Theorem (... ... ... # 12 2.8 12)

Suppose φ_t ... Anosov flow $V \to X$

... $T_\gamma^{\#}$ are lengths primitive closed orbits ...

T_γ^{-1}, ... $E_s(x) \oplus E_u(x)$ ... $\|d\varphi_t|_{E_s}\| \le C e^{-\beta t}$, $t > 0$

$F(\lambda) = \prod_\gamma (1 - e^{i\lambda T_\gamma^{\#}})$ continues ... to $\mathbb{C}$ ... $\mu \in E_s(x)$ ...

... Anosov flows ... 2011 ... 8pp + 14 pp of appendix

... dynamical ... : $\exists L, t_\varepsilon$ $\forall \varepsilon > 0, T > t_\varepsilon$

$\# \{ \ldots : t_0 \le t \le T, \; d(x, \varphi_t(x)) < \varepsilon \}$

$\le C \varepsilon^{n} e^{nLT}$

$n = \dim X$

[redacted] $\ldots \le C e^{L|t|} H(\ldots)$

... estimate

... ($\varepsilon \to 0$)

$N(T) = \# \{ T_\gamma^{\#} \le T \} \le C e^{(2n-1)LT}$

$N(T) \sim \frac{e^{LT}}{T}$

Step 1. $F_P(\lambda)$ is ... of ...

COMMENT: $I - P_\gamma$ ... $(-1)^k$ ...

$F_P(\lambda) = \prod (1 - e^{i\lambda T_\gamma^{\#}}) =$

$= \exp\left(-\sum \sum_{m=1}^{\infty} \frac{1}{m} e^{i\lambda T_\gamma^{\#} m}\right) =$

$= \exp\left(-\sum T_\gamma^{\#} e^{i\lambda T_\gamma} / T_\gamma\right) = \prod_{k=0}^{n-1} \exp\left(-\sum \frac{T_\gamma^{\#} e^{i\lambda T_\gamma} \operatorname{tr} \Lambda^k P_\gamma}{T_\gamma \det(I - P_\gamma)}\right)$

γ all closed orbits

2.2) Smale's used

$$\det(I - P_\gamma) = \sum_{k=0}^{n-1} (-1)^k \operatorname{tr} \Lambda^k P_\gamma$$

$$P_\gamma = d\varphi_{-T_\gamma}\big|_{E_s(x)\oplus E_u(x)} \qquad x \in \gamma \qquad \boxed{\text{the det indep. of } x}$$

Poincaré map.

Orientability assumption $\quad x \mapsto E_s(x)$ is orientable

$$\Longrightarrow \det(I - P_\gamma) = \epsilon\, (-1)^{\dim E_s} |\det(I - P_\gamma)|$$

(can be removed S. Shen)

$$\frac{d}{d\lambda} \log \zeta_R(\lambda) = \frac{1}{i} \sum_\gamma \frac{T_\gamma^\# e^{i\lambda T_\gamma} \operatorname{tr} \Lambda^k P_\gamma}{|\det(I - P_\gamma)|}$$

$$= \frac{1}{i} \int_0^\infty \sum_\gamma \frac{T_\gamma^\# \delta(t - T_\gamma) \operatorname{tr} \Lambda^k P_\gamma}{|\det(I - P_\gamma)|} \qquad u(t) \in \mathcal{D}'(\mathbb{R}_t)$$

Atiyah–Bott '66 ... Guillemin '75

Atiyah–Bott–Guillemin trace formula:

$$\mathcal{E}_0^k = \{ u \in \mathcal{E}^k : \iota_V u = 0 \}$$

$$\Lambda^k T^* X$$

$$P : C^\infty(X, \mathcal{E}_0^k) \to C^\infty(X, \mathcal{E}_0^k), \quad P = \frac{1}{i} \mathcal{L}_V$$

$$\text{"tr"}\, e^{-itP} = \sum_\gamma \frac{T_\gamma^\# \delta(t - T_\gamma) \operatorname{tr} \Lambda^k P_\gamma}{|\det(I - P_\gamma)|} \qquad \boxed{t > 0}$$

$$(\iota_V \mathcal{L}_V u = \iota_V d(\iota_V u) + \iota_V du = \iota_V^2 du = 0)$$

2.3] $\boxed{k=0}$ function

$$\text{"}tr\text{"}\ \varphi_t^{*} \ \text{“}tr\text{” } P = \sum_{\sigma} \frac{T_\sigma^{\#}\,\delta(t-T_\sigma)}{|\det(I-P_\sigma)|}$$

"tr" φ_t^*

$$\varphi_t^* f(x) = f(\varphi_t(x)) = \int_X K(t,x,y)\, f(y)\, dy$$

time dependent; kernel OK

$$\text{"}tr\text{"}\ P_t^* = \int_X K(t,x,x)\, dx = \pi_* \delta^* K$$

diagonal

$$\delta : (t,x) \longrightarrow (t,x,x) \qquad \pi : (t,x) \longrightarrow t$$

$$\mathbb{R}\times X \to \mathbb{R}\times X\times X \qquad \mathbb{R}\times X \longrightarrow X$$

Example $\quad Au(x) = u(\alpha x), \quad x \in \mathbb{R}$

$$Au = \int K_A(x,y)\, u(y)\, dy \qquad K_A(x,y) = \delta(\alpha x - y)$$

$y = \alpha x$

$$WF(K_A) = \{(x,y;\ -\alpha\eta,\ \eta)\}$$

$\delta^* K_A$ well def if $WF(K_A) \cap \{(x,x,\xi,-\xi)\} = \emptyset$; $\quad N^*\Delta$

i.e. $\alpha \neq 1$ $\quad N^*\{y=\alpha x\} = \{(x,y,-\alpha\eta,\eta)\}$

$$\text{"}tr\text{"}\ K_A = \int \delta(\alpha x - x)\, dx = \frac{1}{|\alpha-1|}$$

2.4]

$$K = f_t^* K_{Id}$$

$WF(f^* u) \subset \{ (z, {}^t f'(z) \eta) : (f(z), \eta) \in WF(u) \}$

$$f : \mathbb{R} \times X \times X \to X \times X$$
$$(t, x, y) \to (\varphi_{-t}(x), y)$$

$$WF(K) \subset \{ (t, \varphi_t(y), y, -V(x)\cdot\eta, \xi, -{}^T d\varphi_t(y)\xi) : t \in \mathbb{R},\ y \in X,\ \xi \in T^*_{\varphi_t(y)} X \setminus 0 \}$$

$WF(K) \cap N^* \{(t,x,x)\} = \{(t, x, x, 0, \xi, -\eta)\}$

$\varphi_t(y) = y$, $-V(x)\eta = 0$,

$\xi = +{}^T d\varphi_t \xi$. **But** $I - d\varphi_t|$ invertible t_j ... $t > 0$

Hence $\mathrm{tr}\, \varphi_t^* = \int K_{\varphi_{-t}}(x, x)\, dx$ makes sense as an element $\mathcal{D}'(\mathbb{R}_+)$

Assume we have (proof later)

$$\mathrm{tr}^k \varphi_t^* = \sum_\gamma \frac{T_\gamma^{\#}\, \delta(t - T_\gamma)\, \mathrm{tr}\, \Lambda^k P_\gamma}{|\det(I - P_\gamma)|}$$

$+ \quad \mathcal{E}_0^k$ **Statement**

$$\zeta_R(\lambda) = \exp\left(- \sum_\gamma \frac{T_\gamma^{\#}\, e^{i\lambda T_\gamma}\, \mathrm{tr}\, \Lambda^k P_\gamma}{T_\gamma\, |\det(I - P_\gamma)|} \right)$$

$\zeta_R(\lambda) = \prod_k \zeta_k(\lambda)^{(-1)^k}$

2.5)

$$\partial_\lambda \log \zeta_Q(\lambda) = \frac{1}{i}\int_0^\infty \mathrm{tr}^b\, \varphi_{-t}^* \, e^{i\lambda t}\, dt$$

Here to prove meromorphy of the LHS it is enough to prove that of the RHS with INTEGRAL RESIDUES

FORMAL ARGUMENT:

$$\partial_\lambda \log \zeta(\lambda) = \frac{1}{i}\int_0^\infty e^{\dots}\, \mathrm{tr}^b\, e^{\dots}\, dt$$

$$= \frac{1}{i}\int_0^\infty e^{\dots}\, \mathrm{tr}^b\, \varphi_{t_0}^*\, e^{\dots}\, dt$$

$$" = " \frac{1}{i}\mathrm{tr}^b \int_0^\infty e^{-it(P-\lambda)}\, dt = \mathrm{tr}^b\, \varphi_{t_0}^* (P-\lambda)^{-1}$$

Theorem (essentially Faure–Sjöstrand 2011)

$\exists\, G \in S^{0+}(X)$, $s(\lambda) = m_G(x,\xi)\log|\xi|$ s.t. for

$$H_{sG} = e^{-sG}L^2, \quad D_{sG} = \{u \in \mathcal{D}'(X) : u \in H_{sG} \ \& \ Pu \in H_{sG}\}$$

$$P - \lambda : D_{sG} \longrightarrow H_{sG}, \quad \mathrm{Im}\,\lambda \geq -s/C$$

is a Fredholm operator of index 0. In particular

$$(P-\lambda)^{-1} : H_{sG} \to H_{sG} \quad \text{meromorphic in } \mathrm{Im}\,\lambda \geq \frac{-s}{C}$$

$$\cup \qquad\qquad \cap$$

$$H^{s} \longrightarrow H^{-s}$$

2.6)

Theorem (DZ 2016) Suppose $\ldots$, λ_0

$$(P-\lambda)^{-1} = R_{hol}(\lambda) + \sum_{j=1}^{J} \frac{(P-\lambda_0)^{j-1}\Pi}{(\lambda-\lambda_0)^j}$$

where $R_{hol}(\lambda)$ is hol near λ_0, $\Pi : H_{sG} \to H_{sG}$ (comm. w P)

$$WF'(R_{hol}(\lambda)) \subset \Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*)$$

$$WF'(\Pi) \subset E_u^* \times E_s^*$$

where $\Omega_+ = \{(e^{-tH_p}(x,\xi), x, \xi) : t \geq 0,\ p(x,\xi)=0\}$

E_u^* dual to E_s sink for the lifted flow

E_s^* dual to E_u source for the lifted flow

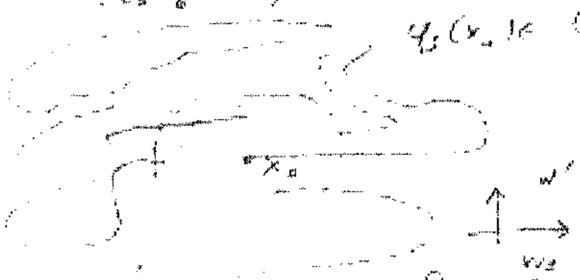
Proof of Guillemin's formula (with skip of details)

$$\int_{\mathbb{R}\times X} K(t,x,x)\,\chi(t,x) = \sum_\gamma \frac{1}{|\det(I-P_\gamma)|} \int_\gamma \chi(\gamma, x)\, dL(\gamma)$$

Local version (hence dependence on x!)

$\chi \in C_c^\infty((t_0-\varepsilon, t_0+\varepsilon) \times U)$

$\varphi_{t_0}(x_0) = x_0$, $\varphi_t(x_0) \notin U$, $|t_0| < \varepsilon$ [illegible] $d = \boxed{\ldots}$ $A \cdot P_\gamma \ldots$



$\varphi_{-t_0}(0, w') = (F(w'), A(w'))$

$F(0) = 0$, $A(0) = 0$

$\varphi_{-t}(w_1, w') = (-t + t_0 + w_1 + F(w'), A(w'))$

2.7) $K(t,x,x) = \delta(w' - A(w'))\, \delta(t - t_0 - F(w'))$

$x \in V$

$$\int_{\mathbb{R}\times V} K(t,x,x)\, \chi(t,x)\, dx\, dt =$$

$$= \int \chi(t_0 - F(w'), (w_1, w'))\, \delta(w' - A(w'))\, dw_1\, dw'$$

$\left[dA(0) = P_\gamma \text{ (conjugate to) } \quad A(w') = w' \text{ has one sol. } \boxed{A(0) = 0} \right]$

$\uparrow$ differ.

$$= \frac{1}{|I - dA(0)|} \int_{-\varepsilon}^{\varepsilon} \chi(t_0, w_1, 0)\, dw_1 =$$

$$= \frac{1}{|I - P_\gamma|} \int_\gamma \chi(T_\gamma, x)\, dL(x).$$

$\left[dL(x) = dt \text{ meas. on } \gamma \right]$