Introduction to Fourier Integral Operators - Lecture 1

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Introductory Workshop

MSRI Microlocal Analysis Program

Overview of Lecture 1

- Examples of Fourier integral operators (FIOs)
- Wavefront (WF) sets and the Hörmander-Sato Lemma
- Conormal distributions
- Oscillatory integrals as distributions and their WF sets

Note: These lectures treat **homogeneous** microlocal analysis, FIOs. Give references at end, including to semi-classical FIOs.

Examples of FIOs: ΨDOs and pull-backs

1. **Pseudodifferential operators**

$$T_1 f(x) := \int\int e^{i(x-y)\cdot\theta} a(x,y,\theta) f(y)\, d\theta\, dy,$$

with $a \in S^m_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$.

2. **Pull-back**: composition with a diffeomorphism

Let X, Y be open subsets of $\mathbb{R}^n$, $\chi : X \to Y$ a C^∞ diffeom.

$T_2(f)(x) = \chi^*(f)(x) := f(\chi(x)) = \int\int e^{i(\chi(x)-y)\cdot\theta}\, 1(\theta)\, f(y)\, d\theta\, dy$

Examples of FIOs: Radon transform

3. **Radon transform $\mathcal{R}$:** hyperplane integrals of a function on $\mathbb{R}^n$

Space $M_{n-1,n}$ of hyperplanes in $\mathbb{R}^n \simeq \mathbb{S}^{n-1} \times \mathbb{R}$,

$$\mathbb{S}^{n-1} \times \mathbb{R} \ni (\omega, s) \leftrightarrow \{y \in \mathbb{R}^n : y \cdot \omega = s\} \in M_{n-1,n}$$

$$T_3 f(\omega, s) = \mathcal{R}f(\omega, s) := \int_{\{y\cdot\omega=s\}} f(y)\, d\sigma(y)$$

$$= c_n \int\int e^{i(y\cdot\omega - s)\theta}\, 1(\theta)\, f(y)\, d\theta\, dy$$

Examples of FIOs: Spherical mean operator

4. **Spherical mean operator:** Let $d\sigma$ = surface measure on $\mathbb{S}^{n-1}$.

   For $t > 0$, define convolution operator $\mathcal{A}_t : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$.

$$T_4(f)(x) = \mathcal{A}_t f(x) := f * d\sigma = \int_{\mathbb{S}^{n-1}} f(x - t\omega)\, d\sigma(\omega)$$

$$= c_{t,n} \int \int e^{i(|x-y|-t)\theta}\, 1(\theta)\, f(y)\, d\theta\, dy$$

   Solution operator for Cauchy problem for the wave equation on $\mathbb{R}^{n+1}$ can be expressed in terms of $\mathcal{A}_t$ and its derivatives in t.

Examples of FIOs: Melrose-Taylor transform

❺ **Melrose-Taylor transform** $\mathcal{R}_{MT}$

Let $\Omega \subset \mathbb{R}^n$ with C^∞, strictly convex boundary $\partial\Omega$

$$T_5(f)(\omega, t) = \mathcal{R}_{MT}(f)(\omega, t) := \int\int_{\{y\cdot\omega = t-s\}\subset \partial\Omega\times\mathbb{R}} f(y, s)$$

$$= \int\int\int e^{i(t-s-y\cdot\omega)\theta}\, 1(\theta)\, f(y, s)\, d\theta\, ds\, dy$$

$T_5 : \mathcal{D}'(\partial\Omega \times \mathbb{R}) \to \mathcal{D}'(\mathbb{S}^{n-1} \times \mathbb{R})$.

Examples of FIOs: Half-wave operator

6. **Half-wave operator** e^{itP}

$P(x,D) \in \Psi^1_{cl}(X)$ elliptic, self-adjoint, 1^{st} order, w/ prin symb $p(x,\xi)$

Ex. Let $P = (-\Delta_g)^{\frac{1}{2}}$ on Riem. (M,g), $p(x,\xi) = |\xi|_g$. For $t \in \mathbb{R}$, w.r.t local coordinates,

$$e^{itP(x,D)} f(x) := \int\int_{X\times\mathbb{R}^n} e^{i\left[\phi(x,y,\theta)+tp(y,\theta)\right]} a(t,x,y,\theta)\, f(y)\, d\theta\, dy$$

with $a \in S^0_{1,0}\big((\mathbb{R}\times X\times X)\times\mathbb{R}^n\big)$

Examples of FIOs: Canonical operators

7. $S(x,\eta)$ $\mathbb{R}$-valued, smooth on $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$, homog deg 1 in η

Assume $\det\left(d^2_{x\eta} S(x,\eta)\right) \neq 0$ and $a \in S^0_{1,0}$. Define

$$
\begin{aligned}
T_7(f) &:= \int e^{iS(x,\eta)}\, a(x,\eta)\, \hat{f}(\eta)\, d\eta \\
&= \int \int e^{i\left(S(x,\eta) - y\cdot\eta\right)}\, a(x,\eta)\, f(y)\, dy\, d\eta
\end{aligned}
$$

Early version of FIOs used by Maslov, Egorov ...

Two symbol classes

- **Def. 1.** For $m \in \mathbb{R}$, $N \in \mathbb{N}$, and X a manifold, the space of (Hörmander) **type** $(1,0)$ symbols is

$$S^m_{1,0}(X \times \mathbb{R}^N) = \Big\{ a(x,\theta) \in C^\infty(X \times \mathbb{R}^N) : \forall \alpha,\ \beta \text{ and } K \subset\subset X,$$

$$|\partial_x^\beta \partial_\theta^\alpha a(x,\theta)| \leq C_{\alpha\beta K} < \theta >^{m-|\alpha|},\ x \in K \Big\}$$

- **Def. 2.** The space of **classical** symbols of order $m \in \mathbb{R}$ on $X \times \mathbb{R}^N$ is

$$S^m_{cl}(X \times \mathbb{R}^N) = \Big\{ a(x,\theta) : a \sim \sum_{j=0}^{\infty} a_j(x,\theta),\ a_j \text{ homog deg } m-j \text{ in } \theta \Big\}$$

Wavefront set: $WF(u)$

- **Cotangent space** $T^*\mathbb{R}^n = \{(x,\xi) : x \in \mathbb{R}^n,\ \xi \in (T_x\mathbb{R}^n)^*\}$, has **zero section**, $\mathbf{0} = \{(x,\xi) : \xi = 0\}$.

- $\Sigma \subset T^*\mathbb{R}^n \setminus \mathbf{0}$ is **conic** if $(x,\xi) \in \Sigma \implies (x,t\xi) \in \Sigma,\ \forall t > 0$

- **Def.** Let $u \in \mathcal{D}'(\mathbb{R}^n)$. We say that $(x_0,\xi_0) \in T^*\mathbb{R}^n \setminus \mathbf{0}$ is **not** in $WF(u)$ if $\exists\ \phi(x) \in C_0^\infty$, $\phi(x_0) \neq 0$, and a conic neighborhood $\Gamma \subset \mathbf{R}^n$ *of* ξ_0 such that

$$|\widehat{\phi u(\xi)}| \lesssim (1+|\xi|)^{-N},\ \forall \xi \in \Gamma.$$

- $WF(u)$ is a **closed, conic** set $\subset T^*\mathbb{R}^n \setminus \mathbf{0}$

- **Ex:** $WF(\delta) = \{(0,\xi), \xi \neq 0\} = T_0^*\mathbb{R}^n \setminus 0$

Wavefront set: $WF(u)$

- If π = the projection from $T^*\mathbf{R}^n$ to $\mathbf{R}^n$, then

$$\pi\left(WF\left(u\right)\right) = \text{ singular support of } u$$

- If $P(x, D)$ is a ΨDO, then P is **pseudolocal:**

$$\text{sing supp}(Pu) \subseteq \text{sing supp}(u))$$

  and even **microlocal:**

$$WF(Pu) \subseteq WF(u)$$

- All of this extends to $WF(u)$ on general C^∞ manifolds.
- **Q.** What do other operators do to WF of functions they act on?

Hörmander-Sato Lemma

- Let X, Y be manifolds, $T : \mathcal{E}'(Y) \to \mathcal{D}'(X)$, with **Schwartz kernel** $K \in D'(X \times Y)$: formally, $Tf(x) = \int K(x,y)\, f(y)\, dy$

- The **wavefront relation** of T is

$$WF_T := \{(x,\xi;y,\eta) : (x,y,\xi,-\eta) \in WF(K)\} \subset T^*X \times T^*Y$$

- **Thm.** Suppose $WF(K) \subseteq \{(x,y,\xi,\eta) : \xi \neq 0 \text{ and } \eta \neq 0\}$, so that $WF_T \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$. Then

$WF(Tu) \subseteq WF_T \circ WF(u) :=$

$$\big\{(x,\xi) : \exists (y,\eta) \text{ s.t.} (x,\xi;y,\eta) \in WF_T \text{ and } (y,\eta) \in WF(u)\big\}.$$

Hörmander-Sato Lemma: composition

- Let X, Y Z be manifolds, $T_1 : \mathcal{E}'(Y) \to \mathcal{E}'(X)$, $T_2 : \mathcal{E}'(Z) \to \mathcal{E}'(Y)$, with Schwartz kernels $K_1 \in D'(X \times Y)$, $K_2 \in D'(Y \times Z)$

  Kernel of $T_1 \circ T_2$ is $K_1 \circ K_2 \in \mathcal{D}'(X \times Z)$

- If $WF_{T_1} \subseteq (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$, $WF_{T_2} \subseteq (T^*Y \setminus \mathbf{0}) \times (T^*Z \setminus \mathbf{0})$,

  then $WF_{T_1 \circ T_2} \subseteq WF_{T_1} \circ WF_{T_2} :=$

$$\Big\{(x, \xi, z, \zeta) : \exists (y, \eta) \in T^*Y \text{ s.t. } (x, \xi, y, \eta) \in WF_{T_1} \text{ and } (y, \eta, z, \zeta) \in WF_{T_2}\Big\}$$

Conormal distributions: conormal bundles

If $Y^{n-k} \subset X^n$, the **conormal bundle** of Y is

$$N^*Y := \{(x,\xi) \in T^*X : x \in Y, \xi|_{T_xY} = 0\} \subset T^*X$$

- **Ex 1:** In $X = \mathbb{R}^n$, write $x = (x', x'')$ with $x' \in \mathbb{R}^k$, $x'' \in \mathbb{R}^{n-k}$

  If $Y = \{(x', x'') : x' = 0\} \simeq \mathbb{R}^{n-k} \hookrightarrow \mathbb{R}^n$,

  $$N^*Y = \{(x', x'', \xi', \xi'') : x' = 0, \xi'' = 0\}$$

  Special case: $Y = \{0\}$, $N^*Y = \{(0,\xi)\} = T_0^*\mathbb{R}^n$

- **Ex 2:** $Y = \{x \in X : \phi_1(x) = \cdots = \phi_k(x) = 0\}$, $\{d\phi_j\}_{j=1}^k$ lin indep

  $$N^*Y = \{(x,\xi) \in T^*X : x \in Y, \xi = \Sigma_{j=1}^k \theta_j d\phi_j(x)\}$$

Conormal distributions: examples

1. $\mathbb{R}^1$: Dirac delta $\delta(x)$ is conormal for $Y = \{0\}$, as are the families x_+^z, x_-^z, $(x + i0)^z$, $(x - i0)^z$, which are are meromorphic, conormal-for-$\{0\}$-valued functions of z.

2. $\mathbb{R}^n$: Dirac delta is again conormal for $\{0\}$, as is

   Newtonian potential, $N(x) = c_n |x|^{2-n}$, $n \geq 3$.

3. $\phi : \mathbb{R}^n \to \mathbb{R}^k$ defining functions for $Y^{n-k} = \{x : \phi(x) = 0\} \subset \mathbb{R}^n$, $u(x)$ a conormal distn on $\mathbb{R}^k$ for $\{t = 0\}$, then $u(\phi) \in \mathcal{D}'(\mathbb{R}^n)$, is conormal for Y. Many examples in Gelfand-Shilov.

   Ex. Using $u = \delta(t) \implies$ a smooth meas μ on Y is conormal for Y.

   Ex. If (X, g) is Riemannian, $s > 0$, $dist(x, Y)^{-k+s}$ is (near Y) conormal for Y.

Conormal distributions: oscillatory integrals

Ex. On $\mathbb{R}^k$: $\quad \hat{\delta}(\theta) \equiv 1 \implies \delta(t) = (2\pi)^{-k} \int_{\mathbb{R}^k} e^{ix\cdot\theta}\, 1(\theta)\, d\theta$

- **Def. 1.** Let $Y \subset X$, $Y = \{x \in X : \phi_1(x) = \cdots = \phi_k(x) = 0\}$ with $\{d\phi_j\}$ lin. indep. Then $u \in D'(X)$ is **conormal** to Y of **order** $m \in \mathbb{R}$ if it can be written as

$$u(x) = \int_{\mathbb{R}^k} e^{i[\Sigma_{j=1}^k \theta_j \phi_j(x)]}\, a(x,\theta)\, d\theta$$

  with $a \in S^m_{1,0}(X \times \mathbb{R}^k)$. [Integral need not converge pointwise.]

- $I^m(X;Y) :=$ class of distributions on X **conormal to** Y of order m

  and $I(X;Y) = \bigcup_{m\in\mathbb{R}} I^m(X;Y)$.

  N.B. This choice of order is useful but disagrees with Hörmander convention for Fourier integral distributions.

Conormal distributions: iterated regularity

- **Prop./Def. 2.** $u \in I(X;Y)$ iff $\exists s_0 \in \mathbb{R}$ s.t. $u \in H^{s_0}_{loc}(X)$ and for every $N \in \mathbb{N}$, and all smooth vector fields $V_1, V_2, \dots, V_N$ on X which are tangent to Y, the **iterated regularity condition** holds:

$$V_1 V_2 \dots V_N u \in H^{s_0}_{loc}(X)$$

- **Ex. 1.** $Y = \{0\} \subset X = \mathbb{R}^n$. The set of vector fields tangent to 0 is a module over $C^\infty(\mathbb{R}^n)$ generated by $V_{jk} = x_j \frac{\partial}{\partial x_k}, 1 \leq j, k \leq n$. Calc:

$$x_j \frac{\partial}{\partial x_k}\Big(\int e^{ix\cdot\theta}\, a(x,\theta)\, d\theta\Big) = \int (\frac{1}{i}\partial_{\theta_j}(e^{ix\cdot\theta}))(i\theta_k a + \partial_{x_k} a)\, d\theta$$

$$= \int e^{ix\cdot\theta}\, b(x,\theta)\, d\theta$$

with $b \in S^m_{1,0}$, same order as $a(x,\theta)$.

Conormal distributions: wavefront set

- **Prop.** If $u \in I(X;Y)$, then $WF(u) \subseteq N^*Y \setminus \mathbf{0}$.

  Thus, sing supp$(u) \subseteq Y$.

- **N.B.** In general $WF(u) \subseteq N^*Y \setminus \mathbf{0} \;\not\Longrightarrow\; u$ is conormal for Y.

- **Ex.** If $\mu \in \mathcal{D}'(X)$ is a smooth density on Y, can write

$$\mu(x) = \int_{\mathbb{R}^k} e^{i[\Sigma_{j=1}^k \theta_j \phi_j(x)]}\, a(x,\theta)\, d\theta, \quad a \in S^0_{1,0}$$

$\Longrightarrow WF(\mu) \subseteq N^*Y \setminus \mathbf{0}$. Same for $dist(x,Y)^{-k+s}$.

Wave front relation of ΨDOs

- ΨDO: $T_1 f(x) = \int e^{i(x-y)\cdot\theta}\, a(x,y,\theta)\, f(y)\, d\theta dy$

  Schwartz kernel: $K_{T_1}(x,y) = \int e^{i(x-y)\cdot\theta} a(x,y,\theta) d\theta$

- WF$(K_{T_1}) \subseteq N^*\{x-y=0\} = \big\{(x,y,\theta,-\theta); x=y,\ \theta\neq 0\big\}$

  $\implies WF_{T_1} \subseteq \big\{(x,\xi;x,\xi) : (x,\xi)\in T^*\mathbb{R}^n\setminus\mathbf{0}\big\} =: \Delta_{T^*\mathbb{R}^n}$,

  the **diagonal** of $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$.

Wave front relation of pull-backs

- Pull-back/composition with a diffeomorphism: on $\mathbb{R}^n$,

  $T_2 f(x) = \int e^{i(\chi(x)-y)\cdot\theta} f(y) d\theta dy,$

  $K_{T_2}(x,y) = \int e^{i(\chi(x)-y)\cdot\theta} 1(\theta) d\theta$

- $\mathsf{WF}(K_{T_2}) = N^*\{\chi(x) - y = 0\} = \{(x, y, D\chi(x)^t\theta, -\theta); \chi(x) = y\}$

  $\implies WF_{T_2} = \left\{(x, \xi; \chi(x), [D\chi(x)^t]^{-1}(\xi)) : (x,\xi) \in T^*\mathbb{R}^n \setminus \mathbf{0}\right\}$

Wave front relation of Radon Transform

- $T_3 f(\omega, s) = \int e^{i(y\cdot\omega - s)\theta} 1(\theta) f(y) d\theta dy,\ K_{T_3} = \int e^{i(y\cdot\omega - s)\theta} 1(x, y, \theta) d\theta$
- WF$(K_{T_3}) = N^*\{s - y\cdot\omega = 0\}$

$$= \left\{(\omega, y\cdot\omega, y; -\theta i^*(y), \theta, -\theta\omega) : \omega \in \mathbb{S}^{n-1},\ y \in \mathbb{R}^n,\ \theta \in \mathbb{R}\setminus 0\right\}$$

where $i^* : T^*_\omega \mathbb{R}^n \hookrightarrow T^*_\omega \mathbb{S}^{n-1}$ = restriction, $\implies$

$$WF_{T_3} = \left\{(\omega, y\cdot\omega, \theta i^*(y), \theta; y, \theta\omega) : \omega \in \mathbb{S}^{n-1},\ y \in \mathbb{R}^n,\ \theta \in \mathbb{R}\setminus 0\right\}$$

Wave front relation of Spherical mean operator

- $T_4 f(x) = c_{t,n} \int e^{i(|x-y|-t)\theta}\, 1(\theta)\, f(y)\, d\theta\, dy$

  $K_{T_4}(x,y) = c_{t,n} \int e^{i(|x-y|-t)\theta}\, 1(\theta)\, d\theta$

- $\mathrm{WF}(K_{T_4}) = N^*\{|x-y| - t = 0\}$

$$= \Big\{ \Big(x, y; \frac{x-y}{|x-y|}\theta, -\frac{x-y}{|x-y|}\theta\Big) : |x-y| = t, \theta \neq 0 \Big\}$$

$$\implies WF_{T_4} = \big\{ (x, \xi, x - t\tfrac{\xi}{|\xi|}, \xi) : (x,\xi) \in T^*\mathbb{R}^n \setminus \mathbf{0} \big\}$$

$=$ graph of canonical transformation $\chi(x,\xi) = \big(x - t\frac{\xi}{|\xi|}, \xi\big)$

Wave front relation of Melrose-Taylor transform

- $T_5 f(\omega, t) = \int e^{i(t-s-y\cdot\omega)\theta} f(y,s)\, d\theta\, ds\, dy$, $y \in \partial\Omega$, $\omega \in \mathbb{S}^{n-1}$

  $K_{T_5} = \int e^{i(t-s-y\cdot\omega)\theta}\, 1(\theta)\, d\theta$

- $\mathrm{WF}(K_{T_5}) = N^*\{t - s - y\cdot\omega = 0\}$

  $$= \left\{(\omega, t, y, s; -\theta i_\omega^*(y), \theta, -\theta j_y^*(\omega), -\theta) : \, t = s + y\cdot\omega\right\}$$

  where $i_\omega^* : T_\omega^*\mathbb{R}^n \to T_\omega^*\mathbb{S}^{n-1}$ and $j_y^* : T_y^*\mathbb{R}^n \to T_y^*\partial\Omega \implies$

  $$WF_{T_5} = \left\{(\omega, t, -\theta i_\omega^*(y), \theta; y, t - y\cdot\omega, \theta j_y^*(\omega), \theta) : \, t = s + y\cdot\omega, \theta \neq 0\right\}$$

Wave front sets of oscillatory Integrals

- **Def.** $\phi(x,\theta)$ is a **phase function** on $X \times (\mathbb{R}^N \setminus \mathbf{0})$ if it is smooth, $\mathbb{R}$-valued, positively homogeneous of degree 1 in θ ($\phi(x,t\theta) = t\phi(x,\theta)$ for $t > 0$) and satisfies

$$(d_x\phi,\, d_\theta\phi) \neq (0,0).$$

- **Prop.** If $\phi(x,\theta)$ is a phase function and $a \in S^m_{1,0}(X \times (\mathbb{R}^N \setminus \mathbf{0}))$, then the oscillatory integral $u(x) = \int e^{i\phi(x,\theta)}\, a(x,\theta)\, d\theta$ is a well defined distribution, $u \in \mathcal{D}'(X)$, defined by

$$\langle u, f\rangle := \int\int e^{i\phi(x,\theta)}\, a(x,\theta)\, f(x)\, d\theta\, dx, \quad \forall f \in C_0^\infty(X).$$

- **Thm.** $\mathrm{WF}(u) \subseteq \Lambda_\phi := \big\{(x, d_x\phi) : d_\theta\phi(x,\theta) = 0,\ \theta \neq 0\big\} \subseteq T^*X \setminus \mathbf{0}$

Prop: Oscillatory integrals $\in \mathcal{D}'(X)$

- $(d_x\phi, d_\theta\phi) \neq (0,0) \implies$ can form

$$L = \sum_j b_j(x,\theta)\frac{\partial}{\partial x_j} + \sum_k c_k(x,\theta)\frac{\partial}{\partial \theta_k} + b_0(x,\theta) \text{ s.t. } L^t(e^{i\phi}) = e^{i\phi},$$

  with $b_0,\, b_j \in S^{-1}_{1,0},\, c_k \in S^0_{1,0}$.

- For $f \in \mathcal{D}(X)$ and $r \in \mathbb{N}$ large, define $< u, f >$ as

$$\int\int e^{i\phi(x,\theta)} a(x,\theta)\, f(x)\, d\theta\, dx := \int\int (L^t)^r(e^{i\phi}) a(x,\theta)\, f(x)\, d\theta\, dx$$

$$:= \int\int e^{i\phi(x,\theta)} L^r\big(a(x,\theta) f(x)\big)\, d\theta\, dx$$

  But $L^r : S^m_{1,0} \to S^{m-r}_{1,0}$, integral converges for $m - r < -N$.

Thm: WF of oscillatory integrals

Let $(x_0, \xi_0) \in T^*X \setminus \Lambda_\phi$, $\psi(x) \in \mathcal{D}(X)$ supported in nhood of x_0

- $\widehat{\psi u}(\xi) = \int \int e^{i(\phi(x,\theta) - x\cdot\xi)} a(x,\theta)\, \psi(x)\, d\theta\, dx$,
- Form vec fld near (x_0, ξ_0): $L = \frac{1}{|d_x\phi - \xi|^2} \sum_j (d_{x_j}\phi - \xi_j)\partial_{x_j}$

  $\implies L(e^{i(\phi(x,\theta) - x\cdot\xi)}) = e^{i(\phi(x,\theta) - x\cdot\xi)}$
- $|d_x\phi(x,\theta) - \xi| \geq c(|\xi| + |\theta|)$ on supp$(a \cdot \psi)$, can integrate by parts

  $\implies \widehat{\psi u}$ rapidly decreasing on conic nhood of ξ_0
- Thus, $(x_0, \xi_0) \notin$ WF(u).

  Lecture 2: Impose 2nd order conditions on ϕ:
  $\implies$ Fourier integral distributions and operators