MSRI LECTURE ON GENERAL RELATIVITY

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ABSTRACT. Rough notes for the lecture on general relativity at the MSRI introductory workshop in Fall 2019.

- Einstein Vacuum Equations: (EVE) $\mathrm{Ein}(g) + \Lambda g = 0$ where $\Lambda \in \mathbb{R}$, g a Lorentzian metric on M and $\mathrm{Ein}(g) = \mathrm{Ric}(g) - \frac{1}{2}\mathrm{scal}_g g$.
  - Upon taking traces, EVE is equivalent to $\mathrm{Ric}(g) - \Lambda g = 0$
- Examples $\Lambda = 0$:
  (1) Minkowski space: $M = \mathbb{R}_t \times \mathbb{R}^3_x$, $g_0 = -dt^2 + dx^2$
    - The local model of any Lorentzian spacetime
  (2) Schwarzschild black hole: $m > 0$ (mass of the black hole), $M = \mathbb{R}_t \times (2m, \infty)_r \times \mathbb{S}^2_\omega$ and $g_m = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2 g_{\mathbb{S}^2}$
    - Unique vacuum spherically symmetric family
    - The apparent singularity in the second term of the metric is only a coordinate singularity and can be removed by a change of coordinates
      * Light cones begin to tilt toward the black hole ($r = 0$) as they approach $r = 2m$
      * Light or massive observer must reach $r = 0$ in finite time once their distance from $r = 0$ is less than or equal to $2m$
    - Trapping occurs at $r = 3m$ (photosphere); in phase space T^*M, we have normally hyperbolic trapping
  (3) Kerr black hole: $m > 0$, $0 \leq a < m$, $g_{m,a} = \ldots$
    - In T^*M, there is normally hyperbolic trapping (Wunsch-Zworski, Dyatlov)
- Examples $\Lambda > 0$:
  (1) De-Sitter space: $g = \dfrac{-d\tau^2 + dy^2}{\tau^2}$ for $\Lambda = 3$
  (2) Schwarzschild de-Sitter:
    - The metric is $g_m = -(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3})dt^2 + (1 - \frac{2m}{r} - \frac{\Lambda r^2}{3})^{-1}dr^2 + r^2 g_{\mathbb{S}^2}$
    - Event horizon at $r \approx 2m$ and cosmological horizon and $r \approx \sqrt{\frac{3}{\Lambda}}$
  (3) Kerr de-Sitter
- The Initial Value Problem:
  - $\mathrm{Ric}(g)_{ij} = g^{kl}\partial_k\partial_l g_{ij} + g\partial^2 g + \mathrm{l.o.t.}$

  - One sees that the first term is $\Box_g g_{ij}$.

Theorem 1. *(Choquet-Bruhat) $\Lambda \in \mathbb{R}$. Given (Σ^3, γ) and Riemannian manifold, K a symmetric two-tensor on Σ satisfying the constraint equations, there exists (M^4, g) such that g satisfies the EVE and $\iota : \Sigma \hookrightarrow M$ such that $\gamma = \iota^* g$ and $K = II^g_{\iota(\Sigma)}$ which is unique up to a diffeomorphism.*

  - Proof sketch:
    * Take $\Lambda = 0$.
    * Guess $M = (\epsilon, \epsilon)_t \times \Sigma$
    * Solve $\mathrm{Ric}(g) = 0$ along with $W(g) = (\text{1-form in } g \text{ and } \partial g) = 0$
    * Typically $W(g) = \mathrm{tr}_g(\nabla^g - \nabla^{g_0}) \sim \Box_{g,g_0} \mathbb{1}$
    * So, $W(g) = 0$ means $\mathbb{1} : (M, g) \to (M, g_0)$ is a wave map
    * Consider the reduced Einstein vacuum equations (REVE):

$$\mathrm{Ric}(g) - \delta_g^* W(g) = 0$$

      where δ_g^* is the symmetric gradient
    * The second term kills off the non-wave term in the EVE
    * REVE is a quasilinear wave equation
    * Given $(\gamma, K) \in (S^2(T^*\Sigma))^2$, cook up $(g^0, g^1) \in (S^2(T^*M))^2$ such that $\gamma = \iota^* g$ and $K = II^g_{\iota(\Sigma)}$ and $W(g^0 + tg^1) = 0$ at $t = 0$
    * Solve REVE using the constraint equations and $\mathcal{L}_{\partial_t} W(g) = 0$ at $t = 0$
    * Use second Bianchi identity to deduce that $\mathrm{Ric}(g) = 0$
- Stability Problems:
  - Minkowski space:
    * $\Sigma = \mathbb{R}^3_x$
    * $|\gamma_{ij} - (g_m)_{ij}| \leq \epsilon r^{-1-\alpha}$, $r \geq 1$, $|m| < \epsilon$
    * $|k_{ij}| \leq \epsilon \langle r \rangle^{-2-\alpha}$
    * Analogous assumptions for derivatives

  Theorem 2. *(Christodoulou-Klainerman, Friedrich, Lindblad-Rodnianski, H-Vasy) There exists a geodesically complete g with $Ric(g) = 0$ on $\mathbb{R}^4$ with*

$$|g_{ij} - (g_0)_{ij}| \leq (1 + |t| + r)^{-1}$$

  Theorem 3. *(H-Vasy) If (γ, K) are polyhomogeneous (phg) on $\overline{\mathbb{R}^3}$ then g is phg on the compactification of $\mathbb{R}^4$.*

  - Kerr stability: (open)
    * Decay rate in Minkowski is polynomial
    * Taturu's work implies decay is t^{-3}
  - $\Lambda > 0$ gives better decay

Theorem 4. *(H-Vasy) (KdS stability)* (γ, K) *close to data of* $g^{KdS}_{m_i,a_i}$ *and* $|a_i| \ll m_i$. *There there exists* g *on* Ω *such that* $Ric(g) - \Lambda g = 0$ *and* (m_f, a_f) *such that* $g = g^{KdS}_{m_f,a_f} + \tilde{g}$ *where* $\tilde{g}$ *decays and* $g^{KdS}_{m_f,a_f}$ *is constant.*

- ∗ Analysis in ${}^bT^*M$ and is as nonlinear waves lecture
- ∗ Sketch of proof:
  - · Solve REVE

    $$\begin{cases} P(g) = (\mathrm{Ric} - \Lambda)(g) - \delta_g^* W(g) = 0 \\ i.c.(g^0, g^1) \end{cases} \tag{1}$$

    which is a quasilinear wave equation
  - · Use iteration scheme
  - · Show for $g^{(k)} = g_{dS} + \tau^\alpha H_b^s$ that $(D_g P)^{-1} \sim (\Box_g + \text{l.o.t.})^{-1} : \tau^\alpha H_b^{s-1} \to \tau^\alpha H_b^s$
  - · Run the iteration scheme
  - · Obtain decaying solution
  - · Suffices to compute the resonances of $D_{g_{dS}} P$
  - · Bad resonances are artifacts of gauge invariance
  - · Change gauge in consistent way to work around bad resonance