) ch is multiplicative, i.e. $\mathrm{ch}(N \otimes N') = \mathrm{ch}(N) \cdot \mathrm{ch}(N')$.

Notes by Ian Coley.

(p3) ch admits an S^1-equivariant refinement, so factors through HC^- negative cyclic homology.

Historically, the first examples were given by [McCarthy, Keller, Ben-Zvi, Nadler] using the functoriality of traces. The second examples by [Toën-Vezzosi] were given using traces of monodromy.

Definition 2. Let $\mathcal{M}$ be a symmetric monoidal category, and $\mathbb{1}_{\mathcal{M}}$ the monoidal unit. An object $L \in \mathcal{M}$ is *dualizable* if there exists $L^\vee \in \mathcal{M}$ with two maps $\mathrm{ev}\colon \mathbb{1}_{\mathcal{M}} \to L \otimes L^\wedge$ and $\mathrm{coev}\colon L^\vee \otimes L \to \mathbb{1}_{\mathcal{M}}$ satisfying the triangle identities.

Definition 3. If $L \in \mathcal{M}$ is dualizable, $f \in \mathrm{End}(L)$, then define $\mathrm{Tr}(f) \in \mathrm{End}(\mathbb{1}_{\mathcal{M}})$ by

$$\mathrm{Tr}(f)\colon \mathbb{1}_{\mathcal{M}} \xrightarrow{\mathrm{ev}} L \otimes L^\vee \xrightarrow{f\otimes \mathrm{id}} L \otimes L^\vee \xrightarrow[\cong]{\mathrm{swap}} L^\vee \otimes L \xrightarrow{\mathrm{coev}} \mathbb{1}_{\mathcal{M}}$$

From Toën-Vezzosi/DAG: let X be a (derived) stack, and consider $\mathcal{L}X := \mathrm{Map}(S^1, X)$ the derived loop space. Here, S^1 is the pushout of $\mathrm{Spec}\, k \leftarrow \mathrm{Spec}\, k \sqcup \mathrm{Spec}\, k \to \mathrm{Spec}\, k$ taken in the category of derived stacks. Then we can also form $\mathcal{L}X$ as the homotopy pullback

$$\begin{tikzcd}
\mathcal{L}X = X \times_{X\times X} X \arrow[r, "p"] \arrow[d, "p"'] & X \arrow[d, "\Delta"] \\
X \arrow[r, "\Delta"'] & X \times X
\end{tikzcd}$$

If $X = \mathrm{Spec}\, A$ is an affine (underived) scheme, then

$$\mathcal{L}X = \mathrm{Spec}(A \otimes^{\mathbb{L}}_{A\otimes^{\mathbb{L}} A^{\mathrm{op}}} A) = \mathrm{Spec}(HH_*(A))$$

Theorem 4. $HH_*(\mathrm{Perf}(X)) = HH_*(X) = \Gamma\mathcal{O}_{\mathcal{L}X} = \mathrm{End}(\mathbb{1}_{\mathrm{Perf}(\mathcal{L}X)})$, where Keller proved the first equality.

Construction of the Chern character: let $E \in \mathrm{Perf}(X)$, which are the dualizable objects in $\mathrm{QCoh}(X)$. Then we can associate

$$E \mapsto (p^*E, \text{monodromy}\colon p^*E \to p^*E \text{ from the } S^1 \text{ action}) \mapsto \mathrm{Tr}(\text{monodromy})$$

where we've obtained an endomorphism of the monoidal unit in $\mathrm{Perf}(\mathcal{L}X)$.

Categorification

	Classical	Categorified
Domain:	stable $(\infty,1)$-category $\mathrm{Perf}(X)$	$(\infty,2)$-category $\mathrm{ShCat}(X)$ of small stable $(\infty,1)$-categories tensored over $\mathrm{Perf}(X)$
Codomain:	$\Gamma\mathcal{O}_{\mathcal{L}X} = HH_*(X)$	$(\infty,1)$-category $\mathrm{Perf}(\mathcal{L}(X))$
Refinement:	through $K(X)$ (nonconnective)	$(\infty,1)$-category of noncommutative motives [Blumberg-Gepner-Tabduada, Robalo]

This is in the realm of 'categorification by delooping'. The righthand column are things of the form $\mathcal{M}$ a symmetric monoidal category with unit $\mathbb{1}_{\mathcal{M}}$, and the corresponding lefthand column is $\mathrm{End}(\mathbb{1}_{\mathcal{M}})$. The case of noncommutative motives categorifying K-theory is a big theorem of BGT, and the other two are (sort of) definitions.

Theorem 5 (HSS)**.** There exists a functor of $(\infty,1)$-categories

$$\mathrm{Ch}\colon i_1\,\mathrm{ShCat}(X)^{\mathrm{dualz}} \to \mathrm{Perf}(\mathcal{L}X)$$

where i_1 means view that $(\infty,2)$-category as an $(\infty,1)$-category, satisfying analogous properties:

- (P1) There exists a unique refinement through the category $\mathrm{NMot}(X)$ of noncommutative motives.
- (P2) It's a symmetric monoidal functor.
- (P3) It factors through $\mathrm{Perf}(\mathcal{L}X)^{hS^1}$, which categorifies negative cyclic homology.

Remark 6. Two remarks:

1. Ch categorifies ch. We can view $\mathrm{Perf}(X)$ as an element in $\mathrm{ShCat}(X)$, and in fact it's the monoidal unit. Therefore $\mathrm{Ch}(\mathrm{Perf}(X))$ by P2 needs to be the monoidal unit in $\mathrm{Perf}(\mathcal{L}X)$, which is $\mathcal{O}_{\mathcal{L}X}$. There's an induced map after taking endomorphisms on both sides, giving

$$\begin{tikzcd}
\mathrm{End}_{\mathrm{ShCat}(X)}(\mathrm{Perf}(X)) \arrow[r, "\mathrm{Ch}\,\mathrm{End}(-)"] \arrow[d, "="'] & \mathrm{End}_{\mathrm{Perf}(\mathcal{L}X)}(\mathcal{O}_{\mathcal{L}X}) \arrow[d, "="] \\
\mathrm{Perf}(X) \arrow[r, "\mathrm{ch}"'] & HH_*(X)
\end{tikzcd}$$

2. We needn't use $\mathrm{Perf}(X)$ for this, but just any stable symmetric monoidal category (of dualizable objects).

Example 7. Let $X = \operatorname{Spec} k$. A model for $\mathrm{ShCat}(X)$ is pretriangulated k-linear dg categories $\mathcal{A}$. Then $\mathrm{Ch}(\mathcal{A}) = HH_*(\mathcal{A}) \in \mathrm{Perf}(\mathcal{L}X)$, and we can identify $\mathrm{Perf}(\mathcal{L}X)$ with complexes of finite dimensional k-vector spaces.

III. Groethendieck-Riemann-Roch

Classical: let $f\colon X \to Y$ be a map of stacks. We get a symmetric monoidal pullback functor $f^*\colon \mathrm{Perf}(Y) \to \mathrm{Perf}(X)$, and a pushforward $f_*\colon \mathrm{Perf}(X) \to \mathrm{QCoh}(Y)$ which not symmetric monoidal and thus fails in general to preserve dualizable objects. But since f^* is symmetric monoidal, it commutes with ch. Does f_*?

Theorem 8 (Grothendieck)**.** If f_* maps into $\mathrm{Perf}(Y)$, then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Perf}(X) \arrow[r, "\mathrm{ch}"] \arrow[d, "f^*"'] & HH_*(X) \arrow[r, "\cong"'] \arrow[d, "\int_f"'] & H^*_{\mathrm{dR}}(X) \arrow[d, "\int_f \vee Td_{X/Y}"] \\
\mathrm{Perf}(Y) \arrow[r, "\mathrm{ch}"'] & HH_*(Y) \arrow[r, "\cong"'] & H^*_{\mathrm{dR}}(Y)
\end{tikzcd}$$

The more classical formulation is the outside square.

Now: let $f\colon X \to Y$ be a map of derived stacks. Then we have the symmetric monoidal $f^*\colon \mathrm{ShCat}(Y) \to \mathrm{ShCat}(X)$ and the not-symmetric-monoidal $f_*\colon \mathrm{ShCat}(X) \to \mathrm{ShCat}(Y)$.

Theorem 9 (HSSS)**.** Let f be a *passable* map. Then we have a commutative square:

$$\begin{tikzcd}
\mathrm{ShCat}(X)^{\mathrm{dualz}} \arrow[r, "\mathrm{Ch}"] \arrow[d, "f_*"'] & \mathrm{Perf}(\mathcal{L}X) \arrow[d, "(\mathcal{L}f)_*"] \\
\mathrm{ShCat}(Y)^{\mathrm{dualz}} \arrow[r, "\mathrm{Ch}"'] & \mathrm{Perf}(\mathcal{L}Y)
\end{tikzcd}$$

The proof is very different from the classical one and involves $(\infty, 2)$-category theory.

Definition 10. $f\colon X \to Y$ is *passable* if:

- The diagonal $X \to X \times_Y X$ is quasi-affine
- $f^*\colon \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ admits a right adjoint f_* which itself admits a right adjoint.
- $f_*(\mathrm{QCoh(X)}) \in \mathrm{QCoh}(Y)$-mod is a dualizable object.

This is actually a mild condition, as all maps between schemes are passable.

Applications:

1. There's a new proof of the classical GRR theorem.
2. McCarthy's and Keller's construction and Toën-Vezzosi's constructions of the classical Chern character we now know must coincide
3. We can recognize the de Rham realization functor is an instance of a categorified Chern character, and thus is a noncommutative invariant (depending only on $\mathrm{Perf}(X)$ as a $\mathrm{Perf}(Y)$-module)

IV. Higher Invariants

We can now iterate this construction, using the factorisation properties P1:

$$\begin{tikzcd}
\mathrm{ShCat}(X)^{\mathrm{dualz}} \arrow[rr, "\mathrm{Ch}"] \arrow[dr] & & \mathrm{Perf}(\mathcal{L}X) \arrow[rr, "\mathrm{ch}"] \arrow[dr] & & HH_*(\mathcal{L}X) = \Gamma\mathcal{O}_{\mathcal{L}^2X} \\
& \mathrm{NMot}(X) \arrow[ur] \arrow[dr] & & K(\mathcal{L}X) \arrow[ur] & \\
& & K(\mathrm{NMot}(X)) \arrow[ur] & &
\end{tikzcd}$$

We can think of $K(\mathrm{NMot}(X))$ as *secondary K-theory*, and the above composition as the *secondary Chern character*. What does it mean? Not sure yet.