Semiclassical propagation estimates

Reference with details:
S. Dyatlov & M. Zworski,
"Mathematical Theory of Scattering Resonances",
AMS, 2019, Appendix E.4

(I.) Quick review of notation: ← PREPARE IN ADVANCE

M compact mfld w/o boundary

$\Psi_h^k(M)$ – the algebra of semiclassical pseudodifferential operators of order k

They have the form $\mathrm{Op}_h(a)$
where $a \in S^k(T^*M)$

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \leq C_{\alpha\beta} \langle \xi \rangle^{k-|\beta|}$$

$$\mathrm{Op}_h(a) = a(x, hD_x), \quad D_x = \frac{1}{i}\partial_x$$

(on manifolds, non-canonically defined by piecing together from local charts)

Principal symbol:

$$A \in \Psi_h^k(M) \mapsto \sigma_h(A) \in \frac{S^k(T^*M)}{hS^{k-1}(T^*M)}$$

$H_h^s(M)$ – semiclassical Sobolev space

$\sigma_h(\mathrm{Op}_h(a)) = [a]$

<u>Elliptic set:</u>

$\mathrm{ell}_h(A) \subset \overline{T}^*M$ open

$\mathrm{ell}_h(A) = \{\sigma_h(A) \neq 0\}$

<u>Wavefront set:</u>

$\mathrm{WF}_h(A) \subset \overline{T}^*M$ closed

$\mathrm{WF}_h(\mathrm{Op}_h(a)) = \text{ess-supp}\, a$

↑ places where $a \neq O(h^\infty \langle \xi \rangle^{-\infty})$

Here $\overline{T}^*M$ is the fiber-radially compactified T^*M, adding S^*M at $|\xi| = \infty$ with $|\xi|^{-1}$ as a b.d.f.

(II.) Propagation of singularities

Setting:

$$(P - iQ)u = f, \quad u, f \in C^\infty(M)$$

$$P \in \Psi_h^1(M),\ Q \in h\Psi_h^0(M),\ P^* = P,\ Q^* = Q$$

$p = \sigma_h(P)$ h-independent, real-valued homogeneous of order 1 for $|\xi| > 1$

$e^{tH_p} : \overline{T}^*M \to \overline{T}^*M$ Hamiltonian flow of p

Theorem [Propagation of singularities]

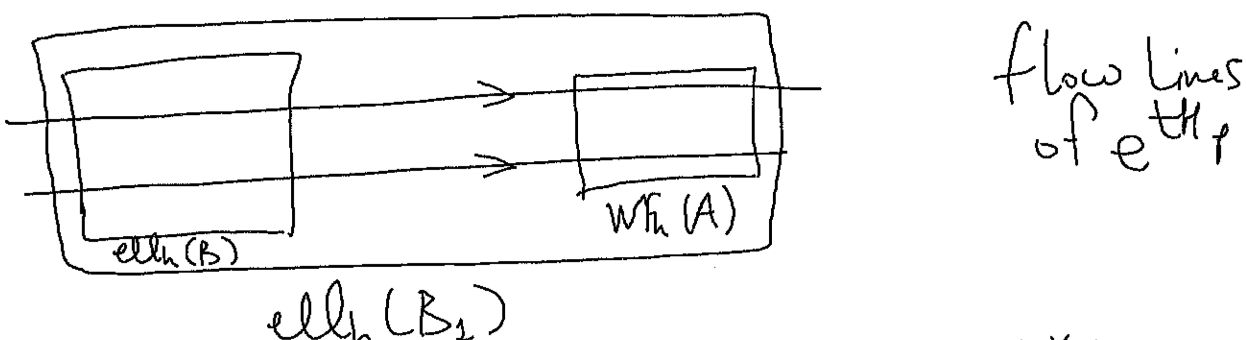
Assume $A, B, B_1 \in \Psi_h^0(M)$ satisfy the <u>control condition</u>:

$\forall \rho \in WF_h(A)\ \exists T \in \mathbb{R}$ such that

- $e^{-TH_p}(\rho) \in ell_h(B)$, and
- $e^{-tH_p}(\rho) \in ell_h(B_1)\ \forall t$ between 0 & T.

Then we have $\forall s, N\ \exists C$

$$(*)\quad \|Au\|_{H_h^s} \leq C\|Bu\|_{H_h^s} + Ch^{-1}\|B_1 f\|_{H_h^s} + Ch^N\|u\|_{H_h^{-N}}.$$



<u>Remarks.</u>

(1) If $f = 0$, this says that $WF_h(u) \subset \overline{T}^*M$ invariant under e^{tH_p}
"Singularities propagate along Hamiltonian flow lines"

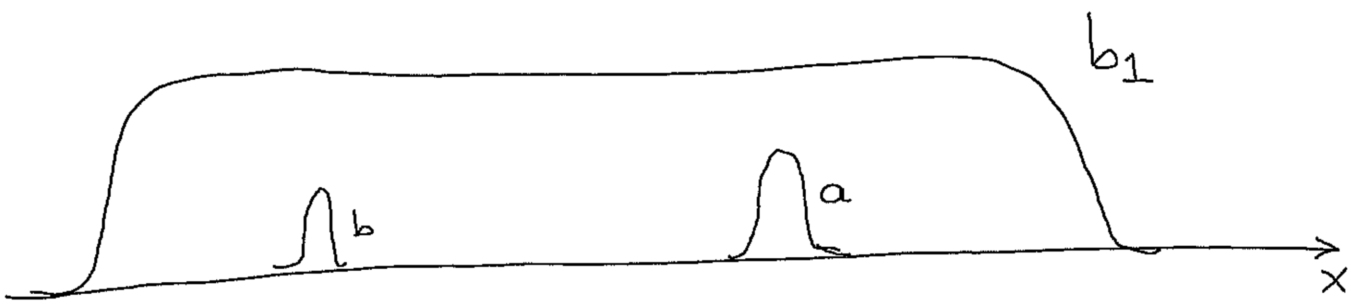
(2) Stronger versions available:
- don't need $u \in C^\infty$
- Can take $Q \in \Psi_h^1$, $\sigma_h(Q) \geq 0$, as long as $T \geq 0$ above.

(III) Proof of PofS in the model case

- $M = \mathbb{R}$ (noncompact but it does not matter)
- $P = hD_x$, $p(x,\xi) = \xi$, $e^{tH_p}(x,\xi) = (x+t,\xi)$
- $Q = hq$, $q \in C_c^\infty(\mathbb{R})$

$(P - iQ)u = f \iff (-ih\partial_x - ihq)u = f$

- $A, B, B_1 \to$ multiplication operators by $a, b, b_1 \in C_c^\infty(\mathbb{R})$
- Assume control condition with $T \geq 0$



- Get the estimate without remainder (just put $s=0$)

$(*)'$ $\|au\|_{L^2} \leq C\|bu\|_{L^2} + Ch^{-1}\|b_1 f\|_{L^2}$.

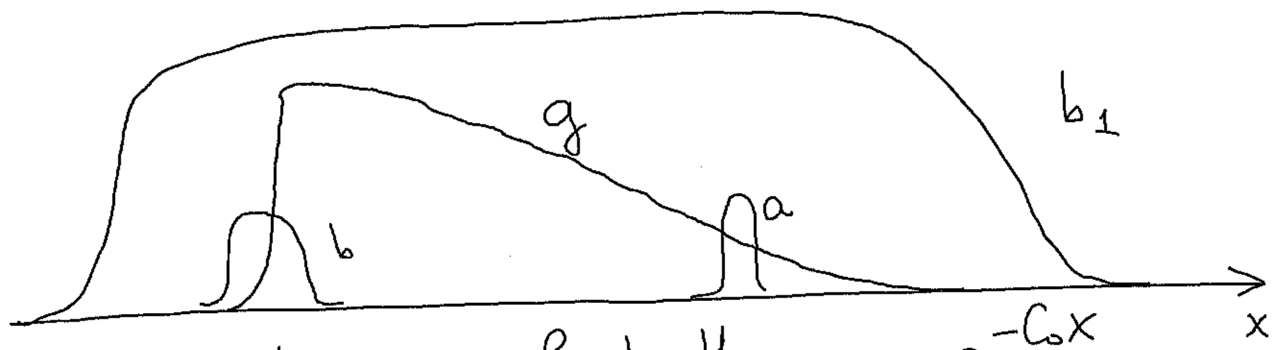
We will use the positive commutator method
(exercise: prove using ODE / FTC)

Step 1: Construction of escape function.

Fix $C_0 > 0$ to be chosen later & take $g \in C_c^\infty(\mathbb{R})$, $g \geq 0$,

(1) $g > 0$ on supp a
(2) $g' = \partial_x g \leq -C_0 g$ near $\{b = 0\}$
(3) supp $g \subset \{b_1 \neq 0\}$

Existence of such g follows from control condition:



Note: can take $g' \leq 0$ first, then $g \cdot e^{-C_0 x}$.

Step 2: the commutator identity

$$\underbrace{\operatorname{Im}\langle f, g^2 u\rangle}_{\text{I}} = \operatorname{Im}\langle (P - iQ)u, g^2 u\rangle = \underbrace{\operatorname{Im}\langle Pu, g^2 u\rangle}_{\text{II}} + \underbrace{-\operatorname{Re}\langle Qu, g^2 u\rangle}_{\text{III}}$$

And
$$\text{II} = \frac{1}{2i}\left(\langle Pu, g^2 u\rangle - \langle g^2 u, Pu\rangle\right) \quad \leftarrow \text{used } P^* = P$$
$$= \frac{1}{2i}\left(\langle g^2 Pu, u\rangle - \langle P g^2 u, u\rangle\right)$$
$$= \frac{i}{2}\langle [P, g^2]u, u\rangle = h\langle g g' u, u\rangle$$

Since $[P, g^2] = -ih[\partial_x, g^2] = -2ihgg'$

Step 3: estimate the terms

$|\text{I}| \leq C\|b_1 f\| \cdot \|gu\|$ (used (3))

$\text{II} \leq -C_0 h\|gu\|^2 + Ch\|bu\|^2$ (used (2))

$\text{III} \leq C_1 h\|gu\|^2$ where $C_1 := \sup|q|$

Put $C_0 := C_1 + 1$, then get

$$h\|gu\|^2 \leq Ch\|bu\|^2 + C\|b_1 f\| \cdot \|gu\|$$

This gives

$$\|gu\| \le C\|bu\| + Ch^{-1}\|b_1 f\|$$

It remains to note that

$$\|au\| \le C\|gu\| \quad \text{(used ①)}$$

to get $(*)'$. $\square$

How to deal with <u>general case?</u>

- Similar proof but now $g^2 \to G^*G$,

  $G = Op_h(g)$,

  $\text{(II)} = \frac{i}{2}\langle [P, G^*G]u, u\rangle$, $[P, G^*G] = -2ih\, Op_h(gH_p g) + O(h^2)_{\Psi_h^{-1}}$

  by the Commutator Rule
- Construct g so that $H_p g \le -C_0 g$ near $\{\sigma_h(B) = 0\}$

  Use this inequality + <u>Sharp Gårding Inequality</u>
- Get a $Ch^{1/2}\|u\|_{H_h^{-1/2}}$ remainder – remove iteratively
- How to get an estimate on H_h^s? Enough to get an L^2 estimate ~~for~~

  $(P - iQ)_s = \langle hD\rangle^s (P - iQ)\langle hD\rangle^{-s}$, here

  $\langle hD\rangle^{\pm s} = Op_h(\langle \xi\rangle^{\pm s}) + O(h)_{\Psi_h^{\pm s - 1}}$

Compute

$$P_s = P + [\langle hD\rangle^s, P]\langle hD\rangle^{-s}$$
$$= P + ihs\,Op_h(\langle\xi\rangle^{-1}H_p\langle\xi\rangle) + h^2\Psi_h^{-1}$$
$$Q_s = Q + h^2\Psi_h^{-1}$$

So $(P-iQ)_s = P - i\tilde{Q} + h^2\Psi_h^{-1}$

where $\tilde{Q} = Q - s\,Op_h(\langle\xi\rangle^{-1}H_p\langle\xi\rangle)$

to which the previous proof applies

(IV) Radial estimates

Theorem [High regularity radial estimate]

Assume P, Q as before, $f = (P-iQ)u$, and

$L \subset \partial \overline{T}^*M$ ← fiber infinity "$\{|\xi| = \infty\}$" cpct set

is a radial source for e^{tH_p}:

- L is e^{tH_p}-invariant
- $\exists$ nbhd U of L in $\overline{T}^*M$: $\forall \rho \in U$, $e^{-tH_p}(\rho) \xrightarrow[t\to\infty]{} L$ uniformly in ρ
- and $|\xi| \to \infty$ exponentially fast on the above trajectories

Then $\exists s_0$ (depending on P, Q) s.t. $\forall B_1 \in \Psi_h^0$ s.t. $L \subset ell_h(B_1)$ $\forall s > s_0$ $(*)$ holds with $B=0$: $\exists A \in \Psi_h^0$, $L \subset ell_h(A)$ and

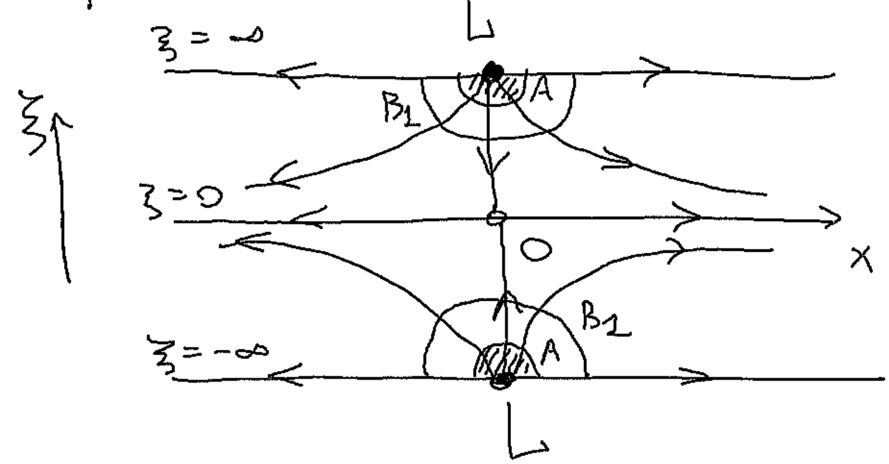
$$(*)_{rad} \quad \|Au\|_{H_h^s} \le Ch^{-1}\|B_1 f\|_{H_h^s} + Ch^N\|u\|_{H_h^{-N}} \quad \forall u \in C^\infty$$

Example: on $M = \mathbb{R}$

$P = x \cdot hD_x - \frac{ih}{2}$ ← to make $P^* = P$

$Q = hq$, $q = \text{const}$

$p = x \cdot \xi$, $H_p = x\partial_x - \xi\partial_\xi$, $L = \{x = 0, \xi = \pm\infty\}$



$x\partial_x + \frac{1}{2} + q$

Special case: $s = 0$, $q > 0$ Then we can take

$A, B_1 \to a, b_1$ multiplication operators

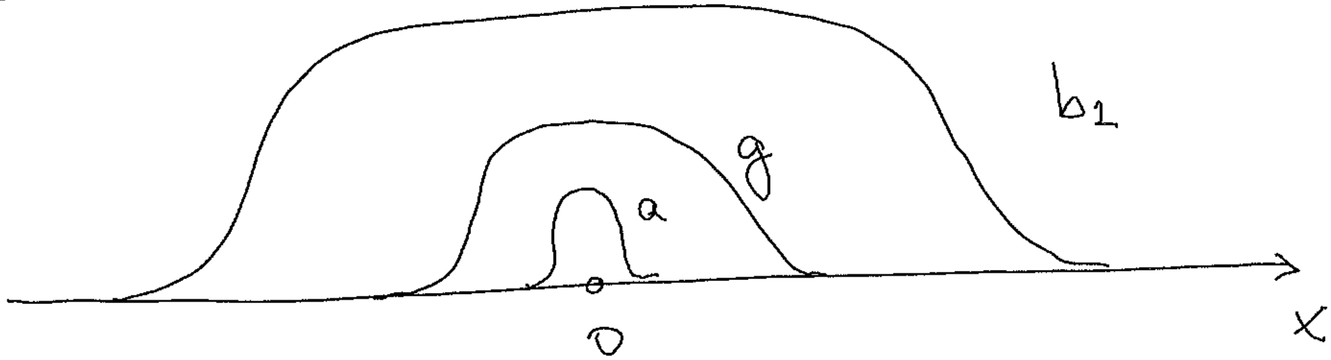
& we set

$(*)'$ $\|au\|_{L^2} \leq C h^{-1} \|b_1 f\|_{L^2}$

Note: $u = x_\pm^{-\frac{1}{2}-q}$ solves $(P - iQ)u = 0$

But $(*)'$ does not apply since $u \notin C^\infty$

Proof of special case: argue as for PofS, with



Now (I) estimated as before,

$$\text{(II)} = h\langle x g g' u, u\rangle \le 0$$

since $xgg' \le 0$

And $\text{(III)} = -hq\|gu\|^2$ and $q > 0$,

So we have

$\text{(II)} + \text{(III)} \le -\frac{h}{c}\|gu\|^2$, the argument still works. $\square$

This special case gives a proof for general P as long as $S=0$,

$$\sigma_h(Q)|_L < 0.$$

Now, proving an estimate on H_h^s is like proving an L^2 estimate for

$$P_s - iQ_s = P - i\tilde{Q} + \cdots,$$

$$\tilde{Q} = Q - s\,Op_h(\langle\xi\rangle^{-1}H_p\langle\xi\rangle).$$

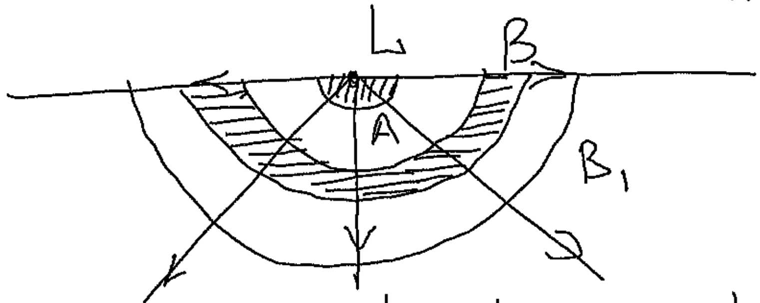
Because L is a radial source, we can arrange $\langle\xi\rangle^{-1}H_p\xi|_L < 0$ (e.g. $p = x\cdot\xi$ model case $|\xi|^{-1}H_p|\xi| = -1$)

So for s large enough, we get $\sigma_h(\tilde{Q})|_L < 0$. This gives the general case

Theorem [Low regularity radial estimate]

Assume P, Q, u, f, L, B_1 as in the high regularity radial estimate. Then $\exists s_1$ (depending on P, Q) and $\exists A, B \in \Psi_h^0$: $L \subset \mathrm{ell}_h(A)$, $WF_h(B) \subset \mathrm{ell}_h(B_1) \setminus L$, and (*) holds when $s < s_1$: $\forall u \in C^\infty$

$$\|Au\|_{H_h^s} \le C\|Bu\|_{H_h^s} + Ch^{-1}\|B_1 f\|_{H_h^s} + Ch^N\|u\|_{H_h^{-N}}.$$



(for the proof see the back)

Note: radial estimates apply also to sinks: L a sink for $p \Leftrightarrow L$ a source for $-p$

Note: the above estimates apply under weaker regularity assumptions:

- PofS & LowRegRad: enough $u \in \mathcal{D}'$, $Bu \in H_h^s$, $B_1 f \in H_h^s$, and we get $Au \in H_h^s$
- HighRegRad: enough $u \in H^{s_0+}$, $B_1 f \in H_h^s$, and we get $Au \in H_h^s$.