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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Ian Coley **Email/Phone:** msri@iancoley.org

Speaker's Name: Dmytro Arinkin

Talk Title: Singular support for categories over a scheme

Date: 3 / 26 / 19 **Time:** 11 : 00 [✓ am] / pm (circle one)

Please summarize the lecture in 5 or fewer sentences:

Singular support is an important tool for studying lci schemes, and measures the difference between perfect and coherent sheaves.

They categorify the issue to address a part of the local geometric Langlands program, where we still get singular support for 'smooth' schemes

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SINGULAR SUPPORT FOR CATEGORIES OVER A SCHEME

DMYTRO ARINKIN

Given in an introductory spirit to introduce the concept and motivate it. Most of this is joint with Dennis Gaitsgory.

Agenda:

1. Review of singular support for coherent sheaves
2. Categorification of (1)
3. Motivation from the local geometric Langlands conjecture

I: Support of X means *locations* where X is nontrivial, but *singular* support of X means *directions* where X is nontrivial. For X = distributions, D-modules, coherent sheaves, categories.

Let X be an affine variety over k, $\operatorname{char} k = 0$ or $k = \mathbb{C}$. Let $\mathcal{F} = \operatorname{Coh}(X)$ (for the moment abelian, not derived). If X is smooth, then $\mathcal{F}$ is perfect, so there shouldn't be such a thing as singular support. If X is just local complete intersection, then singular support should measure how far $\mathcal{F}$ is from being perfect.

So let's fix X lci for a bit. Take $\mathbb{L}X$ the cotangent bundle, and look at $H^{-1}(\mathbb{L}X)$. Explicitly, if $X = \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_m) = \operatorname{Spec} A$,

$$X \times \mathbb{A}^m \supset H^{-1}(\mathbb{L}X) = \{(x \in X, a_1, \ldots, a_m) : \sum_i a_i df_i(x) = 0\}$$

Technically we should say that it's the total space of H^{-1} – see the video for more details. Now given $\mathcal{F} \in \operatorname{Coh}(X)$, we get a conical closed subset $\operatorname{singsupp}(\mathcal{F}) \subset H^{-1}(\mathbb{L}X)$.

Remark 1. $\mathcal{F}$ is perfect if and only if $\operatorname{singsupp}(\mathcal{F})$ is contained in the zero section.

Singular support is defined via the action of $HH^*(X)$ on $\mathcal{F}$ or, classically, via cohomological operators (Gulliksen, Eisenbud, Avramov-Buchweitz, Krause-Iyengar-Benson).

Notes by Ian Coley.

II. Move one categorical level up

Categories over X are analogous to perfect complexes, and we can identify them with A-linear categories if $X = \operatorname{Spec} A$. Categories, of course, means cocomplete dg-categories. So what should correspond to coherent sheaves on the categorical side?

The role of $H^{-1}(\mathbb{L}X)$ will now be played by T^*X categorically. Whereas before we had X is lci, we now want X to be smooth hereafter. The formalism we hope for is that for a conical Lagrangian $Z \subset T^*X$ we get a 2-category $\mathcal{C}$ of categories with singular support contained in Z. When Z is the zero section, $\mathcal{C} \simeq A$-linear categories.

Remark 2. This is very similar to Kapustin-Rozansky-Saulina.

We also can take $\mathcal{C}_{\text{total}} = \operatorname{colim}_Z \mathcal{C}_Z$, so $M \in \mathcal{C}_{\text{total}}$ is like the Ind-completion of coherent categories over X. Then we can take $\operatorname{singsupp}(M) = \bigcap_{M \in \mathcal{C}_Z} Z$.

Example 3. Fix $x \in X$. Consider the dg-scheme $x \times_X x = \operatorname{Spec}(k_x \otimes_A k_x) \cong \operatorname{Spec} k[\eta_1, \ldots, \eta_j]$ where $|\eta_i| = -1$. Fact: A-linear categories such that the support of M is contained in $\{x\}$ is equivalent (as a 2-category) to modules over the monoidal category $\operatorname{Perf}(x \times_X x)$, where the monoidal action is convolution, not tensor.

The above is actually a lie, but the lie disappears when we Ind-complete. The convolution unit isn't actually perfect so we need to pass to the completion to get it in there. Because $x \times_X x$ is not smooth, coherent sheaves is strictly bigger than perfect ones, but since it is lci we can understand the difference via singular support.

Definition 4. $\mathcal{C}_{T^*_x X}$ is defined to be the 2-category of module categories over $\operatorname{Coh}(x \times_X x)$ under the convolution product.

Inside of here are modules over $\operatorname{Perf}(x \times_X x)$, which we identify exactly as the A-linear categories supported at $\{x\}$.

Generalizations:

(1) If $Z \subset T^*X$ is the conormal bundle to some $Y \subset X$, let $\mathcal{C}_Z$ be module categories over $\operatorname{Coh}(Y \times_X Y)$ with the same monoidal structure.

(2) Even more generally, if Y is smooth, $f\colon Y \to X$ proper, then we get an isotropic $Z \subset T^*X$:

$$\begin{tikzcd}
\ker(df) \arrow[r, tail] \arrow[d, two heads] & T^*X \times_X Y \arrow[d, two heads] \\
Z \arrow[r, tail] & T^*X
\end{tikzcd}$$

If Z arises like this, define $\mathcal{C}_Z$ as above. We still need to show that if the same Z arises from two different Y_1, Y_2 then we get the same 2-category, but this is settled:

Theorem 5. If $Y_1 \to X \leftarrow Y_2$ such that $Z_1 \subset Z_2$ (as above), there is a natural 2-full embedding $\mathrm{Coh}(Y_1 \times_X Y_1)$-mod $\to \mathrm{Coh}(Y_2 \times_X Y_2)$-mod.

Final remark: this kind of map should be given by a bimodule – what is it?

III. Motivation Automorphic questions have Galois answers in this program. A global automorphic question is given by local systems on a Riemann surface with Arthur parameters. A local automorphic question is given by local systems on a punctured disc plus Arthur parameters which sit in singular support.

In the global picture: perfect complexes on $X = \mathrm{LS}\,\Sigma$ for Σ a Riemann surface, we correct the issue by using coherent sheaves on X with controlled (nilpotent) singular support. This makes sense because $\mathrm{LS}\,\Sigma$ is lci but not a smooth stack.

In the local picture, consider categories over $X = \mathrm{LS}\,D^\circ$, the punctured disc. We need something to correct it, but that X is smooth ... and there's no singular support on smooth objects. But singular support does still exist by passing to Ind-completion of coherent categories over X with controlled (nilpotent) singular support. Unfortunately, this is not quite an algebraic stack. Nonetheless it's still workable!