MSRI LECTURES ON GEOMETRIC MICROLOCAL ANALYSIS
LECTURE 2

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ABSTRACT. Rough notes for lectures on geometric microlocal analysis at the MSRI introductory workshop in Fall 2019.

- Start with noncompact space and an elliptic operator: (M, g), Δ; $\Delta u = f$
  - Want: $G \in \mathcal{D}'(M \times M)$ which solves $\Delta G = I = G\Delta$.
  - Need to understand G
  - We'll have
    $$u(z) = \int G(z, \tilde{z}) f(\tilde{z}) dv(\tilde{z})$$
    otherwise known as $u = (\pi_L)_*(G\pi_R^* f)$ where π_L and π_R are projections on the left/right from $M \times M \to M$.
  - G is singular along the diagonal of $M \times M$, i.e., when $z = \tilde{z}$.
- Parametrix method allows us to find $\tilde{G}$ which approximates G; $\Delta\tilde{G} = I - R_1$, $\tilde{G}\Delta = I - R_2$ with $R_1, R_2 \in \Psi^{-\infty}$ (smoothing operators), i.e., $R_j(z, \tilde{z}) \in C^\infty(M \times M)$.
  - This gives local regularity but not Fredholmness or mapping properties.
- <u>Better:</u> $R_j : L^2 \to \rho^k H^\infty \implies \Delta$ is Fredholm
  - The kernel decays away from the diagonal
  - $\Delta u = 0$ (u in L^2-null space) $\implies u = R_2 u \implies u \in C^\infty$ and definite rate of decay.
- <u>Test case</u>: $\Delta - s(n-1-s)$ on $\mathbb{H}^n$
  - Set $\lambda = s(n-1-s)$
  - We want to solve for the resolvent $(\Delta - \lambda)^{-1}$
  - G_s solves
    $$\big(\partial_r^2 + (n-1)\frac{\cosh r}{\sinh r}\partial_r + \lambda\big)G_s = 0$$
  - Setting $\rho = e^{-r}$ yields $G_s \sim \rho^s$ or ρ^{n-1-s} as $\rho \to 0$.
  - Specialize to $n = 3$: explicit solution $G_s = \dfrac{e^{-(s-1)\rho}}{\sinh\rho}$ in the UHS model
  - Dilation invariance: $G_s(x, \tilde{x}, y - \tilde{y}) = G_s\big(\beta x, \beta\tilde{x}, \beta(y - \tilde{y})\big)$ for all $\beta > 0$
  - Blow up the submanifold $\{x = 0,\ \tilde{x} = 0,\ y = \tilde{y}\}$ in $M \times M$. and define $M_0^2 = [M \times M, S]$, that is, we are attaching a sphere at each point of the diagonal

  - G is singular along the diagonal and smooth up to the front face
  - $(\Delta - s(n-1-s))u = f \in C_0^\infty \implies u = G_s f \sim x^2 + h.o.t. = a(y)x^s + \dots + b(y)x^{n-1-s}$
- Scattering operator: $S(s) : a(y) \mapsto b(y)$; Scattering "matrix"
  - $s = \dfrac{n-1}{2}$ is a "critical line"; to the left, a decays more and to the right b decays more
- Generalize:
  - $\Delta + V - s(n-1-s)$, $V \in C_0^\infty$ or $V = 0$ in $\mathbb{H}^n \setminus \Omega$
  - Still want G_s with $(\Delta + V - s(n-1-s))G = \delta$
  - G_s is approximated by $\tilde{\chi}_1 G_{in}\chi_1 + \tilde{\chi}_2 G_{out}\chi_2$ as before
  - We get

$$(\Delta + V - s(n-1-s))\tilde{G}_s = \tilde{\chi}_1 L G_{in}\chi_1 + \tilde{\chi}_2 L G_{out}\chi_2 + [L, \tilde{\chi}_1]G_{in}\chi_1 + [L, \tilde{\chi}_2]G_{out}\chi_2$$

  - Want to solve $LG_{in} = I - Q_1$ with $Q \in C^\infty(M \times M)$.
- Claim: $\tilde{G}_s$ is smooth on M_0^2.
  - In fact: $L\tilde{G}_s = I - R_1$ and $\tilde{G}_s L = I - R_2$ with $R_1, R_2 \in C^\infty(M_0^2)$ where C^∞ is being identified with $\mathscr{A}_{phg}$, polyhomogeneous functions with smooth expansions.
- Consider $\mathbb{H}^n/\Gamma$, convex cocompact quotients
  - $M = \mathbb{H}^n/\Gamma \to \bar{M}$, its compactification
  - Look at $M_0^2 = [\bar{M} \times \bar{M}, \partial(\text{diag})]$
  - Consider $\Delta - s(n-1-s)$ on $\mathbb{H}^n/\Gamma$
  - G_s =resolvent$\in \Psi_0^{-2,\epsilon}$ and is meromorphic in s where the subscript 0 refers to the blow-up
  - Consider the vector fields $\nu_0 = \{x\partial_x, x\partial_{y_i}\}$
  - Take $\{U_j\}$ a cover of our manifold
  - For each U_j, choose G_j an inverse for $\Delta - s(n-1-s)$ in $U_j \subset \overline{\mathbb{H}^n}$.
  - $\tilde{G}_s = \sum \tilde{\chi}_j G_j \chi_j$ and $(\Delta - s(n-1-s))\tilde{G}_s = I - R$ for a smoothing operator R
    * $[\Delta, \tilde{\chi}_j]G_j\chi_j$ shows up in the error
    * The commutator is smooth and bounded while each G_j decays like x^s which tells us that the error is supported away from the diagonal.
    * Thus, $(\Delta - s(n-1-s))\tilde{G}_s = I+$compact smoothing operator
  - We want $G_s = \tilde{G}_s(I - R_s)^{-1}$
    * Approximate $(I - R)^{-1} \sim I + R_s + R_s^2 + \dots$
  - $g = \rho^{-2}\tilde{g}$, $\dfrac{dx^2 + h(x,y)}{x^2}$ with $h(x,y) \sim \sum [(h_l)_{ij}(y)dy^i dy^j]x^l$
    * $\Delta_g \sim \Delta_{\mathbb{H}^n}$
    * $\sum a_{j\alpha}(x,y)(x\partial_x)^j(x\partial_y)^\alpha$ the leading term of Δ in $\mathbb{H}^n$.
- Summary: start with (M, g)

- First step:
  - Compactify and form $M_0^2 = [\bar{M} \times \bar{M}, \partial(\text{diag})]$
  - $\tilde{G} = G_0 + G_1 + G_2$ operator does not degenerate after blowup
  - Lift $x\partial_x$, $x\partial_{y_i}$ to M_0^2
  - $\Delta G_0 = I - R_0$ with R_0 smooth localized along blowup
  - $R_0 : x^s L^2 \to x^s H_0^\infty$ where subscript 0 refers to differentiation with respect to the vector fields $x\partial_x$ and $x\partial_{y_i}$
- Second step:
  - Gain extra decay and apply L^2 Arzela-Ascoli