# MSRI LECTURES ON GEOMETRIC MICROLOCAL ANALYSIS LECTURE 3

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ABSTRACT. Rough notes for lectures on geometric microlocal analysis at the MSRI introductory workshop in Fall 2019.

- Categorical Point of View:
  - Local and Global theory:
    - Manifold $M$ with boundary and defining $\Psi^{*,*}(M)$ an algebra of $\Psi$DOs exhibiting a certain degeneracy
  - Starting point: $\mathcal{V}_0 = \{$all $C^\infty$ vector fields on $M$ which vanish at $\partial M\}$ with local coordinates $(x, y_1, ..., y_{n-1})$
    - $\mathcal{V}_0 = \text{span}\{x\partial_x, x\partial_{y_j}\}$
    - Form $\text{Diff}_0^*(M) = \{$locally finite sums of $v_j \in \mathcal{V}_0\}$
    - Prototype: $L = \sum_{j+|\alpha|=m} a_{j\alpha}(x,y)(x\partial_x)^j(x\partial_{y_j})^\alpha$
  - $\Psi_0^{*,*}$ is the quantization of $\text{Diff}_0^*$
  - Observe, under dilations $D_\lambda : (x,y) \mapsto (\lambda x, \lambda y)$, $D_\lambda^* L = \sum a_{j\alpha}(\lambda x, \lambda y)(x\partial_x)^j(x\partial_{y_j})^\alpha$ is "almost" dilation invariant
- Hierarchy of symbols:
  - ${}^0\sigma_m(L)(z, \zeta = \sum a_{j\alpha}(x,y)\xi^j\eta^\alpha$ where $z = (x,y)$ and $\zeta = (\xi, \eta)$ which lines in the zero cotangent bundle, ${}^0T^*M$
  - Full ellipticity means:
    - ${}^0\sigma_m(L)$ is invertible when $\zeta \neq 0$
    - $N(L)$ is invertible
  - $N_p(L) = \sum a_{j\alpha}(x,y)(x\partial_x)^j(x\partial_{y_j})^\alpha$
- Given $M$, form $M^2$ and $\tilde{G} \in \mathcal{D}'(M^2)$.
  - Blow up $M^2$; $M_0^2 = [M^2; \partial(diag)]$
  - Denote by $\beta : M_0^2 \to M^2$ the push-down map
  - $\Psi_0^{*,*} = \{A : \beta^* K_A = k_A \in \mathscr{A}_{phg}(M_0^2, diag)\}$ where $K_A$ is the Schwartz kernel of $A$
  - $(M)$ = conormal distributions on $M = \{u,$ stable regularity with respect to $\mathcal{V}_b(M)\}$ where $\mathcal{V}_b(M) = \{C^\infty$ vector fields tangent to $\partial M\}$.
  - Stable regularity means $u \in E$ implies $v_1 ... v_l u \in E$ for any $v_j \in \mathcal{V}_b$, $j = 1, ..., l$, for any $l$ where $E$ is your favorite function space.

  - $\mathscr{A}_{phg} = \{u \in \mathscr{A}_{L^2} : \forall N \; \exists v_1, ..., v_N \in \mathcal{V}_b$ s.t. $v_1...v_N u \in x^N L^2\}$, i.e., $u \sim x^{\gamma_0} u_0(y) + ... + x^{\gamma_N} u_N(y) + O(x^N)$, $\mathrm{Re}(\gamma_j) \to \infty$
- Denote by $\mathcal{E} = (E_{10}, E_{01}, E_{11})$ the Frobenius indices at the corresponding faces of $M^2$
  - $A \in \Psi_0^{k,\mathcal{E}}$, $B \in \Psi_0^{l,F}$, do we get $A \circ B \in \Psi_0^{k+l,\mathcal{E}+F}$?
- Want composition and mapping properties
  - $A \in \Psi_0^{k,\mathcal{E}} \implies A : x^\delta H_0^s \to x^{\delta'} H_0^{s-k}$ (differentiation with respect to $(x\partial_x, x\partial_y)$).
  - Composition: We can project in three ways from $M^3$, $\pi_L$, $\pi_M$, $\pi_R$ onto the left, middle, or right respectively
  - $k_{A \circ B} = (\pi_M)_*(\pi_L^* k_A \cdot \pi_R^* k_B) = \int A(z, z') \cdot B(z', z'') dz'$
  - See Melrose's Push-Forward theorem
    - Idea of proof: compactify your manifold, blowup and define ΨDOs relative to the blowup
- Asymptotic conic geometry:
  - A manifold $(M, g)$ where $g \sim dr^2 + r^2 h$ with $(Y, h)$ compact
  - Define $x = \frac{1}{r}$ to compactify
  - This yields $g \sim \frac{dx^2}{x^4} + \frac{h}{x^2}$
  - $\mathcal{V}_{sc} = \{x^2 \partial_x, x^2 \partial_{y_i}\} = x\mathcal{V}_b$ scattering vector fields
- Tale of three operators:
  - $\Delta + 1$, $\Delta$, $\Delta - 1$
  - $\Delta = x^2(x^2 \partial_x^2 + 2x\partial_x + \Delta_{h(y)})$ where the term inside parentheses is an elliptic b-operator
  - After blow up, $\Delta - 1$ is invariant while $\Delta + 1$ is not