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Name: Ian Coley **Email/Phone:** msri@iancoley.org

Speaker's Name: Kirsten Wickelgren

Talk Title: An arithmetic enrichment of the degree of a map between smooth schemes and applications to enumerative geometry

Date: 3 / 26 / 19 **Time:** 2 : 00 **am / [✓ pm] (circle one)**

Please summarize the lecture in 5 or fewer sentences:

A^1-homotopy theory provides a new way to think about the degree of a map of smooth varieties and takes values not in Z but in GW(k).

Here they give one concrete application and a few more in progress of new enumerative geometric calculations.

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AN ARITHMETIC ENRICHMENT OF THE DEGREE OF A MAP BETWEEN SMOOTH SCHEMES AND APPLICATIONS TO ENUMERATIVE GEOMETRY

KIRSTEN WICKELGREN

Outline:

I. Degree + enumerative applications with Kass and Solomon (in progress).

II. Arithmetic count of lines meeting 4 lines in $\mathbb{P}^3$ with Srinivasan

I. How do we define degree? It's a function $\deg\colon [S^r, S^r] \to \mathbb{Z}$ on homotopy classes of maps of spheres. So if $f\colon X \to Y$ is a map of d-dimensional smooth oriented manifolds over $\mathbb{R}$, then $\deg f \in \mathbb{Z}$ is computed via local data

$$\deg f = \sum_{x \in f^{-1}(\{y\})} \deg_x f$$

where, if we choose local coordinates near x, y we reinterpret $f\colon \mathbb{A}^d \to \mathbb{A}^d$ then pick a small ball around x and declare

$$\deg_x f = \deg(\partial B(x, \varepsilon) \to \partial B(0,1))$$

of the map $f/|f-y|$. We can also say that

$$\deg_x f = \begin{cases} 1 & \det \mathrm{Jac}_f > 0 \\ -1 & \det \mathrm{Jac}_f < 0 \\ \text{formula of Eisenbud-Levine-Khimshiashvili} & \text{else} \end{cases}$$

Example 1. Here's an application to enumerative geometry. The Wronskian of $f_1, \dots, f_m \in \mathbb{C}[t]$ is

$$\det \left(\frac{\partial}{\partial t}^{j-1} f_i \right) = \mathrm{Wr}(f_1, \dots, f_m)$$

Let $\mathrm{Gr}(m, V)$ be the Grasmannian parametrising m-dimensional subspaces of V an $\mathbb{R}$-vector space. Then

$$\mathrm{Wr}\colon \mathrm{Gr}(m, \mathbb{C}[t]_{\deg \leqslant m+p-1}) \to \mathrm{Gr}(1, \mathbb{C}[t]_{\deg \leqslant mp}) \cong \mathbb{P}^{mp}$$

Notes by Ian Coley.

sending $\mathrm{span}(f_1, \dots, f_m)$ to $\mathrm{span}(\mathrm{Wr}(f_1, \dots, f_m))$. This connects to the following question: if $\gamma\colon \mathbb{A}^1 \to \mathbb{A}^3$ be $\gamma(t) = (t, t^2, t^3)$, how many lines meet $T_{\gamma(r_i)}\gamma$ for $i = 1, 2, 3, 4$ different tangent lines $r_i \in \mathbb{C}$? We have a bijection

$$\{\text{such lines}\} \iff \mathrm{Wr}^{-1}\left(\prod_{i=1}^{4}(t - r_i)\right)$$

for $\mathrm{Wr}\colon \mathrm{Gr}(2, \mathbb{C}[t]_{\leqslant 3}) \to \mathbb{P}^4$. For $\mathrm{span}(f_1, f_2)$ in the Grassmannian, we know that $f_i = A_i \circ \gamma$ for some affine linear $A_i\colon \mathbb{A}^3 \to \mathbb{A}^3$, and then we get the line $\ker A_1 \cap \ker A_2$. So the number of lines is the degree of the Wronskian map, which Schubert calculated to be

$$\frac{1!2! \cdots (p-1)!(mp)!}{m!(m+1)! \cdots (m+p-1)!}$$

which for $m = p = 2$ is just 2.

Example 2. Motivating example: let $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$ be the Kontsevich moduli space parametrising $(C, p_1, \dots, p_n, f)$ where C is a genus 0 curve with at most nodal singularities, $p_i \in C$ smooth points, and $f\colon C \to \mathbb{P}^2$ a degree d stable map. There's an evaluation map $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d) \to (\mathbb{P}^2)^n$. When $n = 3d-1$, the dimension on both sides is the same, so we can take the degree of the map. It tells us about the number of stable rational curves of degree d in $\mathbb{P}^2$ passing through $3d - 1 = n$ points.

But what about fields k that aren't $\mathbb{C}$ or $\mathbb{R}$? Some other people are working already on $\mathbb{R}$, but $\mathbb{A}^1$-homotopy theory of Morel-Voevodsky is the tool for other fields in general.

Morel constructed a degree map $\deg\colon [\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}] \to \mathrm{GW}(k)$, the Grothendieck-Witt group which is the group completion of the isomorphism classes of nondegenerate symmetric bilinear forms on finite dimensional k-vector spaces (when $\mathrm{char}\, k \neq 2$). It's even a ring under $\otimes$.

Presentation: every form is diagonalizable over k, so this ring is generated by $\langle a \rangle\colon k \times k \to k$ by $(x, y) \mapsto axy$ for $a \in k^\times$. It has relations $\langle ab^2 \rangle = \langle a \rangle$ and $\langle a \rangle + \langle b \rangle = \langle ab(a+b) \rangle + \langle a+b \rangle$. This implies that $h = \langle 1 \rangle + \langle -1 \rangle = \langle a \rangle + \langle -a \rangle$ for all $a \in k^\times$. It's very concretely computed.

Example 3. Some compuations:

- If $k^\times/(k^\times)^2 = 0$, then $\mathrm{GW}(k) \cong \mathbb{Z}$ via the rank map.
- $\mathrm{GW}(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$ via the sign×rank map onto its image.
- $\mathrm{GW}(\mathbb{F}_q) \to \mathbb{Z} \times \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$ if $2 \nmid q$ by the rank×discriminant map.

So yes, it's interesting and computable.

We also have transfers: if L/k is a finite separable extension, then $\mathrm{tr}_L^k \colon \mathrm{GW}(L) \to \mathrm{GW}(k)$ by postcomposition by the trace map. Separability ensures that the image of a nondegenerate form is nondegenerate.

We may now (re)arrive at a notion of local degree: $f \colon \mathbb{A}^d \to \mathbb{A}^d$ a map, $y \in \mathbb{A}^d(k)$, $x \mapsto y$ isolated in the fibre $f^{-1}(y)$. Then

$$\deg_x f = \deg\left(\mathbb{P}^d/\mathbb{P}^{d-1} \to \mathbb{P}^d/\mathbb{P}^d\backslash\{x\} \cong U/U\backslash\{x\} \xrightarrow{\overline{f}} \mathbb{A}^d/\mathbb{A}^d\backslash\{y\} \overset{\text{purity}}{\cong} \mathbb{P}^d/\mathbb{P}^{d-1}\right)$$

Computed:

$$\deg_x f = \begin{cases} \mathrm{Tr}_{k(x)/k}\langle \mathrm{Jac}_f(x)\rangle & \mathrm{Jac}_f(x) \neq 0 \\ \omega^{\mathrm{EKL}} & \text{else} \end{cases} \in \mathrm{GW}(k)$$

where ω^{EKL} is a bilinear form on $k[x_1, \ldots, x_d]/(f_1 - y_1, \ldots, f_d - y_d)$ and there's a canonical one. So now let's compute the actual degree of a map $f \colon X \to Y$ of smooth varieties over k both of dimension d.

Definition 4. Such a map f is *relatively orientable* if there exists the data of a line bundle L on X and $L^{\otimes 2} \xrightarrow{\cong} \mathrm{Hom}(\Lambda^d TX, f^*\Lambda^d TY)$.

Definition 5. *Nisnevich coordinates* around $x \in X$ is $x \in U$ a Zariski open subset with $\psi \colon U \to \mathbb{A}^d(k)$ étale such that $k(\psi(x)) \cong k(x)$.

Definition 6. We say that Y/k is $\mathbb{A}^1$-*chain connected* if for all L/k separable extensions, we have the following property: for any $y_1, y_2 \in Y(L)$, there exists a map $g \colon \mathbb{A}^1(L) \to Y(L)$ and $t_1, t_2 \in \mathbb{A}^1(L)$ such that $g(t_i) = y_i$.

Theorem 7. Theorem/Definition of the degree of a map with value in $\mathrm{GW}(k)$. Let $f \colon X \to Y$ be a map of smooth varieties over k of dimension d both admitting Nisnevich coordinates with Tf invertible at at least one point, Y $\mathbb{A}^1$-chain connected with a k-point $y \in Y(k)$. Then

$$\deg f = \sum_{x \in f^{-1}(y)} \deg_x f$$

is well-defined and independent of y, so long as we pick some finite fibre (which we claim always exists).

Example 8. $\deg \mathrm{Wr}_{m=p=2} = \langle 1\rangle + \langle -1\rangle$. Work of Brazelton exists for other m, p.

Example 9. Fix a Galois action on the n-points $\mathrm{Gal}(\overline{k}/k) \to \Sigma_n$. Then let's look at $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)_\sigma$ parametrising stable maps with points permuted by σ. We can then let

$$\mathrm{ev}_\sigma \colon \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)_\sigma \to (\mathbb{P}^2)^n_\sigma$$

For $n = 2$, $\sigma \colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{Gal}(k(\sqrt{a})/k) \to \Sigma_2$ so $\deg(\mathrm{ev}_\sigma) = \langle 1 \rangle$ but if we change orientation we sort of change this value, which is a bit mysterious.

For $n = 3$, it's computable via a computer, but we need more data to be sure.

But here's something finished: How many lines meet 4 (pairwise disjoint) lines $L_1, \ldots, L_4$ in $\mathbb{P}^3$? A: for $k = \mathbb{C}$, consider $L_1 \subset \mathbb{P}^3$ corresponds to the intersection of two hyperplanes $\{f_1 = f_2 = 0\}$, so we could look at the tautological bundle $S \to \mathrm{Gr}(2, 4)$, and call elements of the righthand side W. Now we get a section of $S^* \wedge S^*$ by $\sigma(W) = f_1|_W \wedge f_2|_W$. Then we have that

$$\{\text{lines } L \text{ meeting } L_1\} \leftrightarrow \{W : \sigma(W) = 0\}$$

and the righthand side we have tools to count it. We can do the same thing for L_2, L_3, L_4 and get a total map $\bigoplus_{i=1}^4 S^* \wedge S^* \to \mathrm{Gr}(2, 4)$ which we also call σ and

$$\{\text{lines } L \text{ meeting all } L_i\} \leftrightarrow \{W : \sigma(W) = 0\}$$

We get an Euler class by summing over all the local degrees which gives us $\langle 1 \rangle + \langle -1 \rangle$.

Question: how do we enumeratively interpret the local degree?

Well, $\{L \cap L_i\}$ is four points on $L \cong \mathbb{P}^1_{k(L)}$. Let λ_L be the cross ratio. Then the planes in $\mathbb{P}^3$ containing L are parametrised by $\mathbb{P}^1_{k(L)}$, so if we consider $\langle L, L_i \rangle$ we get four points of $\mathbb{P}^1_{k(L)}$. Let μ_L be their cross ratio. After making some choices...

Theorem 10 (Srinivasan-W)**.** Let $\mathrm{char}\, k \neq 2$, $L_1, \ldots, L_4$ disjoint lines in $\mathbb{P}^3_k$. Then

$$\sum_{L \text{ meets all } L_i} \mathrm{Tr}_{k(L)/k} \langle \lambda_L - \mu_L \rangle = \langle 1 \rangle + \langle -1 \rangle$$

where the terms of the sum is the local degrees up to choices.

There are extensions of this result, including Marc Levine's paper on Welschinger invariants, which also gives us elements in $\mathrm{GW}(k)$ that we are not sure are the same. Also, Mathias Wendt does oriented Schubert calculus and many related important calculations.