MSRI LECTURE ON QUANTUM CHAOS

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ABSTRACT. Rough notes for the lecture on quantum chaos at the MSRI introductory workshop in Fall 2019.

- Billiard flows: $\phi^t : (x, \xi) \mapsto (x + \xi, \xi)$ on compact manifolds (classical)
  - Quantum: Obey Schrödinger equation $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi$
  - Numerical results on spectrum obtained in 80s
- Conjecture: Show spectrum of quantum mechanical system resembles that of large random matrices.
- Quantum Unique Ergodicity Conjecture: sufficiently chaotic implies $|\psi(x)|^2$ is uniform
  - Conjectures meant for $\hbar \to 0$, the semiclassical limit
- M a billiard table, compact Riemannian manifold of dimension d
  - $\Delta \psi_k = -\lambda_k \psi_k$, $\|\psi\|_{L^2(M)} = 1$, $\lambda_k \to \infty$
  - Study weak limits of $|\psi_k(x)|^2 \mathrm{dVol}(x)$
- Quantum Ergodicity Theorem: Assume the action of geodesic flow is ergodic for Liouville measure. Let $a \in C(M)$. Then

$$\frac{1}{N(\lambda)} \sum_{\lambda_k \leq \lambda} \left| \int_M a(x) |\psi_k(x)|^2 \mathrm{dVol}(x) - \int_M a(x) \mathrm{dVol}(x) \right| \to_{\lambda \to \infty} 0$$

  if and only if there exists a subset of $\mathbb{N}$ of density 1 so that

$$\int a(x) |\psi_k(x)|^2 \mathrm{dVol}(x) \to \int a(x) \mathrm{dVol}(x)$$

  .
  - Full statement uses analysis on phase space.
- Why geodesic flow?
  (1) Quantum variance: $\mathrm{Var}_\lambda(K) = \frac{1}{N(\lambda)} \sum_{\lambda_k \leq \lambda} |\langle \psi_k, e^{it\sqrt{\Delta}} K e^{-it\sqrt{\Delta}} \psi_k \rangle_{L^2(M)}|$ for K a bounded operator (later it will be a ΨDO)
  (2) ΨDO of order zero evolved by $e^{it\sqrt{\Delta}} \cdot e^{-it\sqrt{\Delta}}$ is given by geodesic flow plus a remainder
  (3) Control quantum variance by L^2 norm of principal symbol
  (4) Use ergodicity of classical dynamics to conclude

- Quantum Unique Ergodicity Conjecture ('94): On negatively curved manifolds, we have convergence of whole sequence

$$\langle \psi_k, a(x, D_x)\psi_k\rangle_{L^2(M)} \to \int_{(x,\xi)\in SM} a(x,\xi)dxd\xi$$

  - General conjecture believed to hold with Anosov property of geodesic flow.

Theorem 1. *M negatively curved of dimension d. Assume*

$$\langle \psi_k, a(x, D_x)\psi_k\rangle_{L^2(M)} \to \int_{(x,\xi)\in SM} a(x,\xi)d\mu(x,\xi)$$

for some measure μ with strictly positive entropy. Constant negative curvature implies μ has dimension greater than or equal to d.

- Entropy: (X, β, ϕ, μ), $\phi : X \to X$ measurable with $\phi^*\mu = \mu$, $X = S^*M$
- $h_{KS}(\mu) \geq 0$.
- If μ is carried by a periodic point then $h_{KS}(\mu) = 0$.
- h_{KS} is affine.

Proposition 0.1. *X a manifold, $\phi \in C^1$ then $h_{KS}(\mu) \leq \int_X \sum \lambda_i^+(p)d\mu(p)$ where λ_i^+ are the Lyapunov exponents.*

- For Anosov flows: equality in above inequality iff μ is Sinai-Ruelle-Bomn measure

Theorem 2. *If μ is a semiclassical measure, then $h_{KS}(\mu) \geq \frac{1}{2}\int_{S^*M} \sum \lambda_i^+(p)d\mu(p)$*