MSRI LECTURES ON PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. Rough notes for lectures at the MSRI introductory workshop in Fall 2019.

A large part of these notes is a shortened version of lecture notes by Richard Melrose, available at `http://math.mit.edu/~rbm/iml90c2.ps`.

1. PROLOGUE: WHY STUDY PSEUDODIFFERENTIAL OPERATORS?

In these two lectures we will mostly study pseudodifferential operators on $\mathbb{R}^n$. They do also work beautifully well on manifolds.

A very basic example of a pseudodifferential operator is the L^2 inverse to the shifted Laplacian

$$P = \Delta + 1, \quad \Delta := -\sum_{j=1}^{n} \partial_{x_j}^2.$$

This inverse is a Fourier multiplier (here $\mathscr{S}$ denotes Schwartz functions):

$$P^{-1} = Q : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n), \quad \widehat{Qu}(\xi) = \frac{1}{1+|\xi|^2}\hat{u}(\xi).$$

Now, imagine that we instead have a *variable coefficient* operator, e.g.

$$-\sum_{j,k} p_{jk}(x)\partial_{x_j}\partial_{x_k},$$

where p_{jk} is a positive definite matrix depending on x. What would the inverse be? At the end of this lectures we construct an approximate inverse which is a *pseudo-differential operator*. This was the original motivation for studying pseudodifferential operators in the theory of PDE.

Pseudodifferential operators are a general class of operators which include differential operators, Fourier multipliers like Q above, approximate inverses to elliptic differential operators, and a lot more.

2. Special quantization formula

To get the formula for a general pseudodifferential operator, we first look at a differential operator of order m

$$A = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha, \quad a_\alpha \in C^\infty(\mathbb{R}^n),$$

where we henceforth adopt the notation

$$D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \quad D_{x_j} = \frac{1}{i} \partial_{x_j}.$$

Take $u \in \mathscr{S}(\mathbb{R}^n)$ and write by the Fourier inversion formula

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi.$$

Now let's differentiate under the integral sign to obtain

$$D_x^\alpha u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \xi^\alpha \hat{u}(\xi)\, d\xi.$$

From here we get

$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi) \hat{u}(\xi)\, d\xi \tag{1}$$

where $a(x,\xi)$ is the *symbol* of the operator:

$$a(x,\xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha. \tag{2}$$

To obtain a pseudodifferential operator, we simply take a more general function $a(x,\xi)$ in (1), one which is not necessarily a polynomial in the ξ variables. The corresponding operator A is called the *quantization* of a, and we write

$$A = \mathrm{Op}_0(a).$$

Here Op_0 stands for quantization for y-independent symbols (as opposed to general symbols defined later). Note that the Fourier multiplier Q defined above can now be written as

$$Q = \mathrm{Op}_0\Big(\frac{1}{1+|\xi|^2}\Big).$$

To get good properties of $\mathrm{Op}_0(a)$ we need to make certain assumption on the behavior of a at infinity. This brings us to symbol classes.

3. Classical Kohn–Nirenberg symbols

We use the notation

$$\langle\xi\rangle := \sqrt{1+|\xi|^2}.$$

This is asymptotic to $|\xi|$ as $\xi \to \infty$ and is also smooth at $\xi = 0$.

Definition 3.1. *Let $m \in \mathbb{R}$. We say $a(z,\xi) \in C^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ lies in the symbol class $S^m(\mathbb{R}^p;\mathbb{R}^n)$, if for all multiindices α, β*

$$|\partial_z^\alpha \partial_\xi^\beta a(z,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|}$$

Here we allow $p \neq n$ for future use, but in the original formula (1) we have $p = n$.

The derivative bounds above mean the following: $a = \mathcal{O}(\langle\xi\rangle^m)$, differentiation in z does not change the growth of a, but differentiation in ξ gives decay by a power of ξ. As an exercise, you can check that:

- a polynomial of the form (2) lies in $S^m(\mathbb{R}^n;\mathbb{R}^n)$ if we assume that all derivatives of a_α lie in L^∞, and
- the symbol of the operator Q above, $a(x,\xi) = \langle\xi\rangle^{-2}$, lies in $S^{-2}(\mathbb{R}^n;\mathbb{R}^n)$.

4. General quantization formula and the power of integration by parts

We now return to the formula (1). Let $a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$. Recalling the definition of the Fourier transform, we rewrite (1) as

$$Au(x) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}a(x,\xi)u(y)\,dyd\xi, \quad u \in \mathscr{S}(\mathbb{R}^n).$$

(This only makes sense for $m < -n$, more on that later.)

We arrive to the general quantization formula by allowing a to also depend on y. (This will be useful in deriving properties of quantization.) Namely, for $a \in S^m(\mathbb{R}^{2n};\mathbb{R}^n)$ we define

$$\mathrm{Op}(a)u(x) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)\,dyd\xi. \tag{3}$$

For $m < -n$, the integral in (3) converges and we get

$$a \in S^m(\mathbb{R}^{2n};\mathbb{R}^n),\ m < -n \quad\Longrightarrow\quad \mathrm{Op}(a) : \mathscr{S}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n). \tag{4}$$

We now make sense of the oscillatory integral (3) for general a (in particular, for a which is a polynomial of the form (2)), by integrating by parts in y. Let's just do it one time. We write

$$\begin{aligned}\mathrm{Op}(a)u(x) &= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} \left(e^{i(x-y)\cdot\xi}\langle\xi\rangle^2\right)\left(\langle\xi\rangle^{-2}a(x,y,\xi)u(y)\right)dyd\xi\\ &= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} \left((1-\xi\cdot D_y)e^{i(x-y)\cdot\xi}\right)\left(\langle\xi\rangle^{-2}a(x,y,\xi)u(y)\right)dyd\xi\\ &= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}(1+\xi\cdot D_y)\left(\langle\xi\rangle^{-2}a(x,y,\xi)u(y)\right)dyd\xi.\end{aligned} \tag{5}$$

The integration by parts in the third line above does make sense for $a\in S^m$ and $m<-n$. However, in the last line we compute

$$(1+\xi\cdot D_y)\left(\langle\xi\rangle^{-2}a(x,y,\xi)u(y)\right) = \mathcal{O}(\langle\xi\rangle^{m-1}\langle y\rangle^{-n-1})$$

so the last integral actually converges when $m<1-n$, which is better than the original definition of quantization! We can now integrate by parts repeatedly to arrive to

$$a\in S^m(\mathbb{R}^{2n};\mathbb{R}^n),\ \text{any } m \quad\Longrightarrow\quad \mathrm{Op}(a):\mathscr{S}(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n). \tag{6}$$

Some caution is needed here: what we really mean is that the linear operation $\mathrm{Op}(a)$ is extended to the symbol class S^m by continuity from, say, S^{-n-1}. Such an extension is necessarily unique, and we can prove identities for $a\in S^m$ by just proving them for rapidly decaying a and arguing by approximation. (There are some subtleties here regarding"approximating by nice symbols").

We can upgrade (6) further as follows:

$$a\in S^m(\mathbb{R}^{2n};\mathbb{R}^n),\ \text{any } m \quad\Longrightarrow\quad \mathrm{Op}(a):\mathscr{S}(\mathbb{R}^n)\to \mathscr{S}(\mathbb{R}^n). \tag{7}$$

For that we need to apply the operators x_j, D_{x_j} to $\mathrm{Op}(a)$ and use the identities

$$\begin{aligned}x_j\,\mathrm{Op}(a) &= \mathrm{Op}(a)x_j - \mathrm{Op}(D_{\xi_j}a),\\ D_{x_j}\,\mathrm{Op}(a) &= \mathrm{Op}(a)D_{x_j} + \mathrm{Op}(D_{x_j}a + D_{y_j}a)\end{aligned}$$

the first of which is proved by integrating by parts in ξ_j and the second one, by integrating by parts in y_j. We show the first one in a bit more detail since it will be used again later:

$$\begin{aligned}(x_j\,\mathrm{Op}(a)-\mathrm{Op}(a)x_j)u(x) &= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}(x_j-y_j)a(x,y,\xi)u(y)\,dyd\xi\\ &= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} \left(D_{\xi_j}e^{i(x-y)\cdot\xi}\right)a(x,y,\xi)u(y)\,dyd\xi\\ &= -(2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}D_{\xi_j}a(x,y,\xi)u(y)\,dyd\xi.\end{aligned} \tag{8}$$

Since $D_{\xi_j}a, D_{x_j}a, D_{y_j}a$ still lie in S^m, we see that $x_j\,\mathrm{Op}(a), D_{x_j}\,\mathrm{Op}(a):\mathscr{S}(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)$, and iteration gives $x^\alpha D_x^\beta\,\mathrm{Op}(a):\mathscr{S}(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)$, which implies (7).

The above discussion leads to the following statement:

Proposition 4.1. *Assume that* $a \in S^m(\mathbb{R}^{2n};\mathbb{R}^n)$*. Then we may define*

$$\operatorname{Op}(a) : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n), \quad \operatorname{Op}(a) : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n).$$

Here the additional powers of x, y are handled similarly to the previous argument. To get the mapping property on tempered distributions, we use a definition by duality:

$$\langle \operatorname{Op}(a)u, \varphi\rangle_{L^2} := \langle u, \operatorname{Op}(a)^*\varphi\rangle_{L^2}, \quad u \in \mathscr{S}'(\mathbb{R}^n), \quad \varphi \in \mathscr{S}(\mathbb{R}^n)$$

where the adjoint $\operatorname{Op}(a)^*$ has the form (3) with a different symbol (see below) and thus maps $\mathscr{S}(\mathbb{R}^n)$ to itself.

We call the resulting class of operators $\operatorname{Op}(a)$ *pseudodifferential operators.* In particular, we denote by

$$\Psi^m(\mathbb{R}^n)$$

operators of the form $\operatorname{Op}(a)$ where $a \in S^m(\mathbb{R}^{2n};\mathbb{R}^n)$.

Define the residual operator class

$$\Psi^{-\infty}(\mathbb{R}^n) := \bigcap_{m\in\mathbb{R}} \Psi^m(\mathbb{R}^n).$$

One can show (with a bit of annoying technical work) that every element of $\Psi^{-\infty}$ has the form $\operatorname{Op}(a)$ where a lies in the residual symbol class

$$S^{-\infty}(\mathbb{R}^{2n};\mathbb{R}^n) = \bigcap_m S^m(\mathbb{R}^{2n};\mathbb{R}^n).$$

Note that $a \in S^{-\infty}$ simply means that each derivative of a decays like $\mathcal{O}(\langle\xi\rangle^{-\infty})$. Then the integral kernel of (3) converges with all the x, y derivatives, implying that every $A \in \Psi^{-\infty}$ is a *smoothing operator*:

$$Au(x) = \int K(x,y)u(y)\,dy \quad \text{where} \quad K(x,y) \in C^\infty(\mathbb{R}^{2n}).$$

In particular we have the mapping property $A : \mathscr{S}' \to C^\infty(\mathbb{R}^n)$.

5. Reduction to y-independent symbols

We now show that the general quantization procedure Op from (3) actually gives the same class of operators as the special quantization procedure Op_0 from (1), and get a useful asymptotic expansion:

Theorem 1. *Assume that* $a \in S^m(\mathbb{R}^{2n};\mathbb{R}^n)$*. Then there exists* $\tilde a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ *such that*

$$\operatorname{Op}(a) = \operatorname{Op}_0(\tilde a).$$

Moreover, we have the asymptotic expansion

$$\tilde a(x,\xi) \sim \sum_{k=0}^\infty \tilde a_k, \quad \tilde a_k := \frac{1}{k!}(-i\partial_y\cdot\partial_\xi)^k a(x,y,\xi)|_{y=x}, \quad \partial_y\cdot\partial_\xi := \sum_{j=1}^n \partial_{y_j}\partial_{\xi_j} \qquad (9)$$

in the following sense: for each N,

$$\tilde{a}(x,\xi) - \sum_{k=0}^{N-1} \cdots \in S^{m-N}(\mathbb{R}^n;\mathbb{R}^n).$$

Note here that the expansion does make sense: the k-th term in the expansion is in S^{m-k} due to the fact that Kohn–Nirenberg symbols improve by a power of ξ when differentiated in ξ. (This is the first time we use this fact, actually.) Note also that the above is an asymptotic expansion, not a convergent series! Asymptotic expansions like the one above are very common in microlocal analysis.

We will not prove Theorem 1 (see Melrose's notes for a proof). We instead prove a simpler statement (from which the full theorem follows, but after a good amount of annoying technical work):

Proposition 5.1. *For each N we may write*

$$\mathrm{Op}(a) = \mathrm{Op}_0\Big(\sum_{k=0}^{N-1} \tilde{a}_k\Big) + \mathrm{Op}(r_N), \quad r_N \in S^{m-N}(\mathbb{R}^{2n};\mathbb{R}^n)$$

where $\tilde{a}_k$ are the terms in the expansion (9)*.*

Proof. We just show the cases $N=1, N=2$, with higher N obtained similarly. We first do $N=1$. The symbol

$$a(x,y,\xi) - \tilde{a}_0(x,\xi) = a(x,y,\xi) - a(x,x,\xi)$$

vanishes on the partial diagonal $\{x=y\}$. We can then write

$$\begin{aligned} a(x,y,\xi) - \tilde{a}_0(x,\xi) &= \int_0^1 \partial_t\big(a(x,x+t(y-x),\xi)\big)\,dt = \sum_{j=1}^{n}(y_j - x_j)b_j(x,y,\xi), \\ b_j(x,y,\xi) &= \int_0^1 (\partial_{y_j}a)(x,x+t(y-x),\xi)\,dt. \end{aligned} \tag{10}$$

From the definition of b_j we see that $b_j \in S^m(\mathbb{R}^{2n};\mathbb{R}^n)$. We now use the key identity proved by (8):

$$b \in S^m(\mathbb{R}^{2n};\mathbb{R}^n) \quad\Longrightarrow\quad \mathrm{Op}((y_j-x_j)b) = \mathrm{Op}(D_{\xi_j}b). \tag{11}$$

We get then

$$\mathrm{Op}(a) - \mathrm{Op}_0(\tilde{a}_0) = \mathrm{Op}(r_1), \quad r_1(x,y,\xi) := \sum_{j=1}^{n} D_{\xi_j}b_j(x,y,\xi)$$

and r_1 does lie in $S^{m-1}(\mathbb{R}^{2n};\mathbb{R}^n)$ owing to the differentiation in ξ.

To do $N=2$, we iterate this process further, applying it now to the symbol r_1. We see that the next term in the expansion should be the restriction of r_1 to $\{x=y\}$;

indeed, $r_1(x,y,\xi) - r_1(x,x,\xi)$ can be again written in the form (10). It is easy to compute that

$$r_1(x,x,\xi) = -i(\partial_y \cdot \partial_\xi)a(x,y,\xi)|_{y=x} = \tilde{a}_1(x,\xi).$$

So we get

$$\mathrm{Op}(a) - \mathrm{Op}_0(\tilde{a}_0 + \tilde{a}_1) = \mathrm{Op}(r_2), \quad r_2 \in S^{m-2}(\mathbb{R}^{2n};\mathbb{R}^n).$$

□

For $A = \mathrm{Op}(a) \in \Psi^m(\mathbb{R}^n)$, where $a \in S^m(\mathbb{R}^{2n};\mathbb{R}^n)$, we define the *principal symbol* $\sigma^m(A)$ as follows:

$$\sigma^m(A) = [a(x,x,\xi)] \in \frac{S^m(\mathbb{R}^n;\mathbb{R}^n)}{S^{m-1}(\mathbb{R}^n;\mathbb{R}^n)}. \tag{12}$$

The principal symbol will have nice algebraic properties as we will see soon. What we see immediately from Proposition 5.1 is the following statement: if $A \in \Psi^m(\mathbb{R}^n)$, then

$$\sigma^m(A) = 0 \iff A \in \Psi^{m-1}(\mathbb{R}^n).$$

So the principal symbol does determine A modulo a lower order term. One often suppresses the order of the operator in the notation, writing σ instead of σ^m.

6. Adjoints

We now discuss algebraic properties of the classes $\Psi^m(\mathbb{R}^n)$. One algebraic property that we can do easily is closure under adjoints:

Theorem 2. *Assume that $A \in \Psi^m(\mathbb{R}^n)$. Then $A^* \in \Psi^m(\mathbb{R}^n)$ and $\sigma(A^*) = \overline{\sigma(A)}$. Here the adjoint is understood in the following sense:*

$$\langle Au, v\rangle_{L^2} = \langle u, A^*v\rangle_{L^2} \quad \text{for all} \quad u,v \in \mathscr{S}(\mathbb{R}^n).$$

Proof. Let $A = \mathrm{Op}(a)$ where $a \in S^m(\mathbb{R}^{2n};\mathbb{R}^n)$. Then we have the following representation of the adjoint:

$$\mathrm{Op}(a)^*v(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}\overline{a(y,x,\xi)}u(y)\,dyd\xi.$$

From here the result follows immediately since $\mathrm{Op}(a)^* = \mathrm{Op}(a^*)$ where $a^*(x,y,\xi) = \overline{a(y,x,\xi)}$. □