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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Ian Coley **Email/Phone:** msri@iancoley.org

Speaker's Name: Peter Scholze

Talk Title: Condensed Mathematics

Date: 3 / 28 / 19 **Time:** 3 : 30 **am / [✓ pm] (circle one)**

Please summarize the lecture in 5 or fewer sentences:

Proétale sheaves on a point have a nice description that relates to the passage from X to topological X (for one's favourite X).

Here is presented a broad generalization of that idea.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

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CONDENSED MATHEMATICS
OR, TRYING TO UNDERSTAND SHEAVES ON A POINT

PETER SCHOLZE

Joint with Dustin Clausen, work in progress.

Question: How do we do algebra (i.e. brave new algebra) when our rings/modules/groups have a topology? Some first examples, $\mathbb{Z}_p$, $\mathbb{R}$, Banach spaces, $\mathrm{GL}_n(\mathbb{Q}_p)$, or $\mathrm{GL}_n(\mathbb{R})$. Better, how do we do such a thing generally instead of *ad hoc*?

Problems:

- Topological abelian groups is no longer an abelian category. Illustrated, consider $(\mathbb{Z}_p, \text{discrete}) \to (\mathbb{Z}_p, p\text{-adic})$. It's clear a bijection, but it shouldn't be an isomorphism. So what's its kernel? What's its cokernel? We'll have an answer before the end.
- Continuous group cohomology does not admit long exact sequences
- There is no theory of *quasi*coherent sheaves on complex/p-adic analytic spaces. If we have $f\colon \operatorname{Spa} B \to \operatorname{Spa} A$ (Spa = analytic spectrum), and M a topological A-module, we want to base change to $M \widehat{\otimes}_A B$, but what's the completed tensor product in a general setting?

We need a very algebraic way to both keep track of the "topologies" and define "completion".

Recall from Bhatt-Scholze that there's a pro-étale of any scheme X, denoted $X_{\text{proét}}$. The design criterion for this site of that there exists a sheaf of abstract rings $\overline{\mathbb{Q}}_\ell$ on $X_{\text{proét}}$ such that sheaf cohomoloy $H^*(X_{\text{proét}}, \overline{\mathbb{Q}}_\ell)$ is the right thing, e.g. ℓ-adic cohomology. The points $U \in X_{\text{proét}}$ are limits of étale maps $U = \lim(U_i \overset{\text{ét}}{\to} X)$ with, say, affine transition maps $U_i \to U_j$ for easier computation.

Warning: for k algebraically closed, the sheaves on $(\operatorname{Spec} k)_{\text{proét}}$ aren't just sets! This site actually has quite a lot of objects, which is a 'feature rather than a bug':

Notes by Ian Coley.

Definition 1. Consider a set ($= *_{\text{proét}}$) with objects profinite sets S (aka totally disconnected compact Hausdorff spaces), maps continuous maps, and covering families finite disjointly surjective families of maps. A *condensed set* is a sheaf of sets on that site. Specifically, it consists of the data $T\colon \{\text{profinite sets}\}^{\text{op}} \to \{\text{sets}\}$ with an 'underling set' $T(*)$ satisfying

(1) $T(S_1 \sqcup S_2) \overset{\cong}{\to} T(S_1) \times T(S_2)$
(2) If $S' \to S$ is a map, then $T(S) = \operatorname{eq}(T(S') \rightrightarrows T(S' \times_S S'))$

Similarly, one can do condensed whatevers = a sheaf of whatevers on $*_{\text{proét}}$. So for any (∞-)category $\mathcal{C}$, we get a category $\operatorname{Cond}(\mathcal{C})$ of $\mathcal{C}$-values (hypercomplete, due to technical reasons) sheaves on $*_{\text{proét}}$.

Example 2. Let T be a topological space. It defines a condensed set $\underline{T}\colon S \mapsto \operatorname{Map}(S, T)$ the set of continuous functions, where S the profinite set has the inverse limit topology.

Proposition 3. On compactly generated topological spaces, $T \mapsto \underline{T}$ is fully faithful

So as a sub-example, consider $\underline{(\mathbb{Z}_p, \text{discrete})} \to \underline{(\mathbb{Z}_p, p\text{-adic})}$. As condensed abelian groups, this map is injective, but has a really large cokernel Q with $Q(*) = 0$. On S-valued points, we have a short exact sequence

$$0 \to \{\text{locally constant } S \to \mathbb{Z}_p\} \to \{\text{continuous } S \to \mathbb{Z}_p\} \to Q(S) \to 0$$

and since not every continuous function is locally constant, $Q(S)$ has some substance.

Different models of $*_{\text{proét}}$

There is a bigger version and a smaller version which give the same sheaf theory:

Bigger: use the site of *all* compact Hausdorff spaces with covers = surjections. We have the same sheaf because every compact Hausdorff is covered by a totally disconnected one. Actually, it's covered in a pretty natural way: consider a compact Hausdorff space S as a discrete set. Then the Stone-Čech compactification $\beta(S, \text{discrete})$ is totally disconnected and has a natural surjection onto S induced by $(S, \text{discrete}) \to S$.

Smaller: use only the *extremely* disconnected compact Hausdorff spaces (Gleason, 60's).

Definition 4. For S compact Hausdorff, S is extremely disconnected if any surjection $S' \twoheadrightarrow S$ from a compact Hausdorff splits.

So these are something like the "projective objects" in compact Hausdorff spaces. Moreover, there are "enough projectives": consider any totally disconnected $S = \beta S_0$, for S_0 discrete. Then for any map $S' \to S$, we can look at $S_0 \subset S$ and construct a continuous map $S_0 \to S'$ easily because S_0 is discrete. It extends unique to a map $\beta S_0 = S \to S'$, and because of all the naturality involved we can show that this is a section of the original map.

Advantages of the smaller construction:

- $T \mapsto T(S)$ for S extremely disconnected commutes with all limits and connected colimits. In particular, there are no higher cohomology groups.
- As such, we no longer have the 'sheaf condition' so condensed sets are now just the data $T\colon \{\text{extr. disc.}\}^{\text{op}} \to \{\text{sets}\}$ such that $T(S_1 \sqcup S_2) = T(S_1) \times T(S_2)$.

Key question: Does the passage from topological X to condensed X preserve all relevant information?

Example 5. Let S be compact Hausdorff. Then the cohomology groups $H^i(S, \mathbb{Z})$ are important, which we can define always as Čech cohomology $H^i_{\text{Čech}}(S, \mathbb{Z})$ and (in good cases) singular cohomology. Note that if S is profinite, singular cohomology does not work so well.

If we treat S as a condensed set, then it's a sheaf on the étale site so we can compute $H^i_{\text{cond}}(S, \underline{\mathbb{Z}})$ interpreted as sheaf cohomology. Take a hypercover $S_\bullet \to S$ by extremely disconnected sets, and form

$$0 \to \Gamma(S_0, \mathbb{Z}) \to \Gamma(S_1, \mathbb{Z}) \to \cdots$$

where we interpret $\Gamma(S_0, \mathbb{Z}) = \text{Map}(S_0, \mathbb{Z})$.

Theorem 6 (Dyckhoff '75)**.** These constructions canonically agree.

As a particular case, for any set I we have $H^i_{\text{cond}}(\prod_I \mathbb{R}/\mathbb{Z}, \mathbb{Z}) \cong \Lambda^i \left(\bigoplus_I \mathbb{Z}\right)$ where $\mathbb{Z}$ is taken in degree 1. This computation is a fair bit more unpleasant without this framework.

Proposition 7. We have the following computations, where $\mathbb{R}\text{Hom}$ is taken in the category $\text{Cond}(\mathbf{Ab})$ unless noted:

- $\mathbb{R}\text{Hom}(\mathbb{R}, \mathbb{Z}) = 0$

- $\mathbb{R}\mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}_\ell) = 0$ if $p \neq \ell$ and $= \mathbb{Z}_p[0]$ else.
- $\mathbb{R}\mathrm{Hom}(\mathbb{Z}_p, \mathbb{R}) = 0$
- $\mathbb{R}\mathrm{Hom}(\mathbb{R}, \mathbb{R}) = \mathbb{R}[0]$ (reminder: this is taken over $\mathbb{Z}$)
- $\mathbb{R}\mathrm{Hom}(\prod_I \mathbb{R}/\mathbb{Z}, \mathbb{Z}) = \bigoplus_I \mathbb{Z}[-1]$
- $\mathbb{R}\mathrm{Hom}(\prod_I \mathbb{R}, \mathbb{Z}) = \mathbb{R}\mathrm{Hom}(\prod_I \mathbb{R}, \underline{\mathbb{R}\mathrm{Hom}}_{\mathbb{R}}(\mathbb{R}, \mathbb{Z})) = 0$ because $\underline{\mathbb{R}\mathrm{Hom}}_{\mathbb{R}}(\mathbb{R}, \mathbb{Z}) = 0$.
- $\mathbb{R}\mathrm{Hom}(\prod_I \mathbb{Z}, \mathbb{Z}) = \bigoplus_I \mathbb{Z}$
- D^b(locally compact abelian groups) embeds fully faithfully in $D^b(\mathrm{Cond}(\mathbf{Ab}))$.

We know now that $\mathrm{Cond}(\mathbf{Ab})$ is an abelian category, complete and cocomplete, and compactly generated by $\mathbb{Z}$[extremely disconnected] which are projective. Now, we can also have a condensed group G acting on a derived condensed abelian group $C \in D(\mathrm{Cond}(\mathbf{Ab}))$, e.g. the Morava stabilizer group acting on Morava E-theory.

Okay, so we can do $H^i(G, C)$ with structure now in $\mathrm{Cond}(\mathbf{Ab})$ which agrees with continuous group cohomology ... usually. But even if it doesn't agree, we get a long exact sequence.

Completion? We want to make sense of $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ or $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[[t]]$.

Definition 8.
1. The free complete condensed abelian group on S a condensed set is constructed as follows: as S is profinite, write $S = \lim S_i$, then define $\mathbb{Z}[S]^\wedge := \lim \mathbb{Z}[S_i]$, which receives a map from $\mathbb{Z}[S]$.
2. A condensed abelian group M is *complete* if for all condensed sets S, every map $S \to M$ always admits a unique factorisation through $\mathbb{Z}[S]^\wedge$.
3. $C \in \mathbb{D}(\mathrm{Cond}(\mathbf{Ab}))$ is complete if for all condensed sets S, $\mathbb{R}\Gamma(S, C) \overset{\cong}{\leftarrow} \mathbb{R}\Gamma(\mathbb{Z}[S]^\wedge, C)$ is an isomorphism.

Theorem 9.
1. $C \in D(\mathrm{Cond}(\mathbf{Ab}))$ is complete if and only if all $H^i(C)$ are complete in $\mathrm{Cond}(\mathbf{Ab})$.
2. Complete condensed abelian groups is an abelian subcategory of $\mathrm{Cond}(\mathbf{Ab})$, closed under limits and colimits.
3. The same is true for complete (derived) chain complexes in all chain complexes $D(\mathrm{Cond}(\mathbf{Ab}))$.
4. There exists a completion functor and $\widehat{\otimes}_{\mathbb{Z}}$

Example 10. The completed tensor product works as it should: $\mathbb{Z}_p \widehat{\otimes}_{\mathbb{Z}} \mathbb{Z}_\ell = \mathbb{Z}_p$ when $p = \ell$ and is 0 otherwise; $\mathbb{Z}_p \widehat{\otimes}_{\mathbb{Z}} \mathbb{Z}[[t]] = \mathbb{Z}_p[[t]]$.

Final remark: hopefully we can do something like algebraic geometry in this condensed setup, e.g. to a scheme X we associate complete condensed quasicoherent sheaves on X.