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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Chanel Lee **Email/Phone:** chanelclee@gmail.com

Speaker's Name: Elise Goujard

Talk Title: Volumes of principal strata of quadratic differentials and intersection numbers

Date: 08/16/2019 **Time:** 3:30 am / (pm) (circle one)

Please summarize the lecture in 5 or fewer sentences: This lecture covers a formula for the volume of principal strata and the idea of the proof for this formula.

CHECK LIST

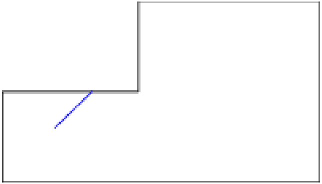
(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **Computer Presentations**: Obtain a copy of their presentation
  - **Overhead**: Obtain a copy or use the originals and scan them
  - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

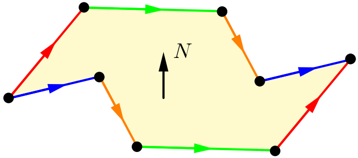
Volumes of principal strata of quadratic differentials and intersection numbers

Elise Goujard – University of Bordeaux

MSRI, Aug. 2019
Connections for Women:
Holomorphic Differentials in Mathematics and Physics



Translation surfaces



Flat metric
Conical angles $(d+1) \cdot 2\pi$

$\updownarrow$

Riemann surface
with a holomorphic 1-form
(Abelian differential)
zeros of degree d

Gauss-Bonnet / Euler-Poincaré:

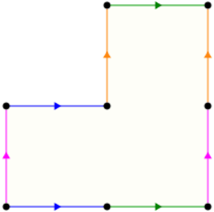
$$\sum_i d_i = 2g - 2$$

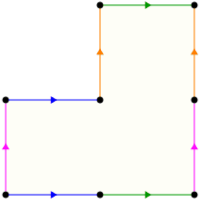
Moduli space of translation surfaces

$$\mathcal{H}_g = \{\text{translation surfaces of genus } g\}/\text{cut and paste} = \bigsqcup_{\underline{d} \vdash 2g-2} \mathcal{H}(\underline{d})$$

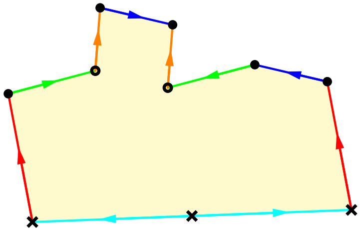
$$\begin{aligned} \mathcal{H}(\underline{d}) &= \mathcal{H}(d_1, d_2, \ldots, d_n) \\ &= \{\text{surfaces in } \mathcal{H}_g \text{ with conical angles } (d_i + 1)2\pi\} \end{aligned}$$

The sides of the polygon representing S, viewed as complex numbers (relative periods of the corresponding holomorphic 1-form) provide local coordinates for the stratum $\mathcal{H}(d)$ around S.





Half-translation surfaces



Flat metric
Conical angles $(k+2)\cdot\pi$

$\updownarrow$

Riemann surface
with a quadratic differential
(at most simple poles)
singularities of order $k \geq -1$

Similarly *n*-differentials on Riemann surfaces produce flat surfaces with conical singularities of angles multiples of $\frac{2\pi}{n}$.

Strata (dimension $2g+n-2$):

$\mathcal{Q}(\underline{k}) = \{$half-trans. surf. with conical angles $(k_i+2)\pi\}/$cut and paste

The moduli spaces

$$\mathcal{Q}_g = \bigsqcup_{\underline{k}\vdash 4g-4} \mathcal{Q}(\underline{k}), \qquad \mathcal{Q}_{g,p} = \bigsqcup_{\underline{k}\vdash 4g-4+p} \mathcal{Q}(\underline{k}, -1^p)$$

identify with the cotangent bundle to $\mathcal{M}_g$ (resp. $\mathcal{M}_{g,p}$)

The anti-invariant part of the relative homology of the orientation double cover provides local coordinates. The Lebesgue measure on these local coordinates give a well defined global $SL(2,\mathbb{R})$-invariant measure on the stratum.
Hypersurface:

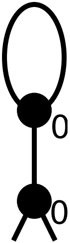
$$\mathcal{Q}_1(\underline{k}) = \{\text{surfaces in } \mathcal{Q}(\underline{k}) \text{ with area } 1\}$$

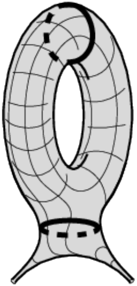
The induced measure on the hypersurface is finite (Masur-Veech measure).

A formula for the volume of principal strata

Stable graphs:

Decorated graphs with legs encoding e.g. the topological type of a simple closed multicurve on a topological surface with punctures.





Let $\mathcal{G}_{g,n}$ be the set of all stable graphs corresponding to a surface of genus g with n punctures ($g =$sum of the vertex markings + first Betti number of the graph, $n =$ number of legs).

A formula for the volume of principal strata

Define the Kontsevich polynomials:

$$N_{g,n}(b_1,\ldots,b_n) = \sum_{\underline{d}\vdash 3g-3+n} \frac{1}{2^{5g-6+2n}\underline{d}!}\langle\psi_1^{d_1}\ldots\psi_n^{d_n}\rangle b_1^{2d_1}\ldots b_n^{2d_n}.$$

For a stable graph Γ in $\mathcal{G}_{g,n}$ define

$$P_\Gamma(\underline{b}) = c_{g,n}\frac{1}{2^{|V(\Gamma)|-1}}\cdot\frac{1}{|\mathrm{Aut}(\Gamma)|}\cdot\prod_{e\in E(\Gamma)} b_e\cdot\prod_{v\in V(\Gamma)} N_{g_v,n_v}(\underline{b}_v).$$

Define the linear operator on polynomials by

$$\mathcal{Z}:\prod_{i=1}^{k} b_i^{m_i}\mapsto\prod_{i=1}^{k}\big(m_i!\cdot\zeta(m_i+1)\big).$$

A formula for the volume of principal strata

Theorem (Delecroix-G-Zograf-Zorich)

$$\mathrm{Vol}_{MV}(\mathcal{Q}_{g,n}) = \mathrm{Vol}_{MV}(\mathcal{Q}(1^{4g-4+n}, -1^n)) = \sum_{\Gamma \in \mathcal{G}_{g,n}} \mathcal{Z}(P_\Gamma).$$

A similar formula holds for Siegel-Veech constants.

Corollary

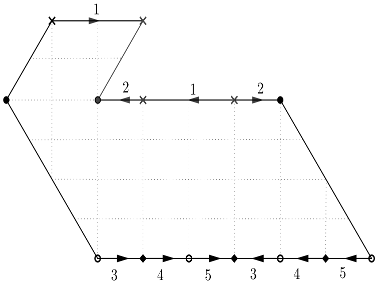
$$\mathrm{Vol}\,\mathcal{Q}_g \geq \sqrt{\frac{2}{3\pi g}} \cdot \left(\frac{8}{3}\right)^{4g-4} \cdot \left(1 + O\left(\frac{1}{g}\right)\right) \text{ as } g \to \infty.$$

Ideas of the proof

1. Evaluate volumes by counting integer points.

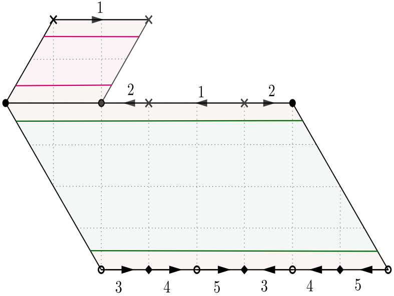
$$\mathrm{Vol}_{MV}(\mathcal{Q}(\underline{k})) = \mu_{MV}\big(\mathcal{Q}_{\leq 1}(\underline{k})\big) = \mu_{MV}\big(\{\text{Surfaces in } \mathcal{Q}(\underline{k}) \text{ of area } \leq 1\}\big)$$
$$= \lim_{N\to\infty} \frac{c}{N^{\dim(\mathcal{Q}(\underline{k}))}} \mathrm{Card}\{\text{Integer points in } \mathcal{Q}_{\leq N}(\underline{k})\}$$

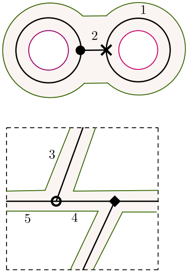
2. Integer points are square-tiled surfaces

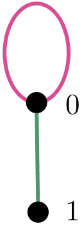


Ideas of the proof

3. Square-tiled surfaces decompose into cylinders.







This decomposition is encoded by stable graphs (vertices: ribbon graphs corresponding to cylinder boundaries, labeling: genus of the ribbon graph, edges: cylinders).

Relation to Mirzakhani's work

Theorem (Mirzakhani)

For any rational multicurve $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$ and any hyperbolic surface $X \in \mathcal{T}_{g,n}$, the number of simple closed geodesic multicurves on X of length at most L of the same topological type as γ is

$$s_x(L,\gamma) \sim B(x) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to \infty$$

where $B(x) = \mu_{Th}(\{\gamma \in \mathcal{ML}_{g,n} \mid l_X(\gamma) \leq 1\}$ is the Thurston measure of the unit ball, and $b_{g,n} = \int_{\mathcal{M}_{g,n}} B(X)dX = \sum_{[\gamma]\in\mathcal{O}} c(\gamma)$.

Theorem

For any $\gamma \in \mathcal{ML}_{g,n}(\mathbb{Z})$, the volume contribution of the associated stable graph is

$$\mathrm{Vol}_{MV}(\Gamma(\gamma)) = const_{g,n} \cdot c(\gamma)$$

so in particular $\mathrm{Vol}_{MV}\, \mathcal{Q}_{g,n} = const_{g,n} \cdot b_{g,n}$.

Asymptotic questions as $g \to \infty$

We consider the case $n = 0$.

- random one-cylinder surfaces / simple closed geodesics

Theorem

$$\frac{c(\gamma_{sep})}{c(\gamma_{nonsep})} \sim \sqrt{\frac{2}{3\pi g}} \cdot \frac{1}{4^g}$$

- random square-tiled surfaces / multicurves

Conjecture

The probability that a random $\gamma \in \mathcal{ML}_g(\mathbb{Z})$ does not separate the underlying topological surface tends to 1 when $g \to \infty$. It is also the probability that all conical points of a random square-tiled surface in $\mathcal{Q}_g$ belong to the same horizontal layer.

Other conjectures on the number of cylinders of random square-tiled surfaces / primitive components of a random multicurve...

Goudard Board Notes

$\dim = 2g + n - 1$

$6\pi \to d = 2$

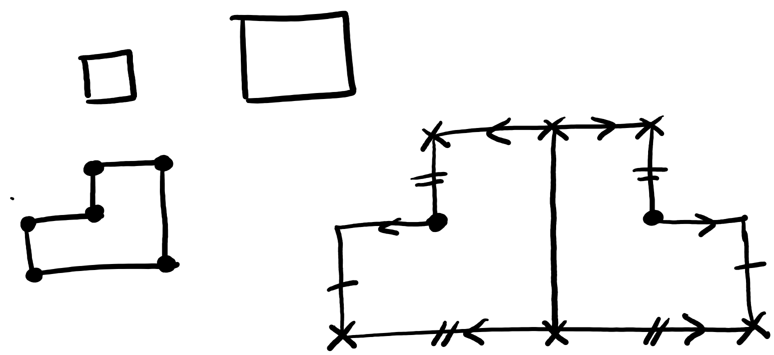
$g = 2 \to \mathcal{H}(2)$

$\pi, \pi, 3\pi, 3\pi$

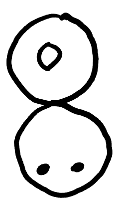
$(k+2)\pi \leftrightarrow k$

$\mathcal{Q}(-1,-1,1,1) \subset \mathcal{Q}_{1,2}$

principal strata



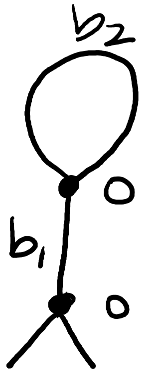
$\mathcal{Q}(1,-1)^5$



$\overline{\Omega}_{1,2}$

$\mathcal{G}_{g,n}$: stable graphs genus g, n legs

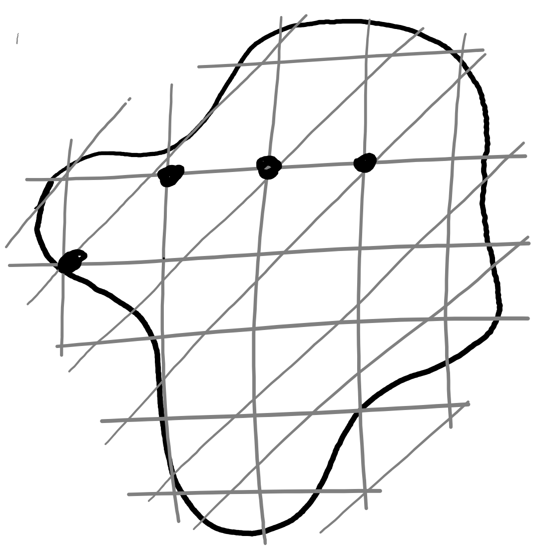
$$\langle \psi_1^{d_1}, \dots, \psi_n^{d_n} \rangle = \int_{\overline{\Omega}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$$



$$\frac{1}{2} \cdot \frac{1}{2} b_1 b_2 N_{0,3}(b_1, b_2, b_2) N_{0,3}(b_1, 0, 0)$$

$$b_1^{m_1} \cdots b_n^{m_n} \longmapsto \prod (m_i)! \cdot \zeta(m_i + 1)$$

$$\frac{1}{4} b_1 b_2 \overset{Z}{\longmapsto} \zeta(2)^2$$



$$N_{g,n}(b_1 \dots b_n) = \# \left\{ \begin{matrix} \text{integer metrics} \\ \text{on } R_{g,n}(b_1, \dots, b_n) \end{matrix} \right\}$$

$$\frac{\log(\dim)}{2} + c$$