MSRI
Mathematical Sciences Research Institute

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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Ian Coley **Email/Phone:** msri@iancoley.org

Speaker's Name: Gabriele Vezzosi

Talk Title: Vanishing cycles, Bloch's conductor conjecture, and non-commutative geometry

Date: 3 / 29 / 19 **Time:** 2 : 00 **am / [✓ pm] (circle one)**

Please summarize the lecture in 5 or fewer sentences:
Based on noncommutative motives, they move to study arithmetic phenomena in DAG. Specifically, using noncomm. motives they can prove new cases of Bloch's conductor conjecture.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- **Computer Presentations**: Obtain a copy of their presentation
- **Overhead**: Obtain a copy or use the originals and scan them
- **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
- **Handouts**: Obtain copies of and scan all handouts

☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.

☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Vanishing cycles, Bloch's Conductor conjecture and non-commutative geometry

Gabriele Vezzosi
Università di Firenze, Italy

MSRI
Berkeley, March 29, 2019

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Plan of the talk

1. ℓ-adic cohomology of dg-categories
2. ℓ-adic cohomology of singularity dg-categories vs. vanishing cycles
3. Chern character and trace formula for dg-categories
4. Categorical Bloch's conductor conjecture
5. A new approach (in progress)
6. Future directions

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i.e. is an ℓ-adic sheaf on $S = Spec\, A$ with $\mathbb{Q}_\ell$ coefficients.

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Note : 1) i is derived lci hence i_* preserves perfect objects.

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3) The above comparison Theorem actually holds before ℓ-adic realization, i.e. at the level of commutative motives: the commutative motive associated to $Sing(X, f)$ is equivalent to Ayoub's motivic tame vanishing cycles.

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$\Longrightarrow$ get a map $\mathbb{H}(B, \mathbb{Q}_\ell) \to \mathbb{H}(B, \mathbb{Q}_\ell)$ of $\mathbb{H}(B, \mathbb{Q}_\ell)$-modules.

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Consider the composition $B \xrightarrow{\Gamma_f} T^{op} \otimes_B T \xrightarrow{ev} B$.
This is equivalent to a perfect B-module denoted $HH(T/B, f)$.
We get, in particular:
- a class $[HH(T/B, f)] \in K_0(B)$ (the HH-theoretic trace of f).

On the other hand,
if T is moreover $\ell^{\otimes}$-admissible, we can apply the same procedure to $\mathbb{H}(f, \mathbb{Q}_\ell)$ in place of f itself
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Let's describe quickly this E_2-version of our the trace formula, then we will move to Bloch's conductor conjecture.

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We will now apply this formula (for $f = id_T$, and an appropriate T) to Bloch's conductor conjecture.

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A kind of arithmetic Gauss-Bonnet: describe the change of Euler characteristic between special and generic fibres.

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  This uses that $\mathcal{H}^*$ of $\mathcal{HH}^{st}_{S'/S}$ is 2-periodic (by HKR + S'/S is locally a hypersurface), $\mathcal{H}^0$ is supported at the closed point, and that each $\mathcal{H}^i(\mathcal{F})$ is coherent over $\mathcal{H}^0(\mathcal{HH}^{st}_{S'/S})$, hence has finite length.

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