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NOTETAKER CHECKLIST FORM

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Name: Ian Coley **Email/Phone:** msri@iancoley.org

Speaker's Name: Dmitry Kaledin

Talk Title: Bökstedt periodicity and Bott periodicity

Date: 3 / 29 / 19 **Time:** 3 : 30 am / [✓ pm] (circle one)

Please summarize the lecture in 5 or fewer sentences:
A new proof of the structure of THH(F_p) yields some new interesting general computational techniques.

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BÖKSTEDT PERIODICITY AND BOTT PERIODICITY

DMITRY KALEDIN

Joint with A. Fonarev.

We want to present a new proof of a known theorem, namely the computation of $THH_*(\mathbb{F}_p)$. Let $A \in D(\mathbb{S})$ be an $\mathbb{E}_1$-ring spectrum, and we want to take M and A-bimodule for coefficients.

Definition 1. $THH(A, M)$ is defined as the homotopy colimit of the following diagram:

$$\begin{tikzcd}M & A \wedge M \arrow[l, shift left] \arrow[l, shift right] & A \wedge A \wedge M \cdots \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2]\end{tikzcd}$$

where the maps are given by the action of A on M.

Specifically, we want to work over $k = \mathbb{F}_p$ or another perfect field of characteristic p. We have an adjunction between $D(k)$ and $D(\mathbb{S})$ which gives us a symmetric lax monoidal comonad Q on $D(k)$. In particular, if A/k is associative and unital and M an A-bimodule, then $A \wedge M = Q(A) \otimes_k M$. Then we can recognise $THH(A, M) = HH_*(Q(A), M)$, where this latter is sometimes called Maclane homology of A with coefficients in M $HM_*(A, M)$. Eilenberg-Maclane knew about the comonad Q in the '50s already.

Uses:

(1) Maclane cohomology also exists, which gives $HM_*(k, k)$ a Hopf algebra structure.
(2) $Q(k)$ is the dual Steenrod algebra, though Q is not k-linear, and there exists some spectra sequence converging to it. In particular, its' something of the form $k[\sigma, A_i, B_i]$ with $|sigma| = 2$, $|A_i| = 2p^i - 1$, $|B_i| = 2p^i$ for $i \geqslant 1$.

If $p > 2$, that spectral sequence is miraculously highly degenerate, and only σ lasts to the E_∞ page, so that $THH(k) = k[\sigma]$ (Bökstedt).

Notes by Ian Coley.

Observe: that σ will always survive to the end. Assume that σ is not nilpotent. Then "we are done", using another spectra sequence $k[\varepsilon, C_i, D_i]$ converging to Maclane cohomology. Because of the Hopf algebra structure, we know that $\langle \sigma^p, \varepsilon^p \rangle = 0$. becuase σ isn't nilpotent, $\varepsilon^p = 0$. Then we conclude $\partial C_1 = \varepsilon^p$ so $\partial A_1 \neq 0$ and B_1 survives. Then go by induction on the index of A, B.

Some reasoning along these lines allows us to do the computation.

Anyway, here's the idea: we need to prove that σ is not nilpotent by constructing some sort of multiplicative map out of THH. There is a general method by which we can do this:

Stabilization: Let I be a pointed small category with finite coproducts, say $I = \Gamma_*$ the category of finite pointed sets. Let $\mathrm{Ho}(I) = \mathrm{Ho}(\mathrm{Fun}(I, \mathbf{Top}_*))$. Since I is pointed, it makes sense to refer to additive functors $F \colon I \to \mathbf{Top}_*$ as those that sends coproducts to products.

If $X \in \mathrm{Ho}(I)$ is (the image of) an additive functor, then $\pi_0 X(*)$ is a commutative monoid. We call it $X(*)$ *grouplike* if it happens to be a group.

Definition 2. An additive functor $F \in \mathrm{Ho}(I)$ is *stable* if for all $i \in I$, F restricted to the collection $\{\coprod i\}$ is grouplike. Let $\mathrm{Ho}^{\mathrm{st}}(I) \subset \mathrm{Ho}(I)$ be the full subcategory of stable functors.

Theorem 3. There is a left adjoint to the inclusion of stable functors, i.e. there is a localization $\mathrm{stab} \colon \mathrm{Ho}(I) \to \mathrm{Ho}^{\mathrm{st}}(I)$. Moreover, if I is monoidal, then for $F \colon I \to \mathbf{Top}_*$ lax monoidal, $\mathrm{stab}\, F$ is still lax monoidal.

Trace functors: If we also have isomorphisms $\tau_{i,i'} \colon F(i \otimes i') \to F(i' \otimes i)$, then $\mathrm{stab}\, F$ also will

Finally, we can move ourselves from $\mathbf{Top}_*$ to k-cdgas and everything still works. So now let I be the category of finite dimensional k-vector spaces.

Example 4. Examples of functors and their stabilizations.

1. $T(M) = M^{\otimes p}$, then $\mathrm{stab}\, T = 0$.
2. $C(M) = (M^{\otimes p})^{\tau}$, where τ is the cyclic permutation on p elements. Then $\mathrm{stab}\, C(M) = \tau^{\leqslant 0} \check{C}^*(C_p, M^{\otimes p})$, the 0-truncation of the Tate cohomology complex. We can see that $\check{H}^i(C_p, M^{\otimes p}) \cong M$ for all i . This functor is lax monoidal and comes with a trace functor on $(M \otimes N)^{\otimes p}$ which twists M by τ but does not twist N. So something like that passes to the stabilization.

Now consider $\psi\colon M \to C(M)$ which sends m to $m^{\otimes p}$. This functor is super not additive. This gives us a map $M \to \operatorname{stab} C(M)$ which is not k-linear.

Lemma 5. The composition

$$M \overset{\operatorname{stab}\psi}{\to} \tau^{\leqslant 0}\check{C}^*(C_p, M^{\otimes p}) \to (M^{\otimes p})_\tau[1]$$

is not zero (as it would be if it were k-linear), and in fact is the same as the Bockstein $\beta\colon M \to M[1]$ post composed with $M[1] \to (M^{\otimes p})_\tau[1]$.

THH also appears as a stabilization of something, but of what? Let $\mathcal{C} \in \mathbf{Cat}$ be a small category, and let $M\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Sets}$ be a functor. Then define the cyclic nerve

$$N_\bullet^{\mathrm{cyc}}(\mathcal{C}, M) \in \mathbf{sSet}$$

with k-simplices $c_0 \to c_1 \to \cdots \to c_k$ with an element $m \in M(c_k, c_0)$ to "loop it". If we postcompose with geometric realization, we get something we can stabilize.

For (A, M) over k, let $P(A)$ be the category of finitely generated projective A-modules. Then let $P(M)\colon P(A)^{\mathrm{op}} \times P(A) \to \{k\text{-vector spaces}\}$ be given by $P \times P' \mapsto \operatorname{Hom}_k(M \otimes_A P, P')$.

Theorem 6. $THH(A, M)$ is the stabilization of the cyclic nerve $N_\bullet^{\mathrm{cyc}}(P(A), P(M))$

We have two additional structures:

1. For any M, $N_\bullet^{\mathrm{cyc}}(P(A), P(-))$ is lax monoidal.
2. There's a trace functor structure that's more obvious if M is finitely generated projective as a left or right A-module. The nextend by taking filtered colimits of such.

Now, consider $\varphi\colon N_\bullet^{\mathrm{cyc}}(P(k), P(M)) \to (M^{\otimes p})_0$ the constant simplicial set. The zero simplices on the left are $P \in P(k)$ with an endomorphism $a\colon P \to P$. We send this to the trace of a^p in $M^{\otimes p}$, and it lands in $(M^{\otimes p})^\tau$ the τ-invariant part.

Now we let $M = k$ and stabilize. We can look at $\operatorname{stab}\varphi$ and get a better idea of the structure of the (a priori) computation that the lefthand side is $THH(k, k)$.