Introduction to Fourier Integral Operators - Lecture 3

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Introductory Workshop

MSRI Microlocal Analysis Program

Overview

1. Symbol calculus
2. Functional and composition calculus
3. Examples and applications
4. Extensions and generalizations of FIO calculus
5. Readings for all three lectures

Symbol calculus of Fourier integral distributions

For $\Lambda \subset T^*X^n \setminus \mathbf{0}$ a smooth conic Lagrangian,

$$I^m(X;\Lambda) = \text{ all locally finite sums of } u \in \mathcal{D}'(X)$$

given by oscillatory integrals

$$u = u(a,\phi) := \int_{\mathbb{R}^N} e^{i\phi(x,\theta)}\, a(x,\theta)\, d\theta, \quad a \in S_{1,0}^{m-\frac{N}{2}+\frac{n}{4}}$$

with $\phi(x,\theta)$ a nondegenerate phase on $X^n \times (\mathbb{R}^N \setminus 0)$

$$\rightsquigarrow Crit_\phi := \{(x,\theta) : d_\theta\phi(x,\theta) = 0\}$$

$$\rightsquigarrow \Lambda_\phi := \left\{(x, d_x\phi) : (x,\theta) \in Crit_\phi\right\} \subset \Lambda.$$

Symbol calculus of Fourier integral distributions

- Define n-form μ_ϕ on $Crit_\phi$ by requiring

$$\mu_\phi \wedge d(\frac{\partial \phi}{\partial \theta_1}) \cdots \wedge d(\frac{\partial \phi}{\partial \theta_N}) = dx_1 \cdots \wedge dx_n \wedge d\theta_1 \cdots \wedge \theta_N$$

- If λ_i are local coord on $Crit_\phi$ then $\mu_\phi = f d\lambda_1 \cdots \wedge d\lambda_n$, with

$$f = \frac{dx_1 \cdots \wedge dx_n \wedge d\theta_1 \cdots \wedge \theta_N}{d\lambda_1 \cdots \wedge d\lambda_n \wedge d(\frac{\partial \phi}{\partial \theta_1}) \cdots \wedge d(\frac{\partial \phi}{\partial \theta_n})}$$

- To obtain an invariantly defined principal symbol, $\sigma_{prin}(u)$, if

$$a^0 := \left[a|_{Crit_\phi} \right] \in S_{1,0}^{m - \frac{N}{2} + \frac{n}{4}} / S_{1,0}^{m - \frac{N}{2} + \frac{n}{4} - 1},$$

Def. The **principal symbol** $\sigma_{prin}(u)$ of $u(a, \phi)$ is the push-forward of the half-density $a^0 \sqrt{\mu_\phi}$ from $Crit_\phi$ to Λ.

Functional calculus: Adjoints

Suppose $A \in I^m(C; X, Y)$. What about (formal) A^*? If

$$K_A(x,y) = \int_{\mathbb{R}^N} e^{i\phi(x,y,\theta)}\, a(x,y,\theta)\, d\theta, \quad a \in S^{m-\frac{N}{2}+\frac{n_X+n_Y}{4}},$$

then

$$K_{A^*}(y,x) = \overline{K_A(x,y)}$$

$$= \int_{\mathbb{R}^N} e^{-i\phi(x,y,\theta)}\, \overline{a}(x,y,\theta)\, d\theta, \quad \overline{a} \in S^{m-\frac{N}{2}+\frac{n_X+n_Y}{4}}$$

$\implies A^* \in I^m(C^t; Y, X)$, where C^t is the transpose relation.

Composition calculus

Suppose $A_1 \in I^{m_1}(C_1; X, Y)$, $A_2 \in I^{m_2}(C_2; Y, Z)$ are properly supported.

- **Q.** Is $A_1 A_2$ an FIO? **No** in general, but **yes** if we impose some geometric conditions.

- Note

$$WF_{A_1 A_2} \subseteq WF_{A_1} \circ WF_{A_2} = WK(K_{A_1})' \circ WF(K_{A_2})'$$

$$\subseteq C_1 \circ C_2 \subset (T^*X \setminus \mathbf{0}) \times (T^*Z \setminus \mathbf{0})$$

  Basic examples show $C_1 \circ C_2$ need not be a smooth canonical relation. However, under a **transversality** or **clean intersection** condition, it is, and the operator theory follows the geometry.

Transverse intersection

- **Def.** $S_1, S_2 \subset M$ intersect **transversally** if $T_m S_1 + T_m S_2 = T_m M$ for all $m \in S_1 \cap S_2$. (Holds $\iff N^*_m S_1 \cap N^*_m S_2 = (0)$.)
  Write $S_1 \overline{\pitchfork} S_2$.

- **Prop.** If $S_1 \overline{\pitchfork} S_2$, then

  (i) $S_3 := S_1 \cap S_2$ is smooth;

  (ii) codim(S_3) =codim(S_1)+codim(S_2); and

  (iii) $TS_3 = TS_1 \cap TS_2$ at all points.

- **Ex.** In $\mathbb{R}^3$: $\{z = 0\} \overline{\pitchfork} \{z = x\}$, but $\{z = 0\} \not\overline{\pitchfork} \{z = xy\}$.

Transverse intersection

- For $C_1 \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$ and $C_2 \subset (T^*Y \setminus \mathbf{0}) \times (T^*Z \setminus \mathbf{0})$,

$$C_1 \circ C_2 = \{(x,\xi,z,\zeta) : \exists (y,\eta)\, s.t.\, (x,\xi,y,\eta) \in C_1,\ (y,\eta,z,\zeta) \in C_2\}$$
$$= (\pi_1 \times \pi_4)\big((C_1 \times C_2) \cap (T^*X \times \Delta_{T^*Y} \times T^*Z)\big)$$

- To have a chance of $A_1 A_2$ being an FIO associated with a smooth canonical relation, need that the intersection set be **smooth.**
- One way to get this is to demand that the intersection be **transverse.**

Transverse intersection calculus

- **Thm. (Hörmander)** Suppose $A_1 \in I^{m_1}(C_1; X, Y)$, $A_2 \in I^{m_2}(C_2; Y, Z)$ are properly supported. If $C_1 \times C_2$ intersects $T^*X \times \Delta_{T^*Y} \times T^*Z$ transversally, then $C_1 \circ C_2$ is a smooth canonical relation and

$$A_1 A_2 \in I^{m_1+m_2}(C_1 \circ C_2; X, Z)$$

- If either C_1 **or** C_2 is a local canonical graph, then $A_1 A_2$ is covered by the $\overline{\pitchfork}$ calculus.

- In particular, $I^m(C; X, Y)$ is closed under composition on the right with $\Psi^0(Y)$ and on the left with $\Psi^0(X)$.

- If C is a canonical graph and $A \in I^m(C; X, Y)$ is properly supported, then $A^*A \in \Psi^{2m}(Y)$, and A elliptic at $(x_0, \xi_0, y_0, \eta_0) \implies A^*A$ elliptic at (y_0, η_0).

Clean intersection calculus

- **Def.** $S_1, S_2 \subset M$ intersect **cleanly** if (i) $S_3 := S_1 \cap S_2$ is smooth; and $TS_3 = TS_1 \cap TS_2$ at all points. The **excess** of the intersection is $e := \text{codim}(S_1) + \text{codim}(S_2) - \text{codim}(S_3) \geq 0$.

  Ex. $S_1 = x$-axis and $S_2 = y$-axis in $\mathbb{R}^3$, with excess $e = 2 + 2 - 3 = 1$.

  Ex. $S_1 = x$-axis and $S_2 = \{y = x^2\}$ do not intersect cleanly in $\mathbb{R}^2$.

- **Thm. (Duistermaat-Guillemin; Weinstein)** If $C_1 \times C_2$ intersects $T^*X \times \Delta_{T^*Y} \times T^*Z$ cleanly with excess e, then $C_1 \circ C_2$ is smooth and

$$A_1 A_2 \in I^{m_1 + m_2 + \frac{e}{2}}(C_1 \circ C_2; X, Z).$$

Clean intersection calculus: flowouts

Ex. Let $\Sigma \subset T^*X^n \setminus \mathbf{0}$ be a conic hypersurface.

- Σ is automatically **co-isotropic:** $(T\Sigma)^\omega \subset T\Sigma$ at all pts.
- Microlocally, can write $\Sigma = \{p(x,\xi) = 0\}$, $p \in C^\infty_{\mathbb{R}}$, homog of deg 1.
- $(T\Sigma)^\omega = \mathbb{R} \cdot H_p$, where H_p is the **Hamiltonian vector field** of p,

$$H_p(x,\xi) = (dp(x,\xi))^\omega = \sum_{j=1}^n \frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}$$

- But $H_p \in T\Sigma$, since $\langle dp, H_p\rangle = \omega(H_p, H_p) = 0$ by skew-symmetry.
- Thus, Σ is foliated by the integral curves of H_p, called the **bicharacteristic curves** of Σ, which are **nonradial** if $H_p \not\parallel \xi \cdot \partial_\xi$. The curve passing through $(x,\xi) \in \Sigma$ is denoted $\Xi_{x,\xi}$.

Clean intersection calculus: flowouts

- **Def.** The **flowout relation** of Σ,

  $$C_\Sigma = \left\{(x,\xi,y,\eta) : (x,\xi) \in \Sigma,\ (y,\eta) \in \Xi_{x,\xi}\right\} \subset (T^*X\setminus\mathbf{0})\times(T^*X\setminus\mathbf{0})$$

  is a smooth, conic canonical relation.
  Note: C_Σ is degenerate. $D\pi_L$, $D\pi_R$ drop rank by 1 **everywhere**.

- $C_\Sigma \circ C_\Sigma$ covered by the clean intersection calc, with excess $e = 1$:

  $$I^{m_1}(C;X,X) \circ I^{m_2}(C;X,X) \subseteq I^{m_1+m_2+\frac{1}{2}}(C;X,X)$$

- Results in a loss of $1/2$ derivs on L^2-based Sobolev spaces:

  Thm. $I^m(C;X,X) : H^s_{comp}(X) \to H^{s-m-\frac{1}{2}}_{loc}(X)$.

Clean intersection calculus: flowouts

- Flowout relations C_Σ describe the propagation of singularities of solutions to $Pu = f$, where $P(x, D) \in \Psi^m_{cl}(X)$.
- **Def.** $P(x, D) \in \Psi_{cl}$ is of **real principal type** if $p(x, \xi) := \sigma_{prin}(P)$ is $\mathbb{R}$-valued, $d_{x,\xi}p \neq (0, 0)$ at $\Sigma = p^{-1}(0)$, and no bicharacteristic $\Xi_{x,\xi}$ of p is trapped over a compact set $K \subset\subset X$.
  (In particular, there are no radial points.)
- **Thm. (Duistermaat-Hörmander)** If $P(x, D)$ is RPT and $Pu = f$, then $WF(u) \setminus WF(f)$ is a union of maximally extended $\Xi_{x,\xi}$. Futhermore, there exists a two-sided parametrix Q, $QP = I - R_1$ and $PQ = I - R_2$ with $R_1, R_2 \in \Psi^{-\infty}(X)$, with $Q \in I^{\frac{1}{2}-m}(C_\Sigma)$ away from Δ_{T^*X}.

Applications: Egorov's Theorem

- Let $\Phi : T^*Y \setminus \mathbf{0} \to T^*X \setminus \mathbf{0}$ be a canonical transformation defined on a conic nhood of (y_0, η_0). Then $C := graph(\Phi)$ is a canonical graph.
- Let $F \in I^0(C; X, Y)$ be an elliptic FIO, and $G \in I^0(C^t; Y, X)$ a parametrix (microlocal inverse mod C^∞), with $C^t = graph(\Phi^{-1})$:

$$GF \equiv I \text{ and } FG \equiv I \mod C^\infty.$$

- **Thm. (Egorov)** If $P(x, D) \in \Psi^m(X)$, then $FPG \in \Psi^m(Y)$, with

$$\sigma_{prin}(FPG)(y, \eta) = \sigma_{prin}(P)(\Phi(y, \eta))$$

- $\implies$ Large literature on reducing ΨDO to normal forms, proving propagation of singularities or local solvability.

Applications: Generalized Radon transforms

Suppose $Z \subset X^{n_X} \times Y^{n_Y}$, codim k. Consider

$$\begin{array}{ccc} \pi_X & Z & \pi_Y \\ \swarrow & & \searrow \\ X & & Y \end{array}$$

- **Def.** Z is a **double fibration** if $\pi_X : Z \to X$ and $\pi_Y : Z \to Y$ are submersions. Then, $\forall x \in X$, $y \in Y$,

$$Y_x := \pi_Y \pi_X^{-1}(\{x\}) \subset Y \text{ and } X^y := \pi_X \pi_Y^{-1}(\{y\}) \subset X \text{ are codim } k$$

- Choice of smooth densities on X, Y, Z induces pair of **generalized Radon transforms**, $\mathcal{R} : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ and $\mathcal{R}^t : \mathcal{E}'(X) \to \mathcal{D}'(Y)$,

$$\mathcal{R}f(x) = \int_{Y_x} f(y)\, dy \text{ and } \mathcal{R}^t g(y) = \int_{X^y} f(x)\, dx.$$

Applications: Generalized Radon transforms

Z is the **incidence relation** of a **generalized Radon transform,** $\mathcal{R}$.

- **Guillemin-Sternberg:** Schwartz kernel of $\mathcal{R} = \delta_Z$, which is a conormal, hence Fourier integral distribution: Locally describe Z as

$$Z = \{(x,y) : \Phi_1(x,y) = \cdots = \Phi_k(x,y) = 0\}.$$

- Writing δ_Z as shorthand for a smooth multiple of $\delta_{\mathbb{R}^k}(\Phi)$,

$$\delta_Z(x,y) = \int_{\mathbb{R}^k} e^{i\sum_{j=1}^k \theta_j \Phi_j(x,y)}\, a(x,y)\, d\theta, \quad a \in S^0_{1,0}(X \times Y \times \mathbb{R}^k)$$

$\Longrightarrow$

$$\mathcal{R} \in I^{0+\frac{k}{2}-\frac{n_X+n_Y}{4}}(C; X, Y),$$

where

$$C = N^*Z' \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$$

and $\mathcal{R}^t \in I^{\frac{n_X+n_Y-2k}{4}}(C^t; Y, X)$.

Applications: Generalized Radon transforms

- If C is a canonical graph, then $\mathcal{R}^*\mathcal{R} \in \Psi^{\frac{n_X+n_Y-2k}{2}}(Y)$, elliptic if $\mathcal{R}$ is.
- $\exists$ parametrix $Q \in \Psi^{-\frac{n_X+n_Y-2k}{2}}(Y), \quad Q\mathcal{R}^*\mathcal{R} \equiv I \mod C^\infty$, and thus $\mathcal{R}f$ determines $f \mod C^\infty, \forall f \in \mathcal{E}'(Y)$.
- **Ex.** Radon transform: $Y = \mathbb{R}^n$, $X = \mathbb{S}^{n-1} \times \mathbb{R}$,

$$Z = \big\{(\omega, s, y) : s - \omega \cdot x = 0\big\}.$$

$\mathcal{R}^*\mathcal{R}f = c_n f * |y|^{1-n}$, which has inverse $c_n(-\Delta)^{\frac{n-1}{2}}$.

The **filtered backprojection** inversion formulae for the Radon transform,

$$f = c_n\big((-\Delta)^{\frac{n-1}{2}}\mathcal{R}^*\big)\mathcal{R}f = c_n\mathcal{R}^*(|\partial_s|^{n-1})\mathcal{R}f,$$

thus generalize ($\mod C^\infty$) to a wide variety of GRTs.

Applications: Generalized Radon transforms

- Suppose $n_X > n_Y$ ($\mathcal{R}$ is **overdetermined**).

$$\dim(T^*X) = 2n_X > \dim(C) = n_X + n_Y > \dim(T^*Y) = 2n_Y.$$

- Then $C = N^*Z'$ is nondegen, i.e., $\pi_L : C \to T^*X$ has maximal rank, iff $D\pi_L$ is injective.

- Clean intersection calculus applies to $\mathcal{R}^*\mathcal{R}$, with excess $e = n_X - n_Y$, but to make sure that $\mathcal{R}^*\mathcal{R}$ is only a ΨDO, need $C^t \circ C \subseteq \Delta_{T^*Y}$.

- **Def. (Guillemin)** $\mathcal{R}$ (or Z or C) satisfies the **Bolker condition** if, in addition to $D\pi_L : TC \to T(T^*X)$ being injective, the map $\pi_L : C \to T^*X$ is injective. I.e., not only is π_L infinitesimally 1-1, it is globally 1-1. (Makes sense for general canonical relations.)

Applications: Generalized Radon transforms

- **Thm. (Guillemin-Sternberg)** Suppose $C \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$ is a canonical relation satisfying the Bolker condition, and

$$F \in I^{m - \frac{n_X - n_Y}{4}}(C; X, Y)$$

is elliptic and properly supported. Then $F^*F \in \Psi^{2m}(Y)$, $+$elliptic.

Hence, u is determined $\mod C^\infty(Y)$ by $Fu \mod C^\infty(X)$, $\forall u \in \mathcal{E}'(Y)$.

- **Ex.** k-plane transform on $\mathbb{R}^n$: $\mathcal{R}_{k,n} \in I^{-\frac{k}{2} - \frac{(k+1)(n-k)-n}{4}}(C; M_{k,n}, \mathbb{R}^n)$

- **Ex.** (M^n, g) a Riemannian manifold without conjugate points has a $(2n-2)$-dimensional space $\mathcal{G}$ of geodesics. The **X-ray transform** on M, defined by $Xf(\gamma) = \int_\gamma f\, ds$, satisfies the Bolker condition, $X^*X \in \Psi^{-1}(M)$, and $Xf \mod C^\infty$ determines $f \mod C^\infty$.

An example where Bolker is violated

- X-ray transform on $(M, g) = (\mathbb{S}^n, g_0)$
- $X \in I^{-\frac{1}{2}-\frac{(2n-2)-n}{4}}(C; \mathcal{G}, M)$ with $C \subset (T^*\mathcal{G} \setminus \mathbf{0}) \times (T^*M \setminus \mathbf{0})$ is nondeg., but $\pi_L : C \to T^*\mathcal{G}$ is 2-1.
- Composition X^*X **is** covered by clean intersection calc, but

$$X^*X \in I^{-1}(\Delta) \cup I^{-1}(\Gamma)$$

where Γ is the graph of the canonical transf induced by antipodal map, and X has a large kernel (all odd distributions).

Paired Lagrangian distributions

- $\exists$ need for distributions [operators] whose wavefront sets [relations] are not a smooth Lagrangian [canonical relation]:
- Duistermaat-Hörmander constructed parametrices Q for RPT operators $P(x, D)$ have

$$WF_Q \subseteq \Delta_{T^*X} \cup C_\Sigma$$

where C_Σ is the flowout of Σ. $\Delta \cap C_\Sigma$ cleanly with excess $e = n - 1$.

- Each of Δ_{T^*X}, C_Σ is smooth, but their union is not, and K_Q is **not** simply a sum in $I^{m_1}(\Delta) + I^{m_2}(C_\Sigma)$.

Paired Lagrangian distributions

- Melrose-Uhlmann-Guillemin-Mendoza introduced classes of Lagrangian-like distributions associated with pairs $\Lambda_0, \Lambda_1 \subset T^*X \setminus \mathbf{0}$ which intersect cleanly in codimension $k = 1, 2, \dots$. Denoted

$$I^{p,l}(X; \Lambda_0, \Lambda_1), \quad p, l \in \mathbb{R}.$$

- Just as $u \in I^m(\Lambda)$ can be characterized either as oscillatory integrals or in terms of iterated regularity, $I^{p,l}$ can be characterized either as

  (i) oscillatory integrals with certain types of product type symbols; or

  (ii) distributions satisfying iterated regularity with respect to $P_j \in \Psi^1_{cl}$ with σ_{prin} vanishing on $\Lambda_0 \cup \Lambda_1$.

- If $u \in I^{p,l}(\Lambda_0, \Lambda_1)$ then microlocally away from $\Lambda_0 \cap \Lambda_1$,

$$u \in I^{p+l}(\Lambda_0 \setminus \Lambda_1) \text{ and } u \in I^p(\Lambda_1 \setminus \Lambda_0).$$

Paired Lagrangian operators

- If C_0, $C_1 \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$ are a cleanly intersecting pair, then

$$I^{p,l}(C_0, C_1; X, Y) = \text{ operators } T \text{ with } K_T \in I^{p,l}(C_0', C_1').$$

- When $Y = X$, $C_0 = \Delta_{T^*X}$: "ΨDO with singular symbols".
- $I^{p,l}$-operators arise in several applications:

  (i) Parametrices for RPT: $Q \in I^{\frac{1}{2}-m,-\frac{1}{2}}(\Delta, C_\Sigma)$ [Melrose-Uhlmann]

  (ii) Parametrices for restricted X-ray transforms [G.- Uhlmann];

  (iii) Linearized inverse probs for seismic, radar imaging [Nolan, Felea];

  (iv) Composing FIOs outside the clean intersection calculus [G.-Uhlmann, Felea].

Paired Lagrangian operators

- **Ex.** $x = (x', x'')$, $C_0 = \Delta_{T^*\mathbb{R}^n}$, $C_1 = N^*\{x' - y' = 0\}$
- **Def. 1.** $K(x,y) = \int e^{i((x'-y')\cdot\xi' + (x''-y'')\cdot\xi'')} a(x,\xi) d\xi' d\xi''$,

  $|\partial_x^\alpha \partial_{\xi'}^\beta \partial_{\xi''}^\gamma a| \leq c_{\alpha\beta\gamma}(1 + |\xi'| + |\xi''|)^{m-|\beta|}(1 + |\xi''|)^{m'-|\gamma|}$
- **Def. 2.** Iterated regularity: $u \in I^{p,l}(C_0, C_1)$ if $P_1 P_2 ... P_N K \in H^{s_0}_{loc}$ where $P_j \in \Psi^1_{cl}$ with $\sigma(P_j)$ vanishing on $C_0 \cup C_1$

Beyond the standard FIO calculus

- **Recall:** Melrose-Taylor Radon transform (T_5 from Lec. I), $\mathcal{R}_{MT} : \mathcal{D}'(\partial\Omega \times \mathbb{R}) \to \mathcal{D}'(\mathbb{S}^{n-1} \times \mathbb{R})$, given by

$$\mathcal{R}_{MT}(f)(\omega, t) = \mathcal{R}_{MT}(f)(\omega, t) := \int\int_{\{y \cdot \omega = t - s\} \subset \partial\Omega \times \mathbb{R}} f(y, s)$$

- $\mathcal{R}_{MT} \in I^{-(n-1)/2}(C)$, with C **not** a canonical graph. Both π_L, π_R have degeneracies of Whitney fold type. Such C called **folding canonical relations**. $T \in I(C)$ lose 1/6 deriv on L^2.

- M-T already observed that the composition $C^t \circ C$ is not a smooth canonical relation, but $\subset \Delta_{T^*(\partial\Omega \times \mathbb{R})} \cup C_1$, where C_1 intersects Δ cleanly in codim 1.

- **Thm. (Nolan-Felea).** If C is a folding canonical relation and $F \in I^m(C; X, Y)$ then $F^*F \in I^{2m,0}(\Delta_{T^*Y}, C_1)$.

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