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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Chanel Lee Email/Phone: chanelclee@gmail.com

Speaker's Name: Xuwen Zhu

Talk Title: Boundary degeneration of Riemann moduli spaces and Weil-Petersson metrics

Date: 08/15/19 Time: 2:00 am / (pm) (circle one)

Please summarize the lecture in 5 or fewer sentences: This talk covers Moduli space of Riemann Surfaces, degeneration of hyperbolic metrics, analysis of singular metrics, and an application of asymptotics of Weil-Petersson metrics.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
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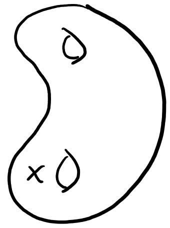
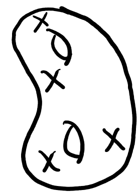
Zhu Board Notes

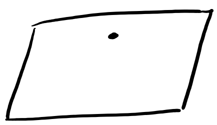


$\Rightarrow \mathcal{M}_{g,1} \xrightarrow{\text{generalize}} \mathcal{M}_{g,n}$

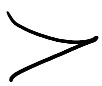
$\mathcal{M}_{g,n-1}$



$\mathcal{M}_{g,1} \to \mathcal{M}_g$



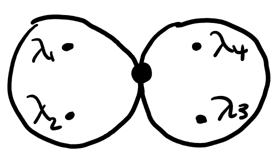
$\mathcal{M}_g$

cusp : 

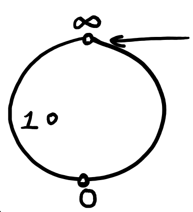
$$\frac{|dz|^2}{(|z|\cdot|g|z||)^2} \quad \text{or} \quad \underset{(r\to\infty)}{dr^2+e^{-2r}d\theta^2} \quad \text{or} \quad \underset{(s\to 0)}{\frac{ds^2}{s^2}+s^2d\theta^2}$$

Normal crossing:

$zw = t \quad (t\to 0)$



$\mathcal{M}_{0,4}$



$T^*\mathcal{M}_g = \{\mu(z)\,dz^2\}$

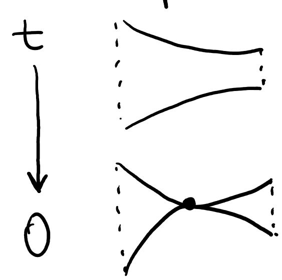
↑ meromorphic

$$g_{WP}(\xi_1,\xi_2) = \int_{fiber} \frac{\mu_1\,dz^2\,\overline{\mu_2\,dz^2}}{g_0\,dz\,d\bar{z}}$$

(g_0 ↖ hyperbolic metric)

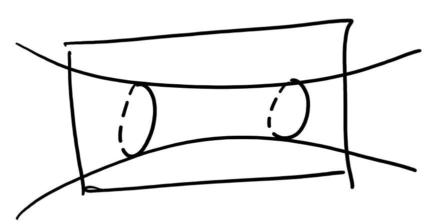
$\xi_i = \mu_i(z)\,dz^2$

Local plumbling model



$\Delta_{g_{pl}} f + R(g_{pl}) = -e^{-2f}$

resolution or bump



Boundary degeneration of Riemann moduli spaces and Weil-Petersson metrics

Xuwen Zhu (MSRI / UC Berkeley)

MSRI Connections for Women:
Holomorphic Differentials in Mathematics and Physics

Joint with Richard Melrose

Outline

1. Moduli spaces of Riemann surfaces
2. Degeneration of hyperbolic metrics
3. Analysis of the singular metrics
4. Application: asymptotics of Weil–Petersson metrics

Riemann surfaces

Consider a compact surface M with genus g.

- A conformal structure on M is given by a smooth Riemannian metric fixed up to multiplication by a positive C^∞ function.
- A complex structure on M is given by an automorphism J of the tangent bundle TM with $J^2 = -\operatorname{Id}$.
- In dimension 2, there is a one-to-one correspondence between the above two structures.

A Riemann surface is a surface with such a structure.

Moduli spaces of Riemann surfaces

The moduli space $\mathcal{M}_g$ is the set of all conformal (complex) structures on a genus g surface M up to diffeomorphism

- The space $\mathcal{M}_g$ is a complex orbifold of dimension $3g - 3$ (when $g \geq 2$)
- In each conformal class, there is a unique metric with constant curvature and finite area (Uniformization theorem)
- The curvature depends on the arithmetic genus (Gauss–Bonnet)
- The metric is hyperbolic when genus $2g - 2 > 0$
- Each point on $\mathcal{M}_g$ represents a diffeomorphism class of hyperbolic metrics on the surface

The pointed moduli space

The pointed moduli space is the set of all conformal structures on a genus g surface M with additional n ordered distinct marked points

- $\mathcal{M}_{g,n}$ is a complex orbifold with dimension $3g - 3 + n$
- Fibration $\mathcal{M}_{g,1} \to \mathcal{M}_g$
- In fact we have fibrations $\mathcal{M}_{g,n} \to \mathcal{M}_{g,n-1}$ by dropping the last marked point
- For the case $2g + n > 2$, on each fiber there is a hyperbolic metric
- The metric in the pointed case has cusps at each marked points
- The fiber metrics vary smoothly

Noncompactness: examples

The moduli space $\mathcal{M}_{g,n}$ is not compact for any (g, n).

Example: $\mathcal{M}_{1,1}$

The moduli space of a pointed torus is the modular surface $\mathcal{M}_{1,1} = \mathbb{H}/SL(2, \mathbb{Z})$.

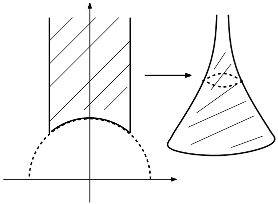


Figure: Moduli space $\mathcal{M}_{1,1}$

Noncompactness: example

The moduli space $\mathcal{M}_{g,n}$ is not compact for any (g, n).

Example: $\mathcal{M}_{0,n}$

When $n = 3$, $\mathcal{M}_{0,3} = \{\text{pt}\}$. When $n = 4$, $\mathcal{M}_{0,4} = \mathbb{CP}^1 \setminus \{0, 1, \infty\}$.
For $n \geq 5$,

$$\mathcal{M}_{0,n} = (\mathbb{CP}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \triangle_{\text{fat}}$$

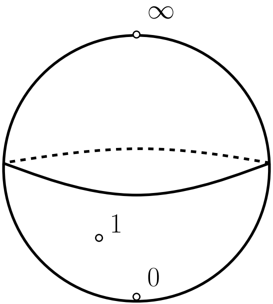


Figure: Moduli space $\mathcal{M}_{0,4}$

Noncompactness

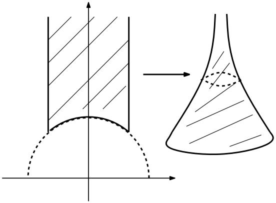


Figure: Moduli space $\mathcal{M}_{1,1}$

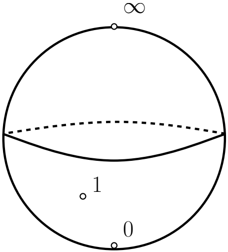


Figure: Moduli space $\mathcal{M}_{0,4}$

The noncompactness comes from two kinds of degenerations:

- Shrinking geodesics
- Separation of “colliding” marked points

The compactification of $\mathcal{M}_{g,n}$ is denoted as $\overline{\mathcal{M}}_{g,n}$.

Degeneration I: pinching geodesics

Take a nontrivial geodesic cycle in M, and let its length go to zero.

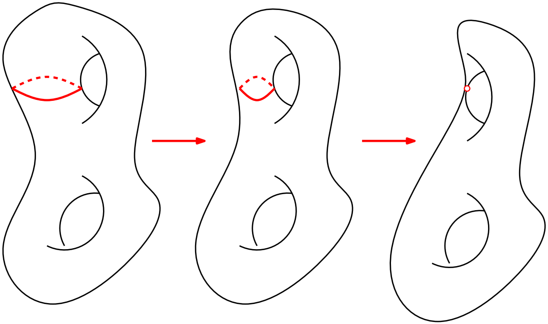


Figure: Degenerating surfaces with a geodesic cycle shrinking to a point

This process can be indexed by a complex parameter $t \in \mathbb{D}$.

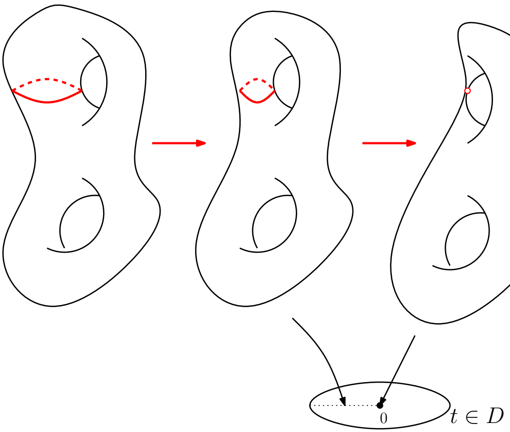


Figure: Degenerating surfaces, indexed by a parameter t

Local geometry: hyperbolic cylinder

Locally the geometry near the shrinking cycle is described by the normal crossing model:

$$(z, w) \in \mathbb{C}^2, \quad zw = t$$

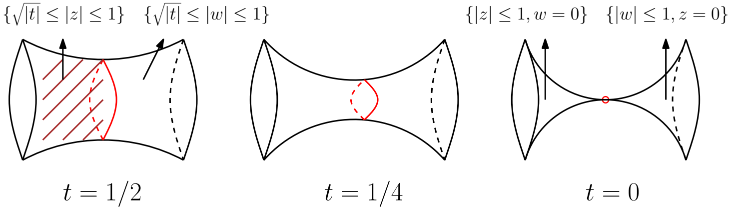


Figure: Local geometry of $zw = t$, with coordinate patch z and w

Nodal crossing divisors

The previous picture might be misleading: the singular surface has a transversal crossing

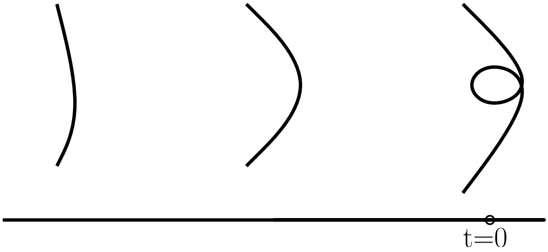


Figure: Transversal crossing of universal curve

Divisors in $\overline{\mathcal{M}_g}$

- The "boundary" $\overline{\mathcal{M}_g} \setminus \mathcal{M}_g$ is a union of normally intersecting, self-intersecting divisors
- Pinching one geodesic gives a pair of nodal points
- If the fiber has k pairs of nodal points, it lies on the intersection of k local divisors, i.e. locally a k-fold intersection of $\mathcal{M}_{g-1,2}$
- The arithmetic genus $\mathcal{G} = 2g + n$ stays the same

Degeneration II: pointed moduli space $\mathcal{M}_{g,n}$

- Another degeneracy: marked points may collide
- Example of $\mathcal{M}_{0,4}$ of $\mathbb{CP}^1$ with 4 points: $\{0, 1, \infty, t\}$ vs $\{0, 1/t, \infty, 1\}$

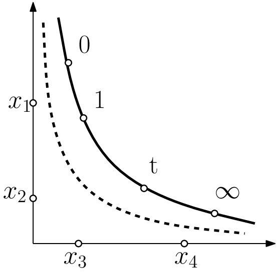


Figure: Degeneration of $\mathbb{CP}^1$ with 4 points

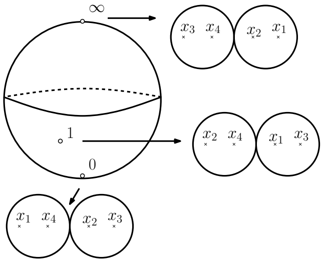
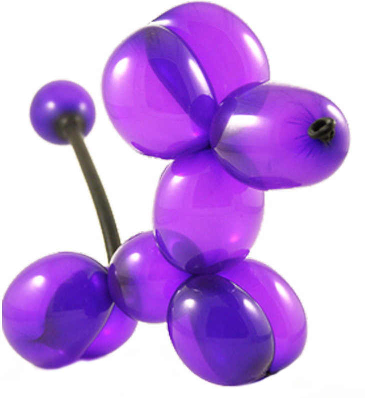


Figure: Compactified moduli space $\mathcal{M}_{0,4}$

Compactification of $\mathcal{M}_{g,n}$

- The compactification separates the “colliding” points by adding nodal spheres
- A divisor in $\overline{\mathcal{M}}_{g,n}$ is represented by a collection of marked surfaces connected by nodal crossings
- Nodal crossing: a pair of cusp points
- Singular fibration of $\overline{\mathcal{M}}_{g,n+1}$ over $\overline{\mathcal{M}}_{g,n}$ by dropping the last point and possibly pinching unstable components

Nodal curves



Picture source: http://www.partyballoonanimals.co.uk/wp-content/themes/alexandria-child/images/balloon-animal.png

Deligne–Mumford–Knudsen compactification

The compactification $\overline{\mathcal{M}}_g$ was introduced by Deligne and Mumford, later $\overline{\mathcal{M}}_{g,n}$ by Knudsen.

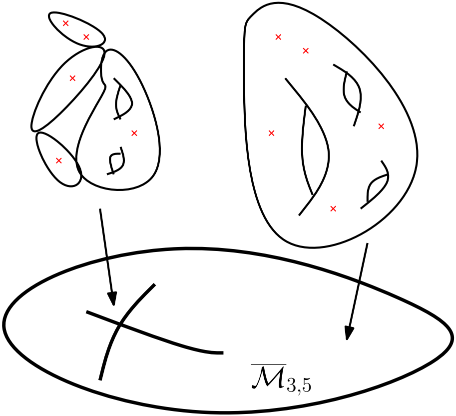


Figure: Fibration over the compactified moduli space $\overline{\mathcal{M}}_{3,5}$

The tangent and cotangent spaces of $\mathcal{M}_{g,n}$

- The tangent space of $\mathcal{M}_g$ is identified with harmonic Beltrami differentials (cf. quasiconformal maps)
- $T\mathcal{M}_g$ can also be identified with transverse traceless tensors
- $T\mathcal{M}_g = \{udx^2 - 2vdxdy - udy^2 | u + iv \text{ is holomorphic } \}$
- The cotangent space of $\mathcal{M}_g$ identified with the holomorphic quadratic differentials

$$T^*\mathcal{M}_g = \{\mu = \zeta(z)dz^2\}$$

- $T^*\mathcal{M}_{g,n}$: meromorphic quadratic differentials, at most simple poles at the punctures
- $T^*\overline{\mathcal{M}}_{g,n}$: at most double poles at the nodes, with matching residues
- Dimension counting: Riemann-Roch

Weil–Petersson metric

There is a natural metric on $\mathcal{M}_{g,n}$ called the Weil–Petersson metric.

- Using this identification, the Weil-Petersson (co-)metric is defined by

$$G_{WP}(\zeta_1, \zeta_2) = \int_{fib} \frac{\zeta_1 \overline{\zeta_2}}{\mu_H}, \ \zeta_1, \ \zeta_2 \in T_p^* \mathcal{M}_{g,n}, \ p \in \mathcal{M}_{g,n}$$

  where μ_H is the area form of the fiber hyperbolic metric and the integrand itself may be identified as a fiber area form.

Understanding the singular geometry

One would like to understand:

Question:

Can we analytically describe the singular fibration $\overline{\mathcal{M}_{g,n}} \to \overline{\mathcal{M}_{g,n-1}}$?

- How does the hyperbolic metric behave when approaching the divisors?
- What does the moduli space look like near the divisors, more specifically, the behavior of the Weil-Petersson metric?

Literature

- Hyperbolic metrics on nodal crossing: [Wolpert, 1990–] [Wolf, 1991–] [Obitsu–Wolpert, 2009]
- Geometry of moduli space: [Bers, 1973, 1974] [Deligne–Mumford, 1979] [Robbin–Salamon, 2006]
- Weil–Petersson metric asymptotics: [Masur, 1976] [Wolpert, 2001–] [Mazzeo–Swoboda, 2015]
- Problems related to Weil–Petersson metric: [Wolpert, 1982–] [Huang, 2003–] [Takhatajan–Zograf, 1991] [Yamada, 2004] [Liu–Sun–Yau, 2004, 2005] [Mirzakhani, 2006, 2007] [Obitsu–To–Weng, 2008] [Burns–Masur–Wilkinson, 2012] [Ji–Mazzeo–Müller–Vasy, 2014] [Wu, 2014–] [Gell-Redman–Swoboda, 2015] [Gell-Redman–Melrose, in progress]

From moduli space to the plumbing model

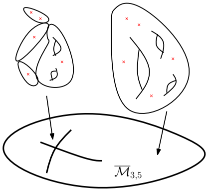


Figure: Compactified moduli space

$\longrightarrow$

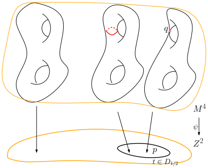


Figure: Lefschetz fibration

$\longrightarrow$

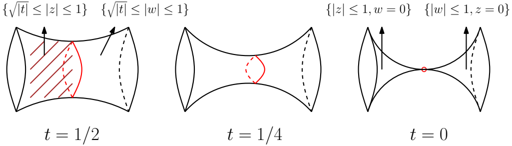


Figure: Local plumbing model

Local model

Near the degenerated fiber, there is a model metric.

Plumbing metric on each fiber

$$g_{pl}^{(t)} = \left(\frac{\pi \log|z|}{\log|t|} \csc \frac{\pi \log|z|}{\log|t|} \right)^2 g_0,$$

$$g_0 = \left(\frac{|dz|}{|z| \log|z|} \right)^2$$

- $g_{pl}^{(t)} \to g_0$ as $t \to 0$.
- Symmetric with the change of $w = t/z$.
- Fiber curvature $= -1$.

Curvature equation on M

Curvature equation for conformal factor: if $g = e^{2f} g_0$, then

$$R(g)e^{2f} = \Delta_{g_0} f + R(g_0),$$

which in our case is

$$\Delta_{g_{pl}} f + R(g_{pl}) = -e^{2f}.$$

The linearization of the curvature operator:

$$(\Delta_{g_{pl}} + 2)f = -1 - R(g_{pl}) + O(f^2).$$

Results on degenerating hyperbolic metric

Theorem(Melrose–Z, 2015)

There exists a resolution of the fibration $\widehat{M} \to \hat{Z}$ such that

- The fiber metric is conformal to a smooth metric on ${}^{L}T\widehat{M}$ a rescaling of the fiber tangent bundle;
- The conformal factor is log-smooth.

Remark: a resolution essentially introduces more smooth variables, in this case, angular variables, $\log|z|$, $\log|w|$, $\log|t|$, and $\log|z|/\log/|w|$.

The metric has the following expansion:

$$g_t = g_{pl}\left(\sum_{k\geq 2} a_k(-\frac{1}{\log|t|})^k\right).$$

Resolved space $\widehat{M}$

We consider the following glued space of $\widehat{M} = (M \setminus P) \cup P_{\text{mr}}$:

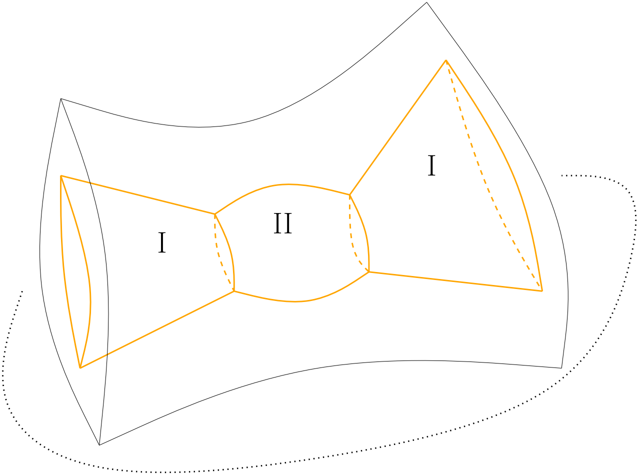


Figure: Final resolved space $\widehat{M}$

Main steps of proof

- Resolve the space by introducing more smooth variables.
- Near the singular fiber we glue the plumbing metric with nearby part to get a model metric g_{pl}. This is a smooth family of Hermitian metrics on the resolved space. It has has constant curvature -1 near the nodal parts and error to second order at the base fiber.
- The inverse family $(\Delta + 2)^{-1}$ on the fibers for this metric is shown to be uniformly bounded on appropriate spaces.
- The prescribed curvature equation for the conformal factor is solved to infinite order at the base fiber.
- The error term is removed using the Implicit Function Theorem to show that the conformal factor to a hyperbolic family exists and is log-smooth.

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Asymptotics of the Weil-Petersson metric

We apply a similar technique to obtain the expansion for the Weil–Petersson metric on the compactified moduli space $\overline{\mathcal{M}}_{g,n}$.

Theorem(Melrose–Z, 2016)

For any g, n with $2g - 2 + n > 0$, there exists a resolution of the moduli space fibration $\widehat{\mathcal{M}}_{g,n+1} \to \hat{\mathcal{M}}_{g,n}$, such that the Weil–Petersson metric lifts to be a log-smooth metric on the rescaled cotangent bundle of $\hat{\mathcal{M}}_{g,n}$.

The metric is of the form

$$g_{WP} = \sum_{i=1}^{k} \pi \left(\frac{ds_i^2}{s_i} + s_i^3 d\theta_i^2 \right) + g'_{WP}$$

- $s_i = -1/\log|t_i| \sim$ length of the shortest geodesic circle;
- g'_{WP} when restricted to the corner is the Weil-Petersson metric on the k-fold intersection of divisors, and $g'_{WP}(\partial_{s_j}, \cdot)$ vanishes at $s_j = 0$.

Corollary: Expansion of shortest geodesics

Corollary

The length of the shortest geodesic under degeneration is a polyhomogeneous function of s.

- In the plumbing model, the shortest geodesic is given by the circle in the middle
- $l_{pl}(s) = 2\pi^2 s$
- Rotational symmetry of the actual hyperbolic metric (up to infinite order)
- $l_{hp}(s) = 2\pi^2 s + s^2 e(s)$ with $e(s)$ log-smooth.
- This implies the leading order of the expansion of g_{WP} under Fenchel-Nielsen coordinates.

The Ricci metric

- The Ricci curvature of the Weil-Petersson metric is itself a Kähler metric on the moduli space
- The quasi-isometry class by [Trapani, 1992]; the leading asymptotics at a divisor [Liu–Sun–Yau, 2004]
- Kähler potential given by $-\log\det(g_{WP})$
- We obtain a "multi-cusp" metric

$$g_{\mathrm{Ri}} = \frac{3}{4}\sum_{j=1}^{k}\left(\frac{ds_j^2}{s_j^2} + s_j^2 d\theta_j^2\right) + h$$

  where h is log-smooth and restricts to the exceptional divisor to be the induced Ricci metric.
- g_{Ri} is complete: spectrum of the Ricci metric

The full curvature tensor of g_{WP}

- The Kähler potential of g_{WP} is of the form (near a single divisor):

$$\phi(z, \bar{z}) + s + s^3 \psi(z, \bar{z}, s)$$

- Implication of the decay of the cross terms

$$\begin{pmatrix} s^3(1 + a' s^2) & s^4 b' \\ s^4 \overline{b'} & h' \end{pmatrix}$$

matches the choice of geodesic coordinates [Ahlfors, 1961]

- Full curvature tensors are computed
- $R_{s\bar{s}s\bar{s}} = O(s^{-1}),\ R_{s\bar{s}z\bar{z}} = O(s^2),\ R_{z\bar{z}z\bar{z}} = O(1)$

Thank you for your attention!