MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Chanel Lee **Email/Phone:** chanelclee@gmail.com

Speaker's Name: Yan

Talk Title: Boundary degeneration of Riemann moduli spaces and Weil-Petersson metrics

Date: 08/15/19 **Time:** 3:30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: This talk covers skein algebras, q-abelization, and WKB foliation.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Q-abelianization for Line Defects (with Andy Neitzke)

Yan

August 15, 2019

A new "invariant" (w/ wall-crossing) for framed links 3-mfd, $M = C \times \mathbb{R}$
C is an oriented surface, valued in $GL(1)$ skein algebra associated with a 3-mfd $\tilde{M} = \tilde{C} \times \mathbb{R}$, where $\tilde{C}$ is a N-fold cover for C.
$F : A \to \tilde{A} \leftarrow$ q-abelianization map
A : algebra of fundamental Wilson lines in $GL(N)$ Chern-Simons thy on M
$\tilde{A}$: algebra of fundamental Wilson lines in $GL(1)$,Chern-Simons thy on $\tilde{M}$
$A = sk(M, GL(N))$, $\tilde{A} = sk(\tilde{M}, GL(1))$
Special Cases:

- L is contained in a 3-ball in M,

$$F(L) = q^{Nw(L)} P_{HOMFLY}, un(L, a = q^N, z = q - q^{-1})$$

  where $w(L)$ = self-linking number and unknot: $\frac{q^N - q^{-N}}{q - q^{-1}}$.

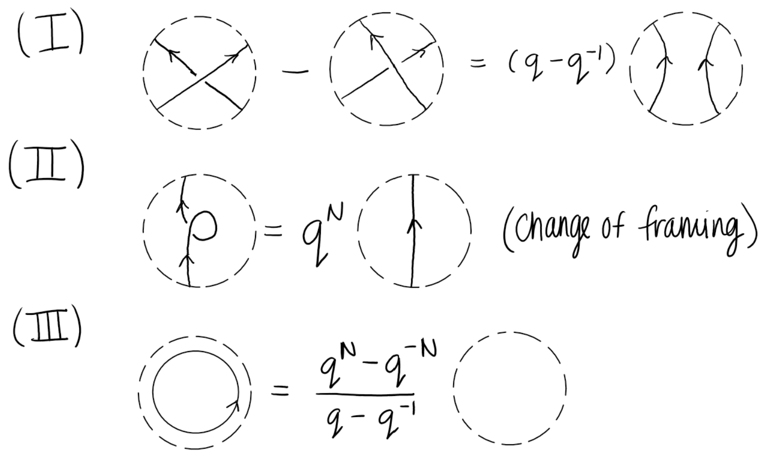
- Proj. of L to C continuous no-crossing and the homology of L is nontrivial. $F(L)$ is generating function for a protected spin character (PSC) encoding the spectrum of framed BPS states associated w/ $\frac{1}{2}$-BPS line defects in a 4d $N = 2$ supersym. thy. [Gaiotto-Moore-Neitzke][Galakhov-Longhi-Moore][Gabella][Allegretti][Kim-Son]

- $N = 2$, [Bonahan-Wang]

1 Skein Algebras

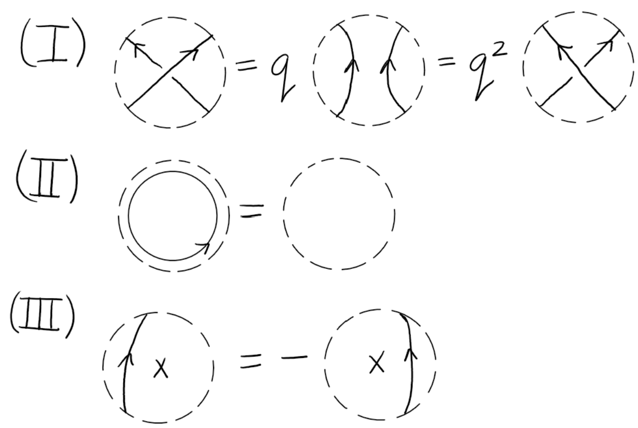
1.1 The $GL(N)$ Skein Module

Def. Given a oriented 3-mfd M, $sk(M, GL(N))$ is free $\mathbb{Z}[q, q^{-1}]$ module gen. by ambient isotopy classes of framed oriented links in M, modulo the sub module gen. by: (skein relations, blackboard framing)



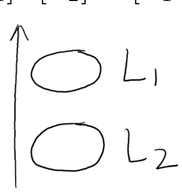
1.2 $GL(1)$ Skein module with branch locus

Def. oriented 3-mfd $\tilde{M}$ with codim-2 branch locus F
$sk(\tilde{M}, GL(1))$ is the free $\mathbb{Z}[q, q^{-1}]$ module gen. by ambient isotopy classes of framed links in $\tilde{M}$, modulo the following relations:



1.3 Skein Algebras: $M = C \times \mathbb{R}$, $\tilde{M} = \tilde{C} \times \mathbb{R}$

$\tilde{C} \xrightarrow{N=1} C$ with branch locus F' counting of simple branch points
$sk(M, GL(N))$ and $sk(\tilde{M}, GL(1))$ are algebras over $\mathbb{Z}[q, q^{-1}]$
$[L_1] \cdot [L_2] = [L_1 L_2]$



Given Γ a lattice with a skew bilinear paring $< \cdot\,, \cdot >$, the quantum torus Q_Γ is a $\mathbb{Z}[q, q^{-1}]$ algebra with basis $\{x_\gamma\}_{\gamma\in\Gamma}$

$$x_{\gamma_1}x_{\gamma_2} = -q^{<\gamma_1,\gamma_2>}x_{\gamma_1+\gamma_2}$$

$sk(\tilde{M}, GL(1)) \tilde{=} Q_\Gamma$, $\Gamma = H_1(\tilde{C}, \mathbb{Z})$ with intersection paring

2 q-abelization $N = 2$

$$F : sk(M, GL(N)) \to sk(\tilde{M}, GL(1))$$
$$[L] \mapsto F([L]) = \sum_{\gamma\in H_1(\tilde{C},\mathbb{Z})} \overline{\Omega}([L], \gamma)x_\gamma$$

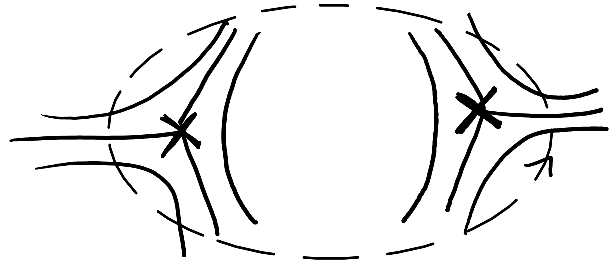
where $[L] \in sk(M, GL(N))$ and $\overline{\Omega}([L], \gamma) \in \mathbb{Z}[q, q^{-1}]$

2.1 WKB foliation (GMN)

$$\tau = \{\lambda|\lambda^N + \sum_{k=1}^{N} \phi_k\lambda^{N-k} = 0\} \subset T^*C$$

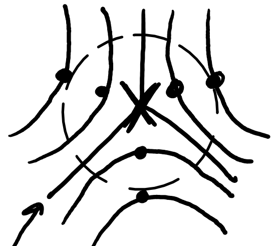
where ϕ_k - mero. diff on C
Locally on C, N 1-forms $\lambda^{(i)}$
Def: Let $w^{(ij)} = \int(\lambda^{(i)} - \lambda^{(j)})$ an ij-leaf with phase θ is oriented on path C, $\mathrm{Im}(e^{-i\theta}\omega^{(ij)}) = 0$

$\mathrm{Re}(e^{-i\theta}w^{(ij)})$ decreases $\quad$ $\mathrm{Re}(e^{-i\theta}w^{(ji)})$



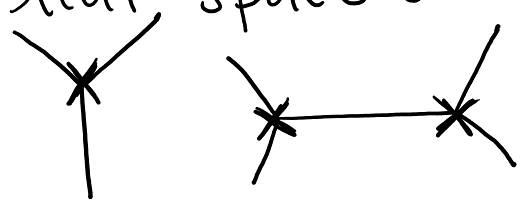
leaf space (C)

N=2:



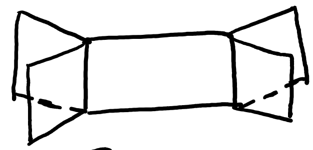
generic leaf

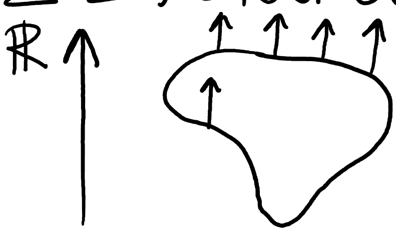
critical leaf



leaf space (M)

2.2) Standard Framing



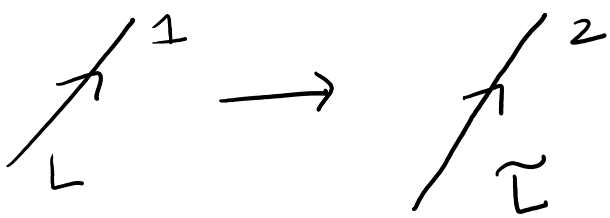


$$F([L]) = \sum_{\tilde{L}} \alpha(\tilde{L})[\tilde{L}]$$

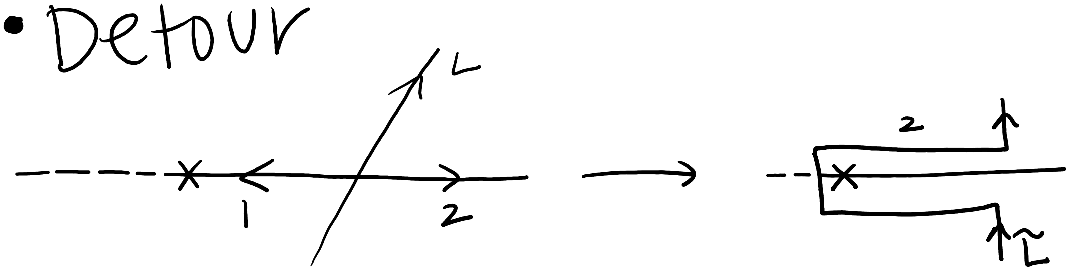
where $\alpha(\tilde{L}) \in \mathbb{Z}[q, q^{-1}]$

Step 1: Enumerate $\tilde{L} \subset \tilde{M}$ build out of:

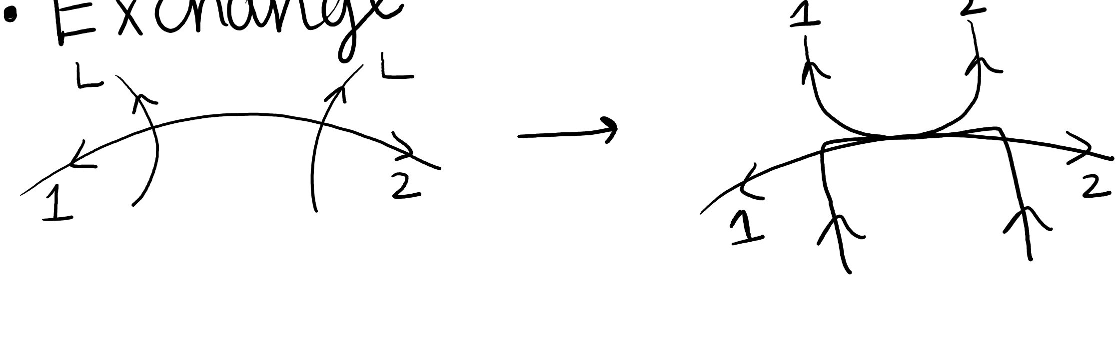
- Direct Lifts



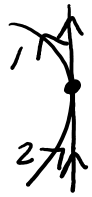
- Detour



- Exchange



Step 2: $\alpha(\tilde{L})$ build out of local factors:



$\to q^{\pm\frac{1}{2}}$

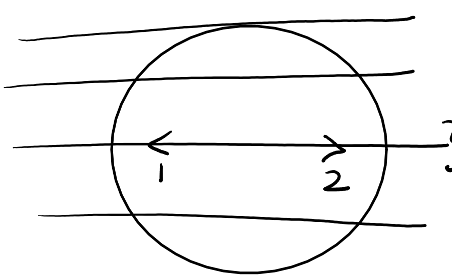
detour: $q^{\pm\frac{1}{2}}$

exchange: $\begin{cases} \pm q^{\pm 1}(q^{-1}-q) \\ \pm(q^{-1}-q) \end{cases}$

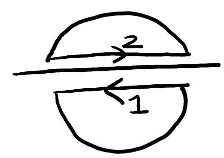
w: winding of $\tilde{L}$, q^{w}

3) Examples

3.1 unknots: ① $C=\mathbb{R}^2$

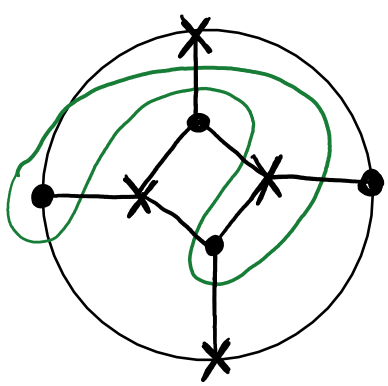


3 lifts $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3$



$$F([L]) = q[\bullet] + q[\bullet] + (q^{-1}-q)[\bullet]$$
$$= (q+q^{-1})[\bullet]$$

C: 4-punctured sphere



$F([L])$ has 33 terms

where $F([L]) \ni \dfrac{(2+q^2+q^{-2})}{1+(1+q^2+q^{-2})} X_\gamma$