MSRI LECTURES ON PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. Rough notes for lectures at the MSRI introductory workshop in Fall 2019.

A large part of these notes is a shortened version of lecture notes by Richard Melrose, available at `http://math.mit.edu/~rbm/iml90c2.ps`

1. Composition formula

We are now ready to prove one of the key properties of pseudodifferential operators: in fancy terms, they form an algebra.

Theorem 1. *Assume that $A \in \Psi^m(\mathbb{R}^n)$, $B \in \Psi^\ell(\mathbb{R}^n)$. Then $AB \in \Psi^{m+\ell}(\mathbb{R}^n)$ and:*

- *Product Rule: $\sigma^{m+\ell}(AB) = \sigma^m(A)\sigma^\ell(B)$, in particular implying that the commutator $[A,B]$ lies in $\Psi^{m+\ell-1}(\mathbb{R}^n)$;*
- *Commutator Rule: $\sigma^{m+\ell-1}([A,B]) = -i\{\sigma^m(A), \sigma^\ell(B)\}$ where $\{\bullet,\bullet\}$ denotes the Poisson bracket:*

$$a, b \in C^\infty(\mathbb{R}^{2n}) \quad \Longrightarrow \quad \{a,b\} = \sum_{j=1}^n (\partial_{\xi_j} a)(\partial_{x_j} b) - (\partial_{x_j} a)(\partial_{\xi_j} b).$$

Remark. It might be useful to check the Commutator Rule on a one-dimensional example (not strictly speaking legal because x is not bounded, but still making sense):

$$A = D_x, \quad B = x, \quad [A,B] = -i.$$

Proof. We use Theorem ?? to write $A = \mathrm{Op}_0(a)$ for some $a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$. Then for $u \in \mathscr{S}(\mathbb{R}^n)$

$$ABu(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi) \widehat{Bu}(\xi)\, d\xi.$$

We now write $\widehat{Bu}$ in terms of u. We first write

$$\widehat{Bu}(\xi) = \langle Bu, e_\xi \rangle_{L^2} = \langle u, B^* e_\xi \rangle_{L^2}, \quad e_\xi(x) = e^{ix\cdot\xi}.$$

We use Theorem 1 of lecture 1 again to write $B^* = \mathrm{Op}_0(\overline{b'})$ for some $b' \in S^\ell(\mathbb{R}^n;\mathbb{R}^n)$. Then we have the following *oscillatory testing* statement:

$$(B^* e_\xi)(x) = (\mathrm{Op}_0(\overline{b}') e_\xi)(x) = \overline{b'(x,\xi)} e_\xi(x). \tag{1}$$

To see (1) we recall that $\widehat{e_\xi}(\eta) = (2\pi)^n \delta(\xi - \eta)$ (by the standard properties of Fourier transform of tempered distributions) and write

$$\begin{aligned}(\mathrm{Op}_0(\overline{b'})e_\xi)(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\eta}\overline{b'(x,\eta)}\widehat{e_\xi}(\eta)\, d\eta \\ &= \int_{\mathbb{R}^n} e^{ix\cdot\eta}\overline{b'(x,\eta)}\delta(\xi-\eta)\, d\eta = e^{ix\cdot\xi}\overline{b'(x,\xi)}.\end{aligned}$$

We now get

$$\widehat{Bu}(\xi) = \langle u(y), \overline{b'(y,\xi)}e^{iy\cdot\xi}\rangle_{L^2} = \int_{\mathbb{R}^n} e^{-iy\cdot\xi} b'(y,\xi)u(y)\, dy.$$

Thus

$$ABu(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} a(x,\xi)b'(y,\xi)u(y)\, dy d\xi. \tag{2}$$

Strictly speaking, the above argument gives (2) for a, b' sufficiently rapidly decaying in ξ, and the general case follows by approximation, where the integral is understood by integration by parts in y as before.

2. Now, (2) gives that $AB \in \Psi^{m+\ell}(\mathbb{R}^n)$ by definition, since $a(x,\xi)b'(y,\xi) \in S^{m+\ell}(\mathbb{R}^{2n};\mathbb{R}^n)$. It remains to check the Product Rule and Commutator Rule. We have from the expansion in Theorem 1 of lecture 1

$$\begin{aligned}\sigma^m(A) &= a \mod S^{m-1}, \\ \sigma^\ell(B) &= b' \mod S^{\ell-1}, \\ \sigma^{m+\ell}(AB) &= ab' \mod S^{m+\ell-1}\end{aligned}$$

which immediately gives the Product Rule.

As for the Commutator Rule, we have to also compute the product BA. We write

$$A = \mathrm{Op}_0(a) = \mathrm{Op}_0(\overline{a'})^*, \quad B = \mathrm{Op}_0(b) = \mathrm{Op}_0(\overline{b'})^*$$

where $a, a' \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ and $b, b' \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ satisfy

$$\sigma(A) = a = a' \mod S^{m-1}, \quad \sigma(B) = b = b' \mod S^{\ell-1}.$$

By the expansion of Theorem 1 from lecture 1 we have

$$a' = a + i(\partial_x \cdot \partial_\xi)a \mod S^{m-2}, \quad b' = b + i(\partial_x\cdot\partial_\xi)b \mod S^{\ell-2}.$$

We have next, by arguing as in part 1 for both AB and BA.

$$[A,B] = \mathrm{Op}(c), \quad c(x,y,\xi) = a(x,\xi)b'(y,\xi) - a'(y,\xi)b(x,\xi).$$

Recalling the expansions for a', b' above we have (modulo $S^{m+\ell-2}$)

$$c(x,y,\xi) = a(x,\xi)b(y,\xi) - a(y,\xi)b(x,\xi) + ia(x,\xi)(\partial_y\cdot\partial_\xi)b(y,\xi) - ib(x,\xi)(\partial_y\cdot\partial_\xi)a(y,\xi).$$

By the expansion of Theorem 1 from lecture 1 we have $[A,B] = \mathrm{Op}_0(\tilde{c})$ where

$$\tilde{c}(x,\xi) = c(x,x,\xi) - i(\partial_y\cdot\partial_\xi)c(x,x,\xi) \mod S^{m+\ell-2},$$

and from here we compute

$$\tilde{c} = -i\{a, b\} \mod S^{m+\ell-2}$$

which gives the Commutator Rule. □

2. Sobolev spaces

It is important to have pseudodifferential operators act on some normed spaces rather than just on $\mathscr{S}$ and $\mathscr{S}'$. The natural spaces are Sobolev spaces $H^s(\mathbb{R}^n)$, defined as subspaces of $\mathscr{S}'(\mathbb{R}^n)$ with the norm

$$\|u\|_{H^s} := \|\langle\xi\rangle^s \hat{u}(\xi)\|_{L^2(\mathbb{R}^n)}.$$

For the sake of time, we give the following important result without proof.

Theorem 2. *Assume that $s, m \in \mathbb{R}$. Then any $A \in \Psi^m(\mathbb{R}^n)$ defines a bounded operator $H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$.*

It follows from the L^2 boundedness theorem of $\Psi^0(\mathbb{R}^n)$ operators (which needs a lot of work!) and then the rest follows from commuting the ξ powers.

3. The scattering calculus

We now introduce a different class of symbols,

$$S^{m,\ell}(\mathbb{R}^n; \mathbb{R}^n)$$

consisting of functions $a(x, \xi)$ satisfying the bounds

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} \langle x\rangle^{\ell-|\alpha|} \langle\xi\rangle^{m-|\beta|}.$$

We see from the above definition that m is the order in ξ (i.e. the differential order of the corresponding operator) and ℓ is the order in x (i.e. the order of growth of the coefficients of the operator, in case we had a differential operator). We have $S^{m,0} \subset S^m$, and the inclusion is strict: for $S^{m,\ell}$ we require differentiation in x to give better decay in x.

We can define *scattering pseudodifferential operators* as those of the form $\mathrm{Op}_0(a)$ where $a \in S^{m,\ell}(\mathbb{R}^n; \mathbb{R}^n)$ (here Op_0 is as before). This class of operators will be denoted by $\Psi^{m,\ell}(\mathbb{R}^n)$.

The basic properties of operators in the scattering calculus are established similarly to the Kohn–Nirenberg calculus (it is again convenient to allow symbols to depend on (x, y, ξ) and there are some details which are different, but we omit all of these here). We have the following:

- The (joint) principal symbol now maps

$$\sigma^{m,\ell} : \Psi^{m,\ell}(\mathbb{R}^n) \to \frac{S^{m,\ell}(\mathbb{R}^n; \mathbb{R}^n)}{S^{m-1,\ell-1}(\mathbb{R}^n; \mathbb{R}^n)}$$

  That is, we get a gain in *both* differential order and decay in x. This is natural to see from the expansion in Theorem 1 from lecture 1 which features powers of $(\partial_y \cdot \partial_\xi)$, and the definition of our symbols classes: ∂_y will give better order in x, ∂_ξ will give better order in ξ.
- Product Rule: if $A \in \Psi^{m,\ell}$, $B \in \Psi^{m',\ell'}$, then $AB \in \Psi^{m+m',\ell+\ell'}$ and $\sigma^{m+m',\ell+\ell'}(AB) = \sigma^{m,\ell}(A)\sigma^{m',\ell'}(B)$.
- Commutator Rule: with A, B as in the Product Rule, $\sigma^{m+m'-1,\ell+\ell'-1}([A,B]) = -i\{\sigma^{m,\ell}(A), \sigma^{m',\ell'}(B)\}$.
- Adjoint Rule: if $A \in \Psi^{m,\ell}(\mathbb{R}^n)$, then $A^* \in \Psi^{m,\ell}(\mathbb{R}^n)$ and $\sigma^{m,\ell}(A^*) = \overline{\sigma^{m,\ell}(A)}$.
- Mapping properties: if $A \in \Psi^{m,\ell}(\mathbb{R}^n)$ then A defines a bounded operator on *weighted* Sobolev spaces

$$A : \langle x \rangle^t H^s(\mathbb{R}^n) \to \langle x \rangle^{t+\ell} H^{s-m}(\mathbb{R}^n), \quad s, t \in \mathbb{R}.$$

- The residual class $\Psi^{-\infty,-\infty} = \bigcap_{m,\ell} \Psi^{m,\ell}$ consists of operators with integral kernels in $\mathscr{S}(\mathbb{R}^{2n})$; they map $\mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$.

A significant change from the Kohn–Nirenberg algebra is the fact that $\langle x \rangle^t H^s(\mathbb{R}^n)$ embeds compactly into $L^2(\mathbb{R}^n)$ for any $s > 0, t < 0$, which means that any element of $\Psi^{m,\ell}(\mathbb{R}^n)$ with $m < 0, \ell < 0$ defines a compact operator on L^2. This is important for applications to geometric scattering theory.

4. The elliptic parametrix

We now present an important application of the pseudodifferential algebra, the elliptic parametrix. We do it in the scattering algebra, the Kohn–Nirenberg analog is very similar.

First though we need to discuss asymptotic expansions, which, unlike series, always converge:

Theorem 3. *Let $A_j \in \Psi^{-j,-j}(\mathbb{R}^n)$, $j = 0, 1, \ldots$. Then there exists $A \in \Psi^{0,0}(\mathbb{R}^n)$ such that $A \sim \sum_{j=0}^{\infty} A_j$ in the following sense: for each N, $A - \sum_{j=0}^{N-1} A_j \in \Psi^{-N,-N}(\mathbb{R}^n)$.*

We do not prove the above theorem but remark that the statement readily translates to the level of full symbols: taking $A_j = \mathrm{Op}_0(a_j)$, $A = \mathrm{Op}_0(a)$, we see that we need to construct a from a_j. This is done by a version of Borel's Theorem.

We now give the elliptic parametrix statement:

Theorem 4. *Assume that $P \in \Psi^{m,\ell}(\mathbb{R}^n)$ is* globally elliptic, *namely the principal symbol $\sigma(P)$ satisfies for some $C > 0$*

$$|\sigma(P)(x,\xi)| \geq C^{-1}\langle x\rangle^m \langle\xi\rangle^\ell \quad \text{when} \quad |x| + |\xi| \geq C.$$

(Note that the above condition is independent of the choice of the representative of the principal symbol.) Then there exists $Q \in \Psi^{-m,-\ell}(\mathbb{R}^n)$ such that

$$PQ = I + \Psi^{-\infty,-\infty}(\mathbb{R}^n), \quad QP = I + \Psi^{-\infty,-\infty}(\mathbb{R}^n).$$

Remarks. 1. A consequence of this is the fact that $P : \langle x\rangle^t H^s(\mathbb{R}^n) \to \langle x\rangle^{t+\ell} H^{s-m}(\mathbb{R}^n)$ is a *Fredholm operator* for any s, t. Indeed, it is invertible modulo $\Psi^{-\infty,\infty}$ which consists of compact operators. (In case someone asks, the index of P need not be 0.)

2. An example of an elliptic operator is given by $\Delta + 1 \in \Psi^{2,0}(\mathbb{R}^n)$, or $\Delta + |x|^2 - E \in \Psi^{2,2}(\mathbb{R}^n)$ where $E \in \mathbb{R}$. And $\Delta + 1$ is elliptic in $\Psi^{2,0}(\mathbb{R}^n)$, with the principal symbol given by $|\xi|^2 + 1$ (it is important that the term 1 is included in the principal symbol of the scattering calculus!). However, $\Delta \in \Psi^{2,0}(\mathbb{R}^n)$ is *not* elliptic in the scattering calculus, though it is elliptic in the Kohn–Nirenberg calculus. This makes sense since $\Delta + 1$ is a Fredholm (in fact, invertible) operator $H^2 \to L^2$, but Δ is not.

Proof. 1. Fix a representative p_0 for $\sigma(P)$. We first define an approximate inverse

$$Q_0 := \mathrm{Op}_0(q_0), \quad q_0 = \frac{1}{p_0} \quad \text{for} \quad |x| + |\xi| \gg 1.$$

Because of the ellipticity condition, we can define q_0 like that and we have $q_0 \in S^{-m,-k}(\mathbb{R}^n;\mathbb{R}^n)$. (The latter does need some work, differentiating p_0^{-1} in x and ξ many times and checking the bounds.)

From the Product Rule we see that $PQ_0 \in \Psi^{0,0}(\mathbb{R}^n)$ and $\sigma^{0,0}(PQ_0) = 1$. Thus

$$R := I - PQ_0 \in \Psi^{-1,-1}(\mathbb{R}^n).$$

2. We now invert $I - R$ by "Neumann series" except it is instead a Neumann asymptotic expansion. Namely we use Theorem 3 to construct $Q_1 \in \Psi^{0,0}(\mathbb{R}^n)$ such that

$$Q_1 \sim \sum_{j=0}^{\infty} R^j.$$

Put $Q := Q_0 Q_1$. Then

$$PQ = (I - R)Q_1 = I + \Psi^{-\infty,-\infty}(\mathbb{R}^n).$$

3. We similarly construct $Q' \in \Psi^{0,0}(\mathbb{R}^n)$ such that

$$Q'P = I + \Psi^{-\infty,-\infty}(\mathbb{R}^n).$$

By looking at the product $Q'PQ$ (and since $\Psi^{-\infty,-\infty}$ is an ideal in the pseudodifferential algebra), we see that $Q - Q' \in \Psi^{-\infty,\infty}(\mathbb{R}^n)$. This finishes the proof. □

5. Manifolds

We now very briefly talk about calculus on manifolds. For a compact manifold M we can define the algebra $\Psi^m(M)$ of pseudodifferential operators with Kohn–Nirenberg symbols. This can be constructed using local charts, where the key fact (which we omit here) is that the class $\Psi^m(\mathbb{R}^n)$ is invariant under coordinate changes (forgetting about $x \to \infty$). The full symbol cannot be invariantly defined, and the principal symbol is now a function on the *cotangent bundle* T^*M. This can already be seen in the case of a vector field X: the corresponding differential operator $P = -iX \in \Psi^1(M)$ has principal symbol

$$\sigma(P)(x, \xi) = \langle \xi, X(x) \rangle.$$

The basic properties still hold. Note that the Poisson bracket makes sense on T^*M: a low-tech way is to say that the formula for it is independent of the choice of local coordinates on M, and a hi-tech way is to use that T^*M is a symplectic manifold.

If M is noncompact, we have to specify what is happening as $x \to \infty$. One approach is to assume that M is a *scattering manifold*, then one can define an analog of the scattering algebra.