an invitation to

Quantum Field Theory on curved spacetimes

MSRI Introductory Workshop: Microlocal Analysis

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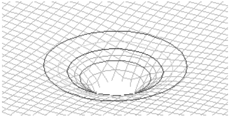
Introduction

Quantum Mechanics

Schrödinger and Dirac operators, semi-classical regime, quantum information theory, ...

General Relativity

Einstein equations, black holes, gravitational waves, (classical) cosmology



Linear QFT

Non-linear QFT

CMI

perturbative QFT, constructive QFT, CFT, cf. stoch. PDEs

Linear QFT on curved spacetimes

Hawking effect, Hadamard states

semi-classical Quantum Gravity

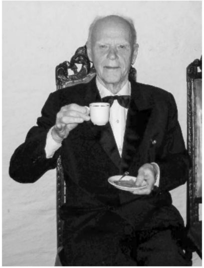
Quantum Gravity?

Loop Quantum Gravity, AdS/CFT, string theory, Liouville gravity, ...

+ effective theories (e.g. non-relativistic QED, bound-state QED)

mathematical QFT and **microlocal analysis**: long-lasting ties

Arthur Wightman



Lars Gårding

Lars Hörmander

Wightman and Gårding co-authored pioneering work on mathematical QFT in the 60s.

Later, Wightman influenced works of Duistermaat and Hörmander (Gårding's former student) on Fourier Integral Operators.

Scalar linear fields on Lorentzian[1] manifold (M, g)

$$(-\Box_g + m^2)\phi(x) = 0$$

$\phi(x)$ with values in operators on Hilbert space.

Scalar products "two-point functions"

$$\langle v, \phi(x_1)\phi(x_2)v\rangle$$

closely related to Schwartz kernels of FIOs.

Non-linear quantities

$$\phi^2(x_1) = \lim_{x_2 \to x_1} \phi(x_1)\phi(x_2) - \text{ singular part}$$

require renormalisation.

[1] pseudo-Riemannian, signature $(1, n-1)$

Quantization

Quantum Mechanics

- Observables are operators on Hilbert space $\mathscr{H}$.
- Typically of the form $a(x, D_x)$ in $\mathscr{H} = L^2(\mathbb{R}^d)$

Quantum Field Theory

- **Particle creation**: number of particles not fixed
- **Causality**: observables commute if localized in *causally disjoint* regions

Given real vector space V and symplectic form $\sigma(\cdot,\cdot)$:

Quantization problem. Find Hilbert space $\mathscr{H}$ and linear $v \mapsto \phi[v]$ (the quantum fields) with values in operators s.t.:

1. $\phi[v]^* = \phi[v]$ for $v \in V$
2. $\exists\, \Omega \in \mathscr{H}$ (the cyclic vector) s.t.

$$\operatorname{span}\left\{\phi[v_1]\ldots\phi[v_m]\Omega \,:\, v_1,\ldots,v_m \in V,\ m \in \mathbb{N}\right\}$$

   is dense in $\mathscr{H}$

3. $\phi[v_1]\phi[v_2] - \phi[v_2]\phi[v_1] = i\sigma(v_1,v_2)\mathbf{1}$ for $v_1, v_2 \in V$

(in case of fermionic fields, $\phi[v_1]\phi[v_2] + \phi[v_2]\phi[v_1] = \langle v_1, v_2\rangle\mathbf{1}$)

Example: If V finite-dimensional with basis $\{e_i\}_{i=1}^{2N}$, then Stone-von Neumann theorem gives unitary equivalence to:

$$\phi[e_{2i}] = x_i, \quad \phi[e_{2i+1}] = D_{x_i}, \qquad i = 1, \ldots, N$$

as unbounded operators on $L^2(\mathbb{R}^{2N})$.

Let $\mathfrak{h}$ a (complex) Hilbert space. The bosonic Fock space is

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n} \mathfrak{h}.$$

For $h \in \mathfrak{h}$, creation/annihilation operators:

$$\begin{aligned} a^*[h]\Psi_n &:= \sqrt{n+1}\; h \otimes_{\mathrm{s}} \Psi_n, \\ a[h]\Psi_n &= \sqrt{n}\,\big(\langle h| \otimes_{\mathrm{s}} \mathbf{1}_{n-1}\big)\Psi_n, \quad \Psi_n \in \otimes_{\mathrm{s}}^{n}\mathfrak{h} \end{aligned}$$

where $\langle h|$ is the map $\mathfrak{h} \ni u \mapsto \langle h, u\rangle \in \mathbb{C}$.

Fock representation $\phi[h] := \frac{1}{\sqrt{2}}\,(a[h] + a^*[h])$ satisfies

$$\phi[h_1]\phi[h_2] - \phi[h_2]\phi[h_1] = i\,\mathrm{Im}\langle h_1, h_2\rangle \mathbf{1} =: i\sigma(h_1, h_2)\mathbf{1}.$$

Cyclic vector $\Omega = (1, 0, 0^{\otimes 2}, 0^{\otimes 3}, \dots)$. **Non-uniqueness**:

$$\text{new scalar product } \langle h_1, h_2\rangle_j = \sigma(h_1, jh_2) + i\sigma(h_1, h_2)$$

provided $(\mathfrak{h}_{\mathbb{R}}, \sigma, j)$ is Kähler, i.e. $j^2 = -\mathbf{1}$ and $\sigma \circ j \geq 0$. New Hilbert space by complexification:

$$(\alpha + i\beta)h := \alpha h + j\beta h, \quad h \in \mathfrak{h}_{\mathbb{R}}, \; \alpha + i\beta \in \mathbb{C}.$$

We focus now on Klein-Gordon fields on Lorentzian (M, g).

Typical assumption:

$P = -\Box_g + m^2$ on $M = \mathbb{R}_t \times S$, $m \in \mathbb{R}$, (M, g) globally hyperbolic, i.e. no closed time-like curves, and S intersected by each maximally extended time-like curve exactly once.

Occasionally for convenience: $g = -dt^2 + h_t$ with h_t Riemannian, $t \in \mathbb{R}$.

Let $u = P_\pm^{-1} f$ be the unique solution of forward/backward problem $Pu = f$, $f \in C_c^\infty(M)$.

$$\sigma(v_1, v_2) = \int_M (v_1 P_+^{-1} v_2 - v_1 P_-^{-1} v_2) d\mathrm{vol}_g$$

defines a symplectic form on $C_c^\infty(M; \mathbb{R}) / P C_c^\infty(M; \mathbb{R})$.

Quantization gives $v \mapsto \phi[v]$, interpreted as operator-valued distribution $\phi(x)$ that solves $P\phi = 0$.

Proposition

Suppose $\Lambda^{\pm} : C^\infty_{\mathrm{c}}(M) \to C^\infty(M)$ satisfies:

$$\boxed{\Lambda^{\pm} \geq 0, \quad \Lambda^+ - \Lambda^- = i(P_+^{-1} - P_-^{-1}), \quad P\Lambda^{\pm} = \Lambda^{\pm} P = 0}.$$

Let $\mathfrak{h}$ be the completion of $C^\infty_{\mathrm{c}}(M)$ w.r.t. $\frac{1}{2}(\Lambda^+ + \Lambda^-)$. Then there exists j such that $(\mathfrak{h}_{\mathbb{R}}, \sigma, j)$ is Kähler and

$$\langle v_1, v_2 \rangle_j = \frac{1}{2} \int_M \overline{v_1} (\Lambda^+ + \Lambda^-) v_2 \, d\mathrm{vol}_g.$$

This gives $(\Omega|\phi[v_1]\phi[v_2]\Omega) = \int_M v_1 \Lambda^+ v_2 \, d\mathrm{vol}_g$, $v_i \in C^\infty_{\mathrm{c}}(M; \mathbb{R})$, hence the name **two-point functions**.

Choosing $\Lambda^{\pm}$ amounts to specifying global **state** of the system, and thus a particle interpretation.

If $P = \partial_t^2 - \Delta_y + m^2$ and $m > 0$, then **vacuum state**:

$$(\Lambda^{\pm}_{\mathrm{vac}} v)(t,y) = \frac{1}{2}\int_{\mathbb{R}} \frac{e^{\pm i(t-s)\sqrt{-\Delta_y + m^2}}}{\sqrt{-\Delta_y + m^2}} v(s,y)ds$$

In general, $\Lambda^{\pm}$ should resemble $\Lambda^{\pm}_{\mathrm{vac}}$ at small distances:

We say that Λ^+, Λ^- define a **Hadamard state** if

$$\Lambda^{\pm} \geq 0, \quad \Lambda^+ - \Lambda^- = i(P_+^{-1} - P_-^{-1}), \quad P\Lambda^{\pm} = \Lambda^{\pm} P = 0,$$

and $\boxed{\mathrm{WF}'(\Lambda^{\pm}) \subset \Sigma^{\pm} \times \Sigma^{\pm}}$ (Hadamard condition).

Here, $\sigma_{\mathrm{pr}}(P)$ is $p(t,y,\tau,\eta) = \tau^2 - \eta \cdot h_t(y)\eta$. Characteristic set:

$$\Sigma = \Sigma^+ \cup \Sigma^-, \quad \Sigma^{\pm} = \left\{ (t,y,\tau,\eta) : \tau = \pm(\eta \cdot h_t(y)\eta)^{\frac{1}{2}}, \ \eta \neq 0 \right\}$$

If $\Gamma \subset T^*M \times T^*M$, $\Gamma' = \left\{ \big((x_1,\xi_1),(x_2,\xi_2)\big) : \big((x_1,\xi_1),(x_2,-\xi_2)\big) \in \Gamma \right\}$.

Theorem

$\Lambda^{\pm}_{\mathrm{vac}}$ are Hadamard

Proof. Use $(i^{-1}\partial_t \pm \sqrt{-\Delta_y + m^2})\Lambda^{\pm}_{\mathrm{vac}} = 0$.

Consequences of Hadamard condition $\boxed{\mathrm{WF}'(\Lambda^{\pm}) \subset \Sigma^{\pm} \times \Sigma^{\pm}}$:

Theorem [Radzikowski '96]

$\Lambda^{\pm}$ is unique modulo op. with $C^{\infty}(M \times M)$ Schwartz kernel.

Proof. If $\widetilde{\Lambda}^{\pm}$ also Hadamard two-point functions, $\Lambda^{+} - \Lambda^{-} = \tilde{\Lambda}^{+} - \tilde{\Lambda}^{-}$, hence $\Lambda^{+} - \tilde{\Lambda}^{+} = \Lambda^{-} - \tilde{\Lambda}^{-}$. These have disjoint wave front sets. Hence $\mathrm{WF}(\Lambda^{\pm} - \widetilde{\Lambda}^{\pm})' = \emptyset$.

Consequences of Hadamard condition $\boxed{\mathrm{WF}'(\Lambda^{\pm}) \subset \Sigma^{\pm} \times \Sigma^{\pm}}$:

We can define non-linear quantities, e.g. Wick square:

$$:\phi^2(x): = \lim_{x \to x'} \phi(x)\phi(x') - \text{parametrix for } \Lambda^{+}(x, x'),$$

similarly quantum stress-energy tensor $:T_{\mu\nu}:$.

So, *"not Hadamard" means infinite energy*. Example:

Theorem Chronology Protection Theorem [Kay, Radzikowski, Wald '96]

If $\Lambda^{\pm}$ Hadamard, $:\phi^2(x):$ blows up at any compactly generated Cauchy horizon.

[Duistermaat, Hörmander '72] gives $\widetilde{\Lambda}^{\pm}$ satisfying $\mathrm{WF}'(\widetilde{\Lambda}^{\pm}) \subset \Sigma^{\pm} \times \Sigma^{\pm}$, $P\widetilde{\Lambda}^{\pm} = \widetilde{\Lambda}^{\pm} P = 0$, $\widetilde{\Lambda}^{\pm} \geq 0$ and $\widetilde{\Lambda}^{+} - \widetilde{\Lambda}^{-} = i(P_{+}^{-1} - P_{-}^{-1})$ <u>modulo C^{∞}</u>

Theorem [Fulling, Narcowich, Wald '79]

Existence of Hadamard states on any globally hyperbolic (M, g).

Also, existence on AdS spacetimes for $\mathrm{WF_b}$ instead of WF [W. '17] (using [Vasy '12]), [Gannot, W. '19]

Theorem [Junker '96], [Junker, Schrohe '02], [Gérard, W. '14]

Hadamard states with Cauchy data in $\Psi(\mathbb{R}^{n-1}) \otimes L(\mathbb{C}^2)$ (more generally, in *bounded geometry* calculus [Gérard, Oulghazi, W. '17])

Especially interesting for wave equation ($m = 0$): note $\sqrt{-\Delta} \notin \Psi(\mathbb{R}^{d-1})$.

Conjecture ✽

Existence of Hadamard states for linearized Einstein equations (at least for perturbations of Minkowski space?)

Maxwell fields: [Furlani '95], [Fewster, Hunt '03], [Dappiaggi, Siemssen '13], [Finster, Strohmaier '15]; Linearized Yang-mills: [Hollands '08] (around 0), [Gérard, W. '15], [Zahn, W. '17]; Linearized gravity: [Benini, Dappiaggi, Murro '14] (problematic topological restrictions)

On real analytic spacetimes:

Theorem [Strohmaier, Verch, Wollenberg '02]

analytic Hadamard condition $\Rightarrow$ **Reeh-Schlieder effect**, i.e.

$$\operatorname{span}\left\{ \prod_{i=1}^{p} \phi[u_i]\Omega : \ p \in \mathbb{N}, \ u_i \in C_{\rm c}^\infty(O) \right\}$$

dense in $\mathscr{H}$ for <u>any</u> open $C_{\rm c}^\infty(O)$.

The general mechanism is that unique continuation theorems for (non-smooth) solutions of $Pu = 0$ imply local-to-global phenomena [Dybalski, W. '19].

Theorem [Gérard, W. '19]

Existence of analytic Hadamard states on any analytic globally hyperbolic (M, g).

Quantum effects induced by geometry

Fundamental principles: **finite speed of propagation**, **causality** and **universal short-distance behaviour**. Unexpectedly, however:

Unruh effect.

Hawking radiation on black hole spacetimes.

Particle creation in scattering situations.

packets on the horizon at late times. If one makes up wave-packets $\{q_{jn}\}$ like the $\{p_{jn}\}$, one finds that a fraction Γ_{jn} penetrates through the potential barrier around the black hole and gets out to $\mathscr{I}^-$ with the same frequency ω that it had on the horizon. This produces a $\delta(\omega-\omega')$ behaviour in $\gamma_{jn\omega'}$. The remaining fraction $1-\Gamma_{jn}$ of the wave-packet is reflected back by the potential barrier and passes through the collapsing body and out onto $\mathscr{I}^-$. Here it will have a similar form to $p^{(2)}_{jn}$. Thus for large ω',

$$|\gamma^{(2)}_{jn\omega'}| = \exp(\pi\omega\kappa^{-1})|\eta^{(2)}_{jn\omega'}| . \tag{4.3}$$

By a similar argument to that used in Section (2) one would conclude that the number of particles crossing the event horizon in a wave-packet mode peaked at late times would be

$$(1-\Gamma_{jn})\{\exp(2\pi\omega\kappa^{-1})-1\}^{-1} . \tag{4.4}$$

For a given frequency ω, i.e. a given value of j, the absorption fraction Γ_{jn} goes to zero as the angular quantum number l increases because of the centrifugal barrier. Thus at first sight it might seem that each wave-packet mode of high l value would contain

$$\{\exp(2\pi\omega\kappa^{-1})-1\}^{-1}$$

particles and that the total rate of particles and energy crossing the event horizon would be infinite. This calculation would, of course, be inconsistent with the

On Cauchy data (ϕ_0, ϕ_1), hermitian form $\langle \phi_0, \phi_1 \rangle_{L^2(S)} - \langle \phi_1, \phi_0 \rangle_{L^2(S)}$ preserved by evolution.

If $C^{\pm} : C^{\infty}_{\rm c}(S)^2 \to C^{\infty}(S)^2$ satisfy $C^{\pm} \geq 0$, $C^+ + C^- = \mathbf{1}$, then $\Lambda^{\pm} = \pm U^* C^{\pm} U$ are two-point functions, where U maps Cauchy data to solutions.

Now suppose ∂_t is a time-like Killing vector field, and rewrite $Pu = 0$ as

$$i^{-1}\partial_t \psi(t) = H\psi(t), \quad \psi(t) = \begin{pmatrix} u(t) \\ i^{-1}\partial_t u(t) \end{pmatrix};$$

for instance $H = \begin{pmatrix} 0 & 1 \\ \sqrt{-\Delta + m^2} & 0 \end{pmatrix}$ on Minkowski.

The Cauchy data of $\Lambda^{\pm}_{\rm vac}$ are $C^{\pm}_{\rm vac} = \mathbf{1}_{\pm[0,\infty)}(H)$.

The Cauchy data of **thermal state** $\Lambda^{\pm}_{\beta}$ (at inverse temperature $\beta > 0$) are $C^{\pm}_{\beta} = (\mathbf{1} - e^{\mp \beta H})^{-1}$

Let $P = -\partial_s^2 + \Delta_y + m^2$ on Minkowski space.

The Killing vector ∂_s is time-like. But in $M_{\mathrm{I}} = \{y \geq |s|\}$, another time-like Killing vector: $X = y\partial_s + s\partial_y$.

Theorem [Fulling '73], [Davies '75], [Unruh '76]

Unruh effect: the vacuum state $\Lambda^{\pm}_{\mathrm{vac}}$ w.r.t. ∂_s restricts to a **thermal state** (with $\beta = 2\pi\alpha/\kappa$) w.r.t. X on M_{I}.

Proof. New coordinates:

$$U = s + y, \quad V = s - y \quad \text{on } M,$$
$$U = e^{\kappa u}, \quad V = e^{-\kappa v} \quad \text{on } M_{\mathrm{I}}.$$

Then $\partial_s = \partial_U + \partial_V$ and $X = \kappa(U\partial_U - V\partial_V) = \partial_u + \partial_v$.
The crucial identity is $\boxed{\mathbf{1}_{\mathbb{R}_+}(D_x) = \chi\big(\tfrac{1}{2}(xD_x + D_x x)\big),}$ where $\chi(\lambda) = (1 + e^{-2\pi\lambda})^{-1}$. Hence $\mathbf{1}_{\mathbb{R}^+}(D_U) = T \circ \chi(D_u) \circ T^{-1}$.

Schwarzschild (exterior of eternal black hole) (M_{I}, g):

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

Kruskal extension (M, g):

$$g = -\frac{16m^3}{r} e^{-r/2m}(dUdV + dVdU) + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

Time-like Killing vector $X = \partial_t$ in M_{I}. But ∂_t not time-like in whole M.

On Schwarzschild (exterior of eternal black hole), time-like Killing vector ∂_t. But ∂_t no longer time-like in whole Kruskal extension.

Geometry enforces **Hawking temperature** $T_{\mathrm{H}} = \frac{\kappa}{2\pi}$, where κ is the horizon's surface gravity: $\mathrm{grad}_g g(\partial_t, \partial_t) = -2\kappa\partial_t$

Theorem [Sanders '15], [Gérard '18] (solving conjecture by [Hartle, Hawking / Israel '76])

Existence of maximally symmetric Hadamard state (Hartle-Hawking-Israel state) on Kruskal extension of Schwarzschild. This state is thermal ($\beta = T_{\mathrm{H}}^{-1}$) w.r.t. ∂_t in exterior region .

Conjecture ❃

The HHI state is analytic Hadamard.

Known in exterior region [Strohmaier, Verch, Wollenberg '02].

On Kerr (exterior of rotating black hole), Killing vector ∂_t not everywhere time-like. Thermal state $\Lambda^{\pm}_{\beta}$ w.r.t. ∂_t not Hadamard. Same for $\partial_t + \Omega\partial_\varphi$.

Conjecture ❀❀ [Kay, Wald '91]

No maximally symmetric Hadamard state on ('Kruskal extension' of) Kerr spacetime.

Partial results [Kay, Wald '91], [Moretti, Pinamonti '12], [Lupo '18], [Pinamonti, Sanders, Verch '19]

Conjecture ❀❀❀ [Unruh '76] [Kay, Wald '91]

Existence of Hadamard state in *union of exterior and interior* of Kerr black hole, asymptotically thermal at past event horizon (Unruh state).

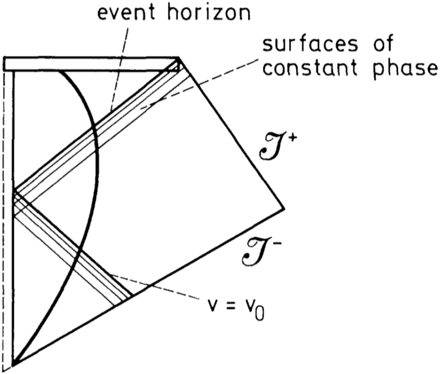
Hadamard states from scattering data: [Moretti '08], [Dappiaggi, Moretti, Pinamonti '09], [Gérard, W. '16], [Vasy, W. '18] using [Baskin, Vasy, Wunsch '15].

Conjecture for *Schwarzschild*: [Dappiaggi, Moretti, Pinamonti '11]

Conjecture for *massless Dirac*: [Gérard, Häfner, W. '19] using [Häfner, Nicolas '04]

Genericity of Hawking radiation spectrum: [Fredenhagen, Haag '90]

Scattering on Kerr(-de Sitter): [Häfner '03], [Vasy '13], [Georgescu, Gérard, Häfner '17] using [Dyatlov '11], [Dafermos, Rodnianski, Shlapentokh-Rothman '18]



More ‘phenomenological’ description of Hawking radiation:

[Bachelot '99], [Melnyk '03], [Häfner '09], [Gérard, Bouvier '14], [Drouot '17]

The case of Klein-Gordon fields on Kerr spacetime is *open*.

Towards semi-classical Quantum Gravity

Einstein equations $\delta(S + S_{\mathrm{matter}}) = 0$

where $S = \int_M (\mathrm{scal}_g + \Lambda) d\mathrm{vol}_g$ Einstein-Hilbert action.

For instance, if matter is a classical Klein-Gordon field,

$$S_{\mathrm{matter}} = \frac{1}{2} \int_M (\nabla_\mu \phi \nabla^\mu \phi + m^2 \phi^2) d\mathrm{vol}_g$$

Relevant term:

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu}(\nabla_\sigma \phi \nabla^\sigma \phi + m^2 \phi^2).$$

Naive derivation of **semi-classical Einstein equations**

$$e^{-S_{\mathrm{eff}}} = \int [d\phi] e^{-\frac{1}{2}\langle \phi, P\phi\rangle} = (\det P)^{-\frac{1}{2}} = e^{-\frac{1}{2}\mathrm{Tr}\log P},$$

$$\text{where } \log P = \lim_{\varepsilon\to 0^+}\left(-\int_{\varepsilon}^{\infty}\frac{e^{-sP}}{s} + (\gamma - \log\varepsilon)\mathrm{id}\right)$$

So one needs to renormalize:

$$S_{\mathrm{eff,ren}} \stackrel{\mathrm{def}}{=} \int_M L_{\mathrm{eff,ren}}(x)\, d\mathrm{vol}_g,$$

$$L_{\mathrm{eff,ren}}(x) = -\int_0^{\infty} ds \frac{1}{s}\frac{1}{(4\pi s)^2} e^{-m^2 s}\sum_{n=3}^{\infty}\alpha_n(x,x)s^n.$$

Quantum stress-energy tensor:

$$:T_{\mu\nu}(x): \stackrel{\mathrm{def}}{=} \frac{2}{\sqrt{|\det g|}}\frac{\delta L_{\mathrm{eff}}}{\delta g^{\mu\nu}} = K_{\mu\nu}[P^{-1}](x,x) - \text{ sing. part}$$

Theorem [Gell-Redman, Haber, Vasy '16], [Vasy '17]

If (M, g) asymptotically Minkowski (or actually globally hyperbolic, non-trapping Lorentzian scattering space), then $P = -\Box_g : \mathcal{X} \to \mathcal{Y}$ is **invertible**.

$$\mathcal{X} = \left\{ u \in H_{\mathrm{b}}^{s, n/2-1}(M) : \; Pu \in \mathcal{X} \right\}, \quad \mathcal{Y} = H_{\mathrm{b}}^{s-1, n/2+1}(M),$$

where $\pm s > \frac{1}{2}$ near sources/sinks (if 0 not a resonance).

Proof by radial estimates ([Melrose '94], [Hassell, Melrose, Vasy '04], [Vasy '13], [Datchev, Dyatlov '13], etc.) for P and $\widehat{N}(P)(\sigma)$ + extra commutator argument.

[Vasy, W. '18]

$$\mathrm{WF}(P_{\mathrm{F}}^{-1})' = (\mathrm{diag}_{T^*M}) \cup \bigcup_{t \leq 0} (\Phi_t(\mathrm{diag}_{T^*M}) \cap \pi^{-1}\Sigma)$$

Theorem [Gérard, W. '16-'19] cf. [Bär, Strohmaier '15]

If (M, g) asymptotically Minkowski, then $P = -\Box_g + m^2 : \mathcal{X} \to \mathcal{Y}$ is **invertible**. Here, $\mathcal{X}, \mathcal{Y}$ defined using spectral projectors $\mathbf{1}_{\pm[0,\infty)}(H)$ at $t = \pm\infty$.

Proof provides 'semi-group generator' $B(t) \in C^\infty(\mathbb{R}_t; \Psi^1(M))$, s.t. $P = (D_t - B(t))(D_t + B^*(t)) \mod \Psi_{\mathrm{sc}}^{-\infty,-1-\delta}(M)$ and $B(t) - \sqrt{-\Delta + m^2} \in \Psi_{\mathrm{sc}}^{1,-\delta}(M)$.

Furthermore,

$$\mathrm{WF}(P_{\mathrm{F}}^{-1})' = (\mathrm{diag}_{T^*M}) \cup \textstyle\bigcup_{t\leq 0}(\Phi_t(\mathrm{diag}_{T^*M}) \cap \pi^{-1}\Sigma)$$

Dereziński conjectured:

If (M, g) asymptotically static then $\Box_g$ is essentially self-adjoint on $C_c^\infty(M)$ and limiting absorption principle

$$(\Box - m^2)_{\mathrm{F}}^{-1} = \lim_{\varepsilon \to 0^+} (\Box_g - m^2 - i\varepsilon)^{-1}.$$

Theorem [Dereziński, Siemssen '18]

If (M, g) static (i.e. $g = -dt^2 + h$) then conjecture is true. Convergence in the sense of strong operator limit $\langle t \rangle^{-s} L^2(M) \to \langle t \rangle^{s} L^2(M)$, $s > \frac{1}{2}$.

The *scattering cotangent bundle* ${}^{\mathrm{sc}}T^*M$ has local basis:

$$\frac{d\rho}{\rho^2}, \frac{dy_1}{\rho}, \ldots, \frac{dy_{n-1}}{\rho}.$$

A Lorentzian sc-metric g is a section of ${}^{\mathrm{sc}}T^*M \otimes_{\mathrm{s}} {}^{\mathrm{sc}}T^*M$ of Lorentzian signature.

Non-trapping if null bicharacteristics at fiber infinity go from *sources* $L^- \subset \overline{{}^{\mathrm{sc}}T}^*_{\partial M}M$ to *sinks* $L^+ \subset \overline{{}^{\mathrm{sc}}T}^*_{\partial M}M$ (or stay within).

Theorem [Vasy '17]

Suppose (M, g) is a non-trapping Lorentzian sc-metric. Then $\Box_g$ is essentially self-adjoint on $C_c^\infty(M)$. Limiting absorption principle for $\Box_g - m^2$ if non-trapping at energy m^2 and Feynman problem invertible.

A few other recent developments in QFT on curved spacetimes:

Perturbative non-linear fields. [Dang '16-'19], [Brouder, Dang, Hélein '16], [Dang, Zhang '17], [Bahns, Rejzner '18], [Dang, Herscovich '19] ; Entanglement entropy. [Witten '18], [Longo, Xu '18], [Hollands '19]; Quantum Energy Inequalities. [Fewster '17], [Fewster, Kontou '19]; Expansions at de Sitter boundary. [Hollands '16], [Vasy, W. '18] [W. '19]; Relationship between fields and geometry. [Dang '19], cf. [Strohmaier, Zelditch '18]

Further reading:

- S. A. Fulling: *Aspects of Quantum Field Theory in Curved Spacetime* (1989)
- R. M. Wald: *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (1994)
- S. Hollands, R. M. Wald: *Quantum Fields in Curved Spacetime* `arXiv:1401.2026` (2014)
- C. Gérard: *Microlocal Analysis of Quantum Fields on Curved Spacetimes* `arXiv:1901.10175` (2019)