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17 Gauss Way   Berkeley, CA 94720-5070   p: 510.642.0143   f: 510.642.8609   www.msri.org

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Name: Ian Coley   **Email/Phone:** msri@iancoley.org

Speaker's Name: Vesna Stojanoska

Talk Title: Dualizing spheres for p-adic analytic groups with applications to chromatic homotopy theory

Date: 3 / 25 / 19   **Time:** 2 : 00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences:
A specific phenomenon arising in chromatic homotopy theory turns out to generalize nicely. We can recognise the dualizing sphere to a p-adic analytic
some concrete applications to those particular cases

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DUALIZING SPHERES FOR p-ADIC ANALYTIC GROUPS WITH APPLICATIONS TO CHROMATIC HOMOTOPY THEORY

VESNA STOJANOSKA

Joint with Agnes Beaudry, Paul Goerss, and Mike Hopkins.

This subject arose for them in exploring a particular case in chromatic homotopy theory, but it turned out to work more broadly with p-adic analytic groups.

Example 1. (of groups of interest)

- $\mathrm{GL}_n(\mathbb{Z}_p)$
- $\S_n$ the Borel stabilizer group, which is $\cong \mathcal{O}_D^\times$ where D is a division algebra of invariant $1/n$ over the p-adic numbers, also $\cong$ Aut(formal group law of height n)
- $\mathbb{G}_n \cong \mathbb{S}_n \rtimes \mathrm{Gal}$ the 'big stabilizer' (details unimportant)
- Less interesting, $M_n(\mathbb{Z}_p)$ or $\mathbb{Z}_p^d$ under addition

What's in common between these groups? They have a open subgroup satisfying Poincaré duality.

Definition 2. Let Γ a pro-p group. Γ is *uniformly powerful* (u.p.) if:

(1) Γ/Γ^p or Γ/Γ^4 if $p = 2$ is abelian.
(2) Γ is topologically finitely generated by some $\{a_1, \dots, a_d\}$
(3) The lower p-series

$$\Gamma = \Gamma_0 \supset \cdots \supset \Gamma_i \supset \Gamma_{i+1} := \overline{\Gamma_i^p[\Gamma_i, \Gamma]} \supset \cdots$$

has between successive quotients a p-power map $\Gamma_i/\Gamma_{i+1} \to \Gamma_{i+1}/\Gamma_{i+2}$. This is an isomorphism. In fact, in our situation $\Gamma_{i+1} = \Gamma_i^p$ so we can just as well require

$$\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}/p\{a_1^{p^i}, \dots, a_d^{p^i}\}$$

for the generators as in (2).

Do these actually exist?

Notes by Ian Coley.

Theorem 3 (Lazard)**.** Any p-adic analytic group G has an open normal subgroup Γ that is u.p. (and conversely).

Theorem 4 (Lazard/Serre)**.** If Γ is u.p. of rank d (i.e. the number of generated from (2)), then Γ is a Poincaré duality group of dimension d. Moreover,

$$H^*_{\text{cts}}(\Gamma; \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p} \operatorname{Hom}_{\mathbb{F}_p}(\Gamma/\Gamma^\pi, \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(a_1^*, \dots, a_d^*)$$

where $\pi = p$ if p is odd and $\pi = 4$ if $p = 2$ and a_i^* is a dual basis of the generators.

For what comes next, here is a construction from page 24 of Serre's Galois cohomology book:

Serre's construction of duals: Let A be a finite abelian group on which Γ acts, then for (say) our Γ_i as above we can construct $\operatorname{colim}[H^*_{\text{cts}}(\Gamma_i, A)]^\vee$ where the colimit is corestriction along the $\Gamma_i \supset \Gamma_{i+1}$.

Well, what if we did that for $\mathbb{Z}_p$ with Γ giving the trivial action. This should be the dualizing module. Except $\mathbb{Z}_p$ isn't finite, but this still works okay.

Topological and covariant analogue: Again consider the filtration Γ_i. To get more topological, consider $B\Gamma_i$ classifying spaces of profinite groups, so they satisfy

$$B\Gamma_i = \operatorname*{holim}_j B(\Gamma_i/\Gamma_{i+j})$$

We still get maps $B\Gamma_{i+1} \to B\Gamma_i$ so get a chain $B\Gamma \leftarrow \cdots \leftarrow B\Gamma_i \leftarrow \cdots$. There's no corestriction but there's a stable map that will do what we want, which gives us transfer maps after applying Σ^∞ of the sort $\text{tr}\colon \Sigma^\infty B\Gamma_i \to \Sigma^\infty B\Gamma_{i+1}$. Thus we can take a homotopy colimit to define

$$\omega_G := \operatorname*{hocolim}_{i,\text{tr}} (\Sigma^\infty B\Gamma_i)^\wedge_p$$

Note: since we're imagining our Γ as an open subgroup of some p-adic analytic group G, we could take $\Sigma^\infty BG$ on to the end of that homotopy colimit, but it doesn't affect the final. Also, this colimit is supposed to be denoted with a 'blackboard bold' ω but computers have not caught up to that quite yet in a satisfying way.

Lemma 5. ω_G is equivalent to $(S^d)^\wedge_p$, where d is the rank of Γ.

If we don't need to even put $\Sigma^\infty BG$ in the notation, why even bother with G in the notation of ω_G? Well, everything in that homotopy colimit is a normal subgroup of G, so G acts on $B\Gamma_i$ by conjugation, and the transfers are conjugation-equivariant, so G acts on ω_G.

However, how should G act on $(S^d)^\wedge_p$? Trivially is the only natural option, but ω_G almost certainly does not carry a trivial G action. Thus the above equivalence is non-equivariant.

Motivation. Let $\mathbb{G}_n = \mathrm{Aut}_k(\text{formal group law of height } n)$, where k is a finite field. $\mathbb{G}_n$ acts on E_n the Lubin-Tate spectrum, where $E_n^{h\mathbb{G}_n} \simeq S^0_{K(n)}$ the $K(n)$-local sphere spectrum, which (collectively) are the building blocks of chromatic homotopy theory.

So what's the "$K(n)$-local Spannier-Whitehead dual"? It would be

$$D_n E_n := \mathrm{Hom}(E_n, S^0_{K(n)}) \overset{\star}{\simeq} \omega_{\mathbb{G}_n}^{-1} \wedge E_n$$

where $\omega_{\mathbb{G}_n}$ turns out to be dualizable, so $\omega_{\mathbb{G}_n}^{-1} = \mathrm{Hom}(\omega_{\mathbb{G}_n}, S^\wedge_p)$. That equivalence $\star$ – which is a $\mathbb{G}_n$-equivariant equivalence – and the dualizability is a theorem of BGHS.

Remark 6. Nonequivariantly, this reduces to $D_n E_n \simeq \Sigma^{-n^2} E_n$, which was proven by Goerss-Hopkins or Strickland.

But how can we get at ω_G practically? It's a homotopy colimit, so we can get it its p-local homology without much trouble a priori, but not much else.

Linearization: Again, in our general setup let $\Gamma \subset G$ open normal u.p. subgroup of rank d.

Definition 7. The Lie algebra of G, $\mathfrak{g}$, as a set is $(\Gamma, +, [,])$ where $x + y := \lim_k (x^{p^k} y^{p^k})^{1/p^k}$ using the homeomorphism $\Gamma \to \Gamma_k$ by the p^k-power map. We don't care about the bracket so won't define it.

As an abelian group $\mathfrak{g} \cong \mathbb{Z}_p^d$. Also, G acts on $\mathfrak{g}$ by conjugation $\mathbb{Z}_p$-linearly by the adjoint representation.

Definition 8. Let $S^{\mathfrak{g}} := \operatorname*{hocolim}_{i,\mathrm{tr}} (\Sigma^\infty B p^i \mathfrak{g})^\wedge_p$, i.e. $\omega_{\mathfrak{g}}$.

This comes with a potentially nontrivial G-action, though $\mathfrak{g}$ acts on $S^{\mathfrak{g}}$ trivially. Still, nonequivariantly we have that $S^{\mathfrak{g}} \simeq (S^d)^\wedge_p$ and $\mathfrak{g}$ is independent of the choice of $\Gamma \subset G$

Linearization hypothesis: $\omega_G \simeq_G S^{\mathfrak{g}}$ are G-equivariantly equivalent, and it suffices because of the particular G-action just to check for a $G/Z(G)$-equivariant equivalence.

Thus it definitely holds if G is abelian, but it's much more interesting in other cases. It is expected to hold in full generality, but the slight hiccup is that there's not actually a map between ω_G and $S^{\mathfrak{g}}$ which makes it more tricky to check in all cases.

Remark 9. Both $\mathfrak{g}$ and $S^{\mathfrak{g}}$ are quite explicit: if $G = \mathrm{GL}_n(\mathbb{Z}_p)$, then $\mathfrak{g} \cong M_n(\mathbb{Z}_p)$. If $G = \mathbb{G}_n$, then $\mathfrak{g}$ is the covariant Dieudonné module of the formal group.

Theorem 10 (BGHS)**.** If $H \subset G$ is a finite subgroup such that the p-Sylow subgroup of $V = H/(H \cap Z(G))$ is an elementary abelian p-group, then $\omega_G \simeq_H S^{\mathfrak{g}}$.

Consequences: $\mathbb{G}_1$ is abelian, so we have relatively few worries and $D_1 E_1 \simeq \Sigma^{-1} E_1$ equivariantly. For $\mathbb{G}_2$, the theorem holds for all p for all finite subgroups, thus

$$D_2(E_2^{hH}) \simeq (S^{-\mathfrak{g}} \wedge E_2)^{hH}$$

which allows us to compute $K(2)$ Spannier-Whitehead duals like $D_2 \,\mathrm{TMF} \simeq \Sigma^{44} \,\mathrm{TMF}$.

Proof. The ingredients are all ≤1990's, mostly from the mid-1980's.

(1) $\omega_G, S^{\mathfrak{g}}$ can both be viewed in $[BV, B\mathrm{GL}_1(S^0_p]$ as stable sphere bundles with V-action. Actually, we can view them as stable sphere bundles without p-completing, so in $[BV, B\mathrm{GL}_1(S^0)]$. Using Lannes' T-functor technology we can identify these homotopy classes as

$$\mathrm{Hom}_{\mathcal{A}\text{-alg}}(H^* B\mathrm{GL}_1(S^0), H^* BV)$$

maps over the Steenrod algebra between those cohomologies. We know a lot about H^*BV because (by assumption) V is elementary abelian. We know a lot about $B\mathrm{GL}_1(S^0)$ by work of many people, including Madsen, May, Milgram...

In turn, that set can be identified further as

$$\mathrm{Hom}_{\mathcal{A}\text{-alg}}(H^*(BU_p^{h\mu}, H^* BV)$$

whose lefthand side is $BO_2^\wedge$ when $p = 2$ and $(BU_p^\wedge)^{hC_{p-1}}$ otherwise. This set of maps is well understood in terms of characteristic classes.

(2) Show that $\omega_G, S^{\mathfrak{g}}$ give rise to the same characteristic classes, so they must be equivalent.

□