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# NOTETAKER CHECKLIST FORM

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**Name:** Ian Coley **Email/Phone:** msri@iancoley.org

**Speaker's Name:** Hélène Esnault

**Talk Title:** Arithmetic subloci of rank one local systems

**Date:** 3 / 28 / 19 **Time:** 11 : 00 [✓ am] / pm (circle one)

**Please summarize the lecture in 5 or fewer sentences:**

A theorem of Simpson on rank one local systems in \C has an analogue in the p-adic world, achieving some interesting corollaries.

They also discuss an extension to local systems of higher rank.

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# ARITHMETIC SUBLOCI OF RANK ONE LOCAL SYSTEMS

HÉLÈNE ESNAULT

Caveat: this talk won't be very derived, but it will be $p$-adic. The following is new work with Moritz Kerz lately available on the arXiv (https://arxiv.org/abs/1902.02961).

Two solid motivations for the following work:

**Motivation 1:** A theorem of Simpson in Hodge Theory. Let $X$ be a smooth projective variety over $\mathbb{C}$, and let $\mathrm{Char}(X(\mathbb{C})) = \mathrm{Hom}(H_1(X), \mathbb{C})$ be the group of local systems. It's essentially a torus, endowed with a Riemann-Hilbert correspondence in a complex-analytic way:

$$RH\colon\ \mathrm{Char}(X(\mathbb{C})) \xrightarrow[\mathbb{C}\text{-analytic}]{\cong} R^{\nabla}_{C}(X_{\mathbb{C}})$$

where the righthand side is the group if isomorphism classes of rank 1 integrable connections. Consider closed and algebraic local system, and assume that the corresponding $RH(S)$ is still closed and algebraic.

**Theorem 1** (Simpson)**.**

(1) The irreducible components of $S$ have the form $a+T$ for $T$ a subtorus.

(1') If $S, RH(S)$ are defined over $\overline{\mathbb{Q}}$, then $a$ above may be taken to be torsion.

(2) These tori are "motivic", i.e. they correspond to quotient Hodge structures of $H_1$; there exists $\psi\colon X \to A$ to an abelian variety with $T = \psi^* \mathrm{Char}(A(\mathbb{C}))$

*Problem:* (with Kerz) What's the $p$-adic arithmetic analogue? Let $X/F$, $F \subset \mathbb{C}$ of finite type over $\mathbb{Q}$ and let $G_F = \mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of $F$. For any ring $A$, we have a functor from $A$-algebras to groups which sends $B$ to $\mathrm{Hom}(H_1, B^\times)$. This is representable by an algebra we will call $\mathrm{Char}_A(X(\mathbb{C}))$.

For $p$ a prime, consider $A = \overline{\mathbb{Q}}_p$. Then $\mathrm{Char}_{\overline{\mathbb{Q}}_p}(X(\mathbb{C})) = \mathrm{Char}_{\overline{\mathbb{Z}}_p \times \mathbb{Q}}(X(\mathbb{C}))$. We can then consider the composite, letting $\pi^{\mathrm{ab}} = \pi_1^{\mathrm{ab}}(X_{\mathbb{C}})$,

$$\varphi\colon\ \mathrm{Hom}_{\mathrm{cts}}(\pi^{\mathrm{ab}}; \overline{\mathbb{Q}}_p^{\times}) = \mathrm{Hom}_{\mathrm{cts}}(\pi^{\mathrm{ab}}; \overline{\mathbb{Z}}_p^{\times}) = \mathrm{Char}_{\overline{\mathbb{Z}}_p}(\overline{\mathbb{Q}}_p) \to \mathrm{Char}_{\overline{\mathbb{Q}}_p}(\overline{\mathbb{Q}}_p) \to \mathrm{Char}_{\overline{\mathbb{Q}}_p}(X(\mathbb{C}))$$

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Notes by Ian Coley.

The LHS of that is endowed with a Galois action, let $S \subset \mathrm{Char}_{\overline{\mathbb{Q}}_p}$ be closed and consider $\varphi^{-1}(S) \subset \mathrm{Hom}_{\mathrm{cts}}(\pi^{\mathrm{ab}}, \overline{\mathbb{Q}}_p^\times) \circlearrowleft G_F$. If $\varphi^{-1}(S)$ is nonempty, then we could call $S$ *integral.*

**Theorem 2** (E-K)**.** If $S$ is as above, then

(1) The integral components of $S$ are of the form $a + T$ for $T$ a subtorus and $a$ a torsion point.
(2) These tori are motivic under the following a geometric assumption: $X$ is an algebraic variety over $\mathbb{C}$ with weights of $H_1$ all negative. $X$ is smooth or normal is enough.

**Remark 3.** The theorem is sharp. Consider a two-point intersection of two rational curves. This has weight 0, but the components of $S$ which are integral are not torsion, e.g. $\widehat{\mathbb{Z}} \to \overline{\mathbb{Z}}_p^\times$.

**Corollary 4.** If $S$ is 0-dimensional, the components of $S$ which are integral are torsion.

**Motivation 2:** companions. Let $p, p'$ two primes that might be the same, and let $\iota \colon \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_{p'}$ be an abstract isomorphism (not a continuous one when $p \neq p'$). A corollary of the theorem is the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{cts}}(\pi^{\mathrm{ab}}, \overline{\mathbb{Q}}_p^\times) \arrow[r, "\varphi"] \arrow[d, dashed, "\text{NO MAP}"'] & \mathrm{Char}_{\overline{\mathbb{Q}}_p} \arrow[d, "\iota"', "\cong"] \\
\mathrm{Hom}_{\mathrm{cts}}(\pi^{\mathrm{ab}}, \overline{\mathbb{Q}}_{p'}^\times) \arrow[r, "\varphi'"] & \mathrm{Char}_{\overline{\mathbb{Q}}_{p'}}
\end{tikzcd}$$

where we can't fill in that righthand side because $\iota$ isn't continuous (in general). However, we know that $\iota(S)$ is closed if $S$ were closed, and we now have the following:

**Corollary 5.** If $S$ is closed, integral, and Galois-invariant, then so is $\iota(S)$.

Why is this interesting? We want to extend the above diagram past $\overline{\mathbb{Q}}_p^\times = \mathrm{GL}_1 \overline{\mathbb{Q}}_p$ to higher $\mathrm{GL}_r$. In this case, taking the form

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{cts}}(\pi(X_{\mathbb{C}}), \mathrm{GL}_r \overline{\mathbb{Q}}_p) \arrow[r, "\varphi"] \arrow[d, dashed, "\text{NO MAP}"'] & \mathcal{M}_B^{\mathrm{irred}}(r) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathrm{cts}}(\pi(X_{\mathbb{C}}), \overline{\mathbb{Q}}_{p'}^\times) \arrow[r, "\varphi'"] & \mathcal{M}_B^{\mathrm{irred}}(r)
\end{tikzcd}$$

where $\mathcal{M}_B^{\text{irred}}(r)$ is the moduli space of rank $r$ local systems over $X(\mathbb{C})$ over $\overline{\mathbb{Q}}_p$ or $\overline{\mathbb{Q}}_{p'}$. So if $S$ is closed, integral, and Galois-invariant do we have the same conclusion for arbitrary $r$?

Well, what if we try the case $S = *$. Then we would get Simpson's integrality conjecture in full generality. Additionally, joint work with Groechenig gives some cohomological rigidity results under some assumptions.

**Vague Motivation 3:** Jumping loci

Let $\mathcal{F} \in D_c^b(X(\mathbb{C}); \mathbb{C})$ be a constructible sheaf and $i, j \in \mathbb{Z}$. Consider

$$\Sigma^{i,j}(\mathcal{F}) := \{L \in \operatorname{Char}_{\mathbb{C}}(X(\mathbb{C})) : h^i(X, \mathcal{F} \otimes L) \geqslant j\}$$

**Corollary 6.** If $\mathcal{F}$ is arithmetic, then $\Sigma^{i,j}(\mathcal{F})$ is a union of $a+T$ torsion plus motivic tori.

where *arithmetic* means that $\mathcal{F}$ descends to a number field $K/\mathbb{Q}$ and there exist infinitely many primes $p$ such that $\mathcal{F} \otimes \overline{\mathbb{Q}}_p$ is integral, Galois-invariant, and lies in $D_c^b(X(\mathbb{C}); \overline{\mathbb{Q}}_p)$. Technically this is just a sufficient condition for arithmetic.

**On the proof of the main theorem:**

**Theorem 7.** Let $S$ be closed, integral, Galois-invariant.

1. $S$ has dimension 0 if and only if its integral components are torsion.
2. If $S$ is higher dimension, then torsion points are dense in integral components.

Assume that $X$ is smooth and projective for this illustration. Using a theorem of Bogomalov, adjusted by Litt, there exists $\sigma \in G_F$ such that $\sigma$ acts on $H^1(X(\mathbb{C}); \overline{\mathbb{Q}}_p)$ as a homothety by a factor $\alpha \in \mathbb{Q}_p^\times$ such that $|1 - \alpha| < 1$.

**Proposition 8** (Key Proposition)**.** Let $S'$ be the union of the integral components of $S$. Let $\xi$ be an integral point of $S'$ and consider a residual representation of it. We have the following picture, where the vertical map is specialization:

$$\begin{tikzcd}
\operatorname{Hom}_{\text{cts}}(\pi^{\text{ab}}, \overline{\mathbb{Q}}_p^\times) \arrow[r, "\varphi"] \arrow[d, "\text{sp}"'] & \operatorname{Char}_{\overline{\mathbb{Q}}_p} \\
\operatorname{Hom}_{\text{cts}}(\pi^{\text{ab}}, \overline{\mathbb{F}}_p^\times) &
\end{tikzcd}$$

Then $\text{sp}^{-1}(\xi) \cap \varphi^{-1}(S')$ contains a torsion point

Note: the proposition implies the density result above. For all nonempty $U \subset S'$, we need a residual representation all lifts of which are in $U$. This is a problem for geometry, so easy.

*Proof.* Let $\xi \mapsto [\xi]$ be a Teichmüller lift from $\overline{\mathbb{F}}_p^\times$ to $W(\overline{\mathbb{F}}_p^\times) \subset \overline{\mathbb{Q}}_p^\times$. It might be that $\sigma$ doesn't stabilize $\xi$, but some power of $\sigma$ does. Thus we should replace $S'$ by $[\xi^{-1}]S'$ and thereby assume $[\xi] = 1$.

Now we have $\varphi^{-1}(p^n[\xi^{-1}]S') = [p^n]\varphi^{-1}([\xi^{-1}]S')$. We have at our disposal a log map which takes a small ball around 1 to a polydisc $B$ in $H^1(X(\mathbb{C}), \overline{\mathbb{Q}}_p)$ around 0. So transport $S$ over to $B$. $\sigma$ acts (repeatedly) on $B$ by producing lines on $S$, so they have to go to zero as $\alpha^n$ approaches infinity. Thus we can conclude that $S$ must approach 1 so $1 \in S$.

If $1 \in S$, then there's a $\mathbb{G}_m$-action on the cone of 1 coming from $\sigma$. The lines that are produced in $B$ come back to our small ball around 1, giving us linearity.

Alternatively, apply Mordell-Lang on tori using results of M. Laurent. □