INTRODUCTORY WORKSHOP ON MICROLOCAL ANALYSIS

Microlocal Analysis and Inverse Problems

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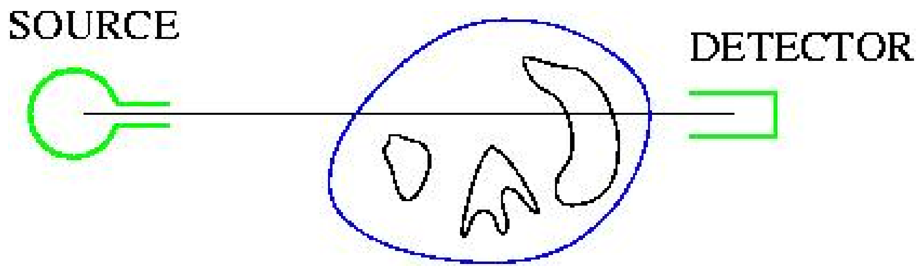
Inverse Boundary Problems

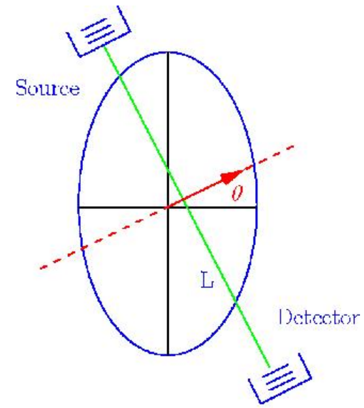
Can one determine the internal properties of a medium by making measurements outside the medium (non-invasive)?

X-ray tomography (CAT-scans)



Problem: Can we recover the density from attenuation of X-rays?

Radon (1917) $n = 2$



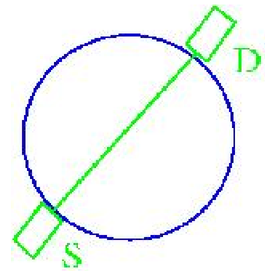
$f(x) =$ Unknown function

$$I_{detector} = e^{-\int_L f} I_{source}$$

$$Rf(s,\theta) = g(s,\theta) = \int_{\langle x,\theta\rangle = s} f(x)\,dH = \int_L f$$

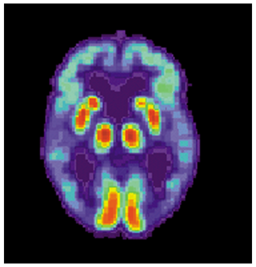
$$f(x) = \frac{1}{4\pi^2}\ p.v. \int_{S^1} d\theta \int \frac{\frac{d}{ds} g(s,\theta)\,ds}{\langle x,\theta\rangle - s}$$

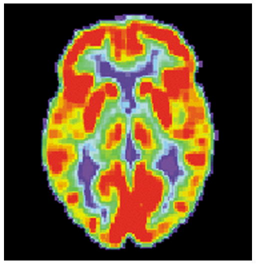
LINEAR (No Scattering)



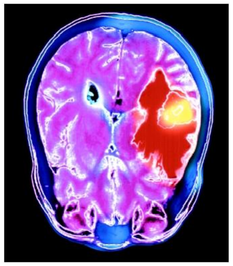
X-ray tomography (CT)

PET

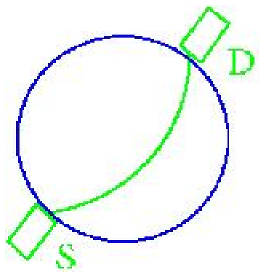




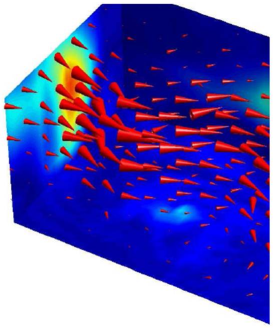
MRI



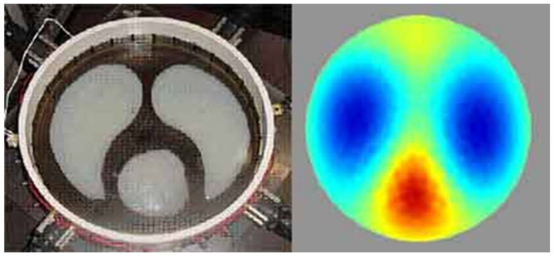
NONLINEAR (Scattering)



Ultrasound



Electrical
Impedance
Tomography
(EIT)



Hybrid Methods

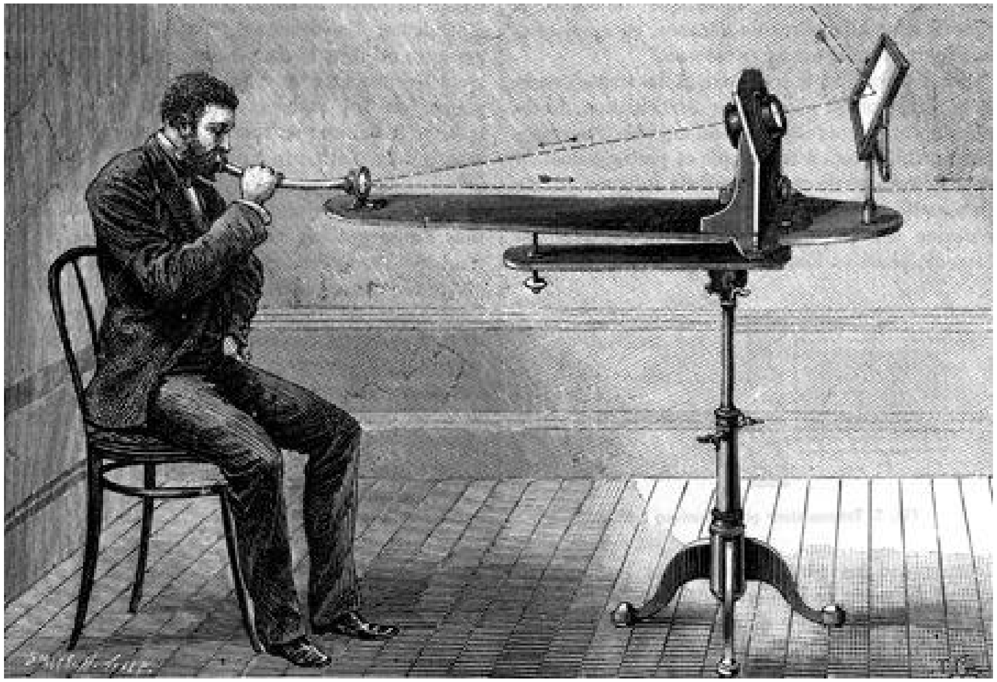
Superposition of 2 images each obtained with a single wave

One single wave in sensitive only to a given contrast

	Ultrasound to bulk compressibility
Photoacoustic Imaging	Optical wave to dielectric permittivity
Thermoacoustic Imaging	LF Electromagnetic wave to electrical impedance, conductivity.

Photoacoustic Tomography

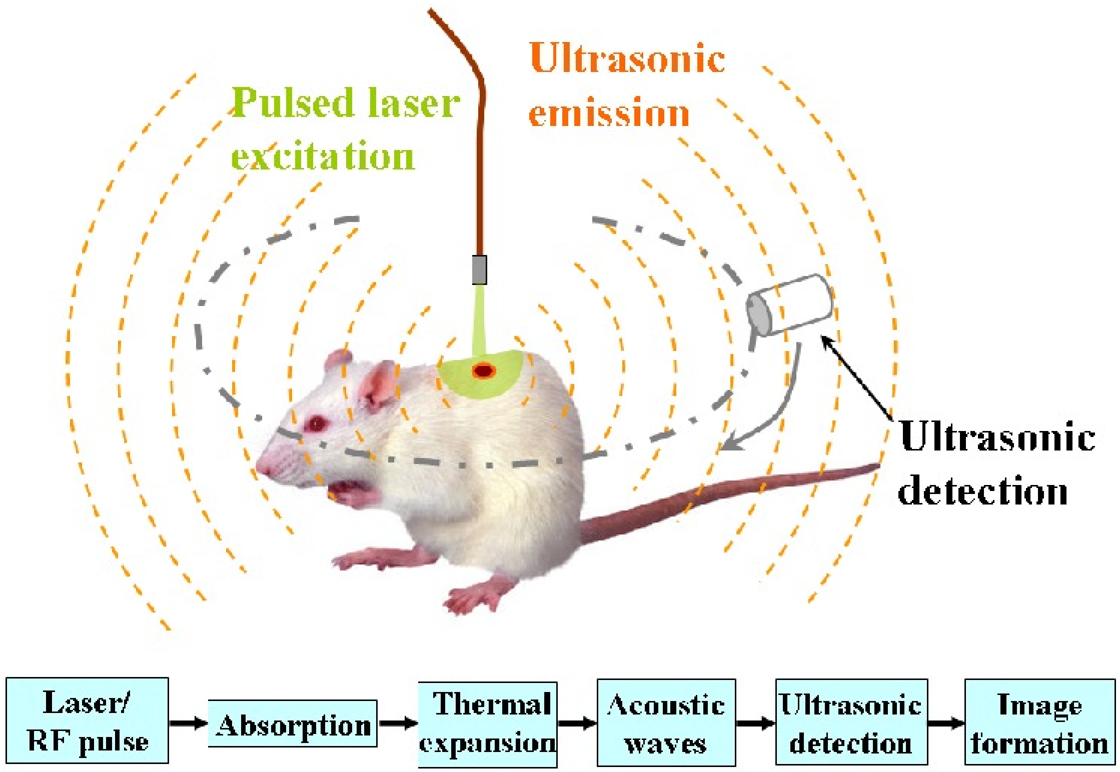
Photoacoustic Effect: **The sound of light**



Picture from Economist
(The sound of light)

Graham Bell: When rapid pulses of light are incident on a sample of matter they can be absorbed and the resulting energy will then be radiated as heat. This heat causes detectable sound waves due to pressure variation in the surrounding medium.

Thermoacoustic Tomography



Wikipedia

(Loading Melanoma3DMovie.avi)

Lihong Wang (Washington U.)

Mathematical Model

First Step: in PAT and TAT is to reconstruct $H(x)$ from $u(x,t)|_{\partial\Omega\times(0,T)}$, where u solves

$$(\partial_t^2 - c^2(x)\Delta)u = 0 \quad \text{on } \mathbb{R}^n \times \mathbb{R}^+$$
$$u|_{t=0} = \beta H(x)$$
$$\partial_t u|_{t=0} = 0$$

Second Step: in PAT and TAT is to reconstruct the optical or electrical properties from $H(x)$ (internal measurements).

PROGRESSING WAVES

Let $q \in C_0^\infty(\mathbb{R}^n)$, supp $q \subset \{x \in \mathbb{R}^n : |x| < R\}$

Let $\omega \in S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

$$CP \quad \begin{cases} ((\partial_t^2 - \Delta) + q)u = & 0 \text{ on } \mathbb{R}_x^n \times \mathbb{R}_t \\ u = & \delta(t - x\cdot\omega),\ t < -R \end{cases}$$

$$\langle \delta(t - x\cdot\omega), \varphi\rangle = \int_{x\cdot\omega=t} \varphi(x)\, dH, \qquad \varphi \in C_0^\infty(\mathbb{R}^n)$$

PROGRESSING WAVES

$\delta(t - x \cdot \omega)$ solves

$$\Box\delta(t - x \cdot \omega) = 0$$

where $\Box = \partial_t^2 - \Delta$ is the D'Alembertian.

$$(\Box + q)\delta(t - x \cdot \omega) = q\delta(t - x \cdot \omega)$$

Next try

$$u_1(t, x, \omega) = \delta(t - x \cdot \omega) + a_1(x, \omega)H(t - x \cdot \omega)$$

$$H(t - x \cdot \omega) = \begin{cases} 1 & t > x \cdot \omega \\ 0 & t < x \cdot \omega \end{cases}$$

$$\Box H(t - x \cdot \omega) = 0$$

PROGRESSING WAVES

$$(\Box + q)u_1 = (q(x) + 2\nabla a_1 \cdot \omega)\delta(t - x \cdot \omega)$$
$$+ (q(x)a_1 - \Delta a_1)H(t - x \cdot \omega)$$

To eliminate main singularity, we choose

$$\nabla a_1 \cdot \omega = -\frac{q(x)}{2}$$

$$a_1(x, \omega) = -\frac{1}{2}\int_{-\infty}^{x \cdot \omega} q(x + (s - x \cdot \omega)\omega)ds$$

PROGRESSING WAVES

If $x \cdot \omega > R$,

$$a_1(x,\omega) = \text{ X-ray transform of } -q/2$$

$$If(x,\omega) = \int f(x+s\omega)ds, \quad f \in C_0^\infty(\mathbb{R}^n)$$

Next try

$$u_2 = \delta(t - x\cdot\omega) + a_1(x,\omega)H(t-x\cdot\omega) + a_2(x,\omega)(t-x\cdot\omega)_+$$

where $\quad s_+^k = \begin{cases} s^k & s > 0 \\ 0 & s < 0 \end{cases}$ and $\quad a_2 \in C^\infty(\mathbb{R}^n \times S^{n-1})$

$$\nabla a_2 \cdot \omega = -\frac{1}{2}(q(x)a_1 - \Delta a_1)$$

PROGRESSING WAVES

$$(\Box + q)u_2 = (q(x) + \delta a_2)(t - x \cdot \omega)_+$$

The solution u of CP is

$$u = \delta(t - x \cdot \omega) + \sum_{j=0}^{N} a_{j+1}(x,\omega)(t - x \cdot \omega)_+^j$$

$$+ C^{N-2}(\mathbb{R}_x^n \times \mathbb{R}_t)$$

Using Borel type lemma (see [MU])

$$(*) \qquad u = \delta(t - x \cdot \omega) + \sum_{j=0}^{\infty} a_{j+1}(x,\omega)(t - x \cdot \omega)_+^j$$

$$+ \text{ smooth error}, \qquad a_j \in C^\infty(\mathbb{R}_x^n \times S_\omega^{n-1})$$

$(*)$ is a conormal distribution to hypersurface $\{t = x \cdot \omega\}$.

PROGRESSIVE WAVE EXPANSION

$$CP: \quad u = \delta(t - x\cdot\omega) + a_1(x,\omega) H(t - x\cdot\omega) + \sum_{j=1}^{\infty} a_{j+1}(x,\omega)(t - x\cdot\omega)_+^j + \text{ smooth}$$

Another representation:

$$u(t,x,\omega) = \int e^{i(t - x\cdot\omega)\cdot\rho}\chi(\rho)\left(1 + \sum_{j=1}^{\infty} c_j \frac{a_j(x,\omega)}{\rho^j}\right) d\rho + \text{ smooth}$$

PROGRESSIVE WAVE EXPANSION

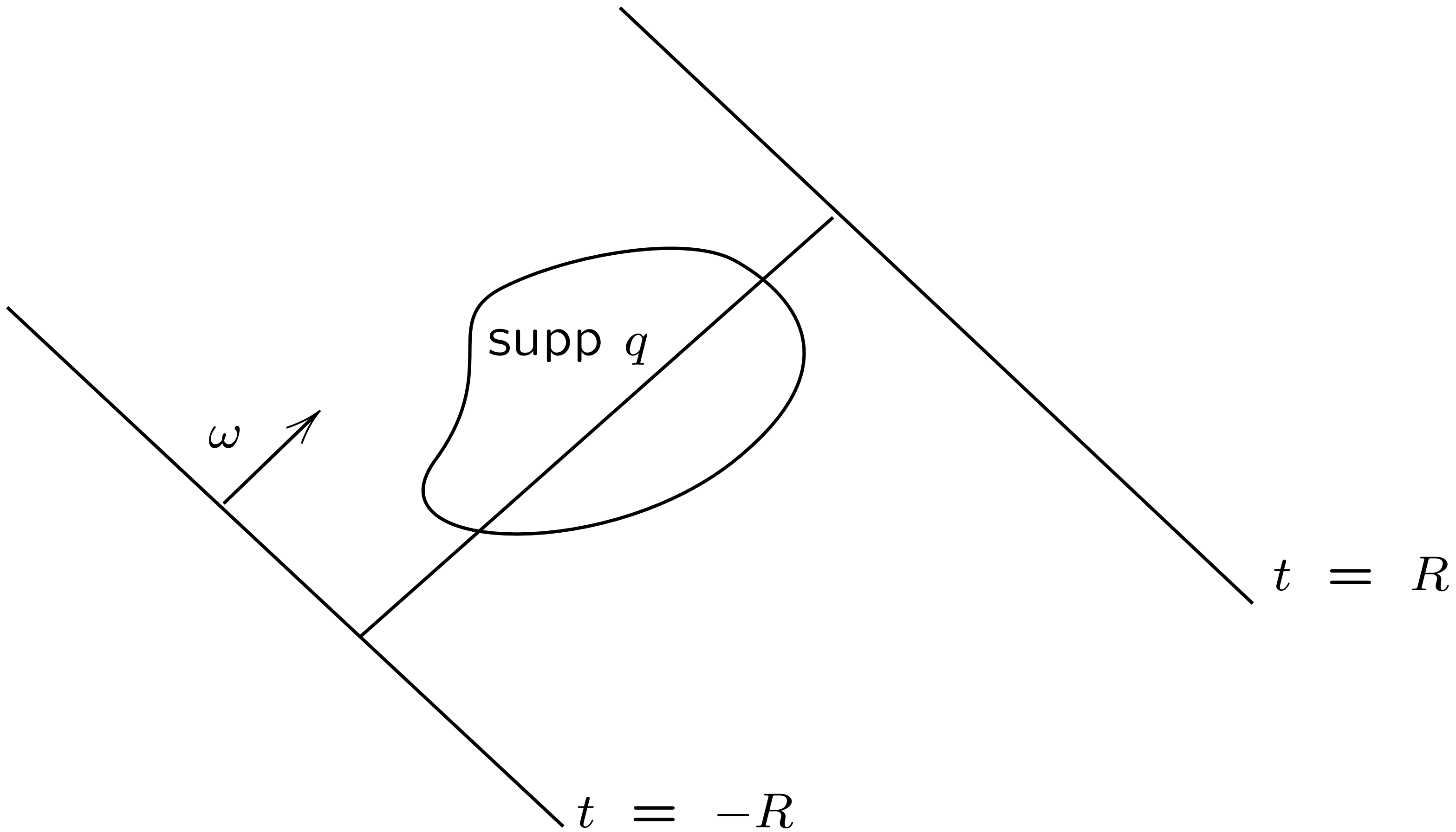
$$\chi(\rho) = \begin{cases} 0, & |\rho| \leq 1/2 \\ 1, & |\rho| \geq 1 \end{cases}, \qquad \chi \in C^\infty(\mathbb{R})$$

c_j some constants

Amplitude : $\chi(\rho)\left(1 + \frac{a_1(x,\omega)}{\rho} + \sum_{j=1}^{\infty} \frac{a_{j+1}(x,\omega)}{\rho^{j+1}}\right)$

From the principal symbol of $u(t,x,\omega) - \delta(t - x\cdot\omega)$ for any $t > R$, we can determine $\int q(x+s\omega)ds = Iq(x,\omega)$, which is the X-ray transform of q, and therefore q.

X-RAY TRANSFORM

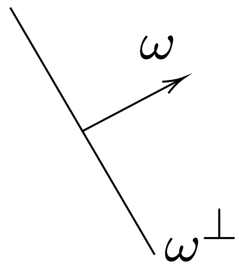


X-RAY TRANSFORM

$$\begin{aligned} If(x,\omega) &= \int f(x+s\omega)\,ds \\ &= \int f(x-(x\cdot\omega)\omega + r\omega)\,dr \end{aligned}$$

$$(x-(x\cdot\omega)\omega)\cdot\omega = 0, \quad x-(x\cdot\omega)\omega \in \omega^{\perp} = \{x\in\mathbb{R}^n : x\cdot\omega = 0\}$$

$$If(x,\omega) = \int f(x+r\omega)\,dr \qquad x\in\omega^{\perp}$$



X-RAY TRANSFORM

Formal Transpose

$$I^* g(x) = \int_{S^{n-1}} g(x - (x \cdot \omega)\omega, \omega)\, d\omega$$

$$g \in C^\infty(T), \qquad T = \{(x, \omega) : x \in \omega^\perp\}$$

Exercise:

$$I^* I f(x) = c_n \int \frac{f(y)}{|x-y|^{n-1}}\, dy \qquad f \in C_0^\infty(\mathbb{R}^n)$$

This extends to $f \in \mathcal{E}'(\mathbb{R}^n)$.

X-RAY TRANSFORM

$I^*I = (-\Delta)^{-1/2}$, a pseudodifferential operator of order -1.

Inversion formula:

$$\boxed{(-\Delta)^{1/2} I^* I f = f} \qquad f \in \mathcal{E}'(\mathbb{R}^n)$$

Non-local inversion formula

X-RAY TRANSFORM

$$
\begin{aligned}
(-\Delta)^{1/2}(-\Delta)^{1/2} I^* I f &= (-\Delta)^{1/2} f \\
(-\Delta) I^* I f &= (-\Delta)^{1/2} f
\end{aligned}
$$

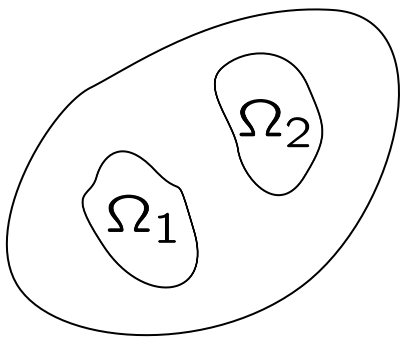
$$\text{WF } (-\Delta)^{1/2} f = \text{WF } f$$

We can recover singularities of f with a local inversion formula (local tomography [FRS])

X-RAY TRANSFORM

Example

$$f = \sum_{i=1}^{2} a_i(x)\chi_{\Omega_i}$$



Ω_i smooth domains, $a_i \in C^\infty(\mathbb{R}^n)$.

ACOUSTIC WAVE EQUATION

$$c(x) \in C^\infty(\mathbb{R}^n), \quad c(x) > 0, \quad c(x) = 1,\ |x| > R$$

$$CP \quad \begin{cases} (\partial_t^2 - c^2(x)\Delta)u = & 0 \text{ on } \mathbb{R}_x^n \times \mathbb{R}_t \\ u = & \delta(t - x \cdot \omega),\ t < -R \end{cases}$$

$$\begin{aligned} u(t,x,\omega) =& A_0(x,\omega)\delta(t - \varphi(x,\omega)) + A_1(x,\omega)H(t - \varphi(x,\omega)) \\ &+ \sum_{j=0}^{\infty} A_{j+1}(x,\omega)(t - \varphi(x,\omega))_+^j + \text{smooth} \end{aligned}$$

EIKONAL EQUATION

Highest order singularity,

$$(1 - c^2(x)|\nabla_x \varphi|^2)\delta''(t - \varphi(x, \omega))$$

Eikonal equation (Non-linear, first order) :

$$\begin{cases} |\nabla_x \varphi|^2 = & \frac{1}{c^2(x)} \\ \varphi(x, \omega) = & x \cdot \omega, \quad x \cdot \omega < -R \end{cases}$$

TRANSPORT EQUATION

Eliminate singularity $\delta'(t-\varphi(x,\omega))$

$$TE \quad \begin{cases} 2c(x)\nabla_x\varphi\cdot\nabla A_0 - c^2(x)\Delta\varphi A_0 = 0 \\ A_0 = 1, \quad x\cdot\omega < -R \end{cases}$$

First order linear PDE

EIKONAL EQUATION

$$\begin{cases} |\nabla_x \varphi|^2 = & \frac{1}{c^2(x)} \\ \varphi(x,\omega) = & x \cdot \omega, \quad x < -R \end{cases}$$

Hamilton - Jacobi theory

Hamiltonian $\qquad H(x,\xi) = \frac{1}{2}(c^2(x)|\xi|^2 - 1)$

Want $\xi = \nabla_x \varphi(x,\omega)$ for some φ.

HAMILTON-JACOBI THEORY

Hamiltonian is given by

$$H_c(x,\xi) = \frac{1}{2}\Big(c^2(x)|\xi|^2 - 1\Big)$$

$X_c(s,X^0) = \Big(x_c(s,X^0), \xi_c(s,X^0)\Big)$ be bicharacteristics,

sol. of $$\frac{dx}{ds} = \frac{\partial H_c}{\partial \xi}, \qquad \frac{d\xi}{ds} = -\frac{\partial H_c}{\partial x}$$

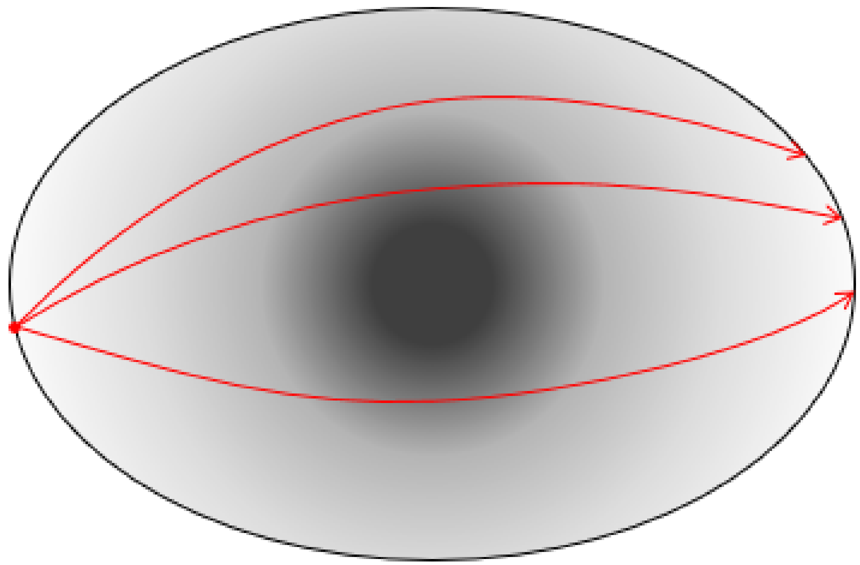
$x(0) = x^0$, $\xi(0) = \xi^0$, $X^0 = (x^0, \xi^0)$, where $\xi^0 \in \mathcal{S}_c^{n-1}(x^0)$

$$\mathcal{S}_c^{n-1}(x) = \big\{\xi \in \mathbb{R}^n;\ H_c(x,\xi) = 0\big\}.$$

Geodesics Projections in x: $x(s)$.

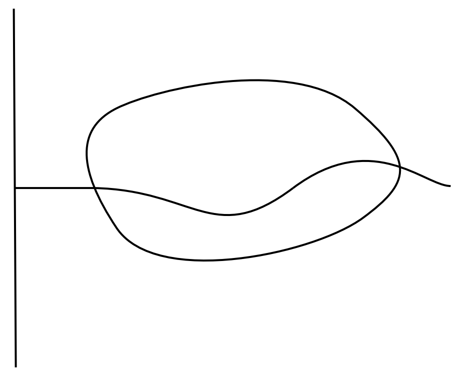
GEODESICS (RAYS)

Geodesics minimize length (time) locally, $\frac{ds}{c}$.



Geodesics in a medium with a slow region in the center

EIKONAL EQUATION



$x \cdot \omega \;=\; -R$

Flow-out from (x_0, ω_0) such that $x_0 \cdot \omega_0 = -R$ by bicharacteristics

Exercise ([MU]) : Flowout $(x(s), \xi(s))$ is a Lagrangian submanifold of $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$. Locally, it is given by $(x, \nabla_x \varphi)$, $\varphi(x, \omega) = x \cdot \omega$, $|x| > R$.

LAGRANGIAN MANIFOLDS

Lagrangian Λ is an n-dimensional submanifold such that the symplectic form $\omega = \sum d\xi_i \wedge dx_i$ vanishes on Λ.

$$\omega(t, \tilde{t}) = 0, \qquad t, \tilde{t} \in T_x\Lambda$$

$$H_c(x, \xi) = \frac{1}{2}(c^2(x)|\xi|^2 - 1)$$

Bicharacteristics stay in $H_c = 0$.

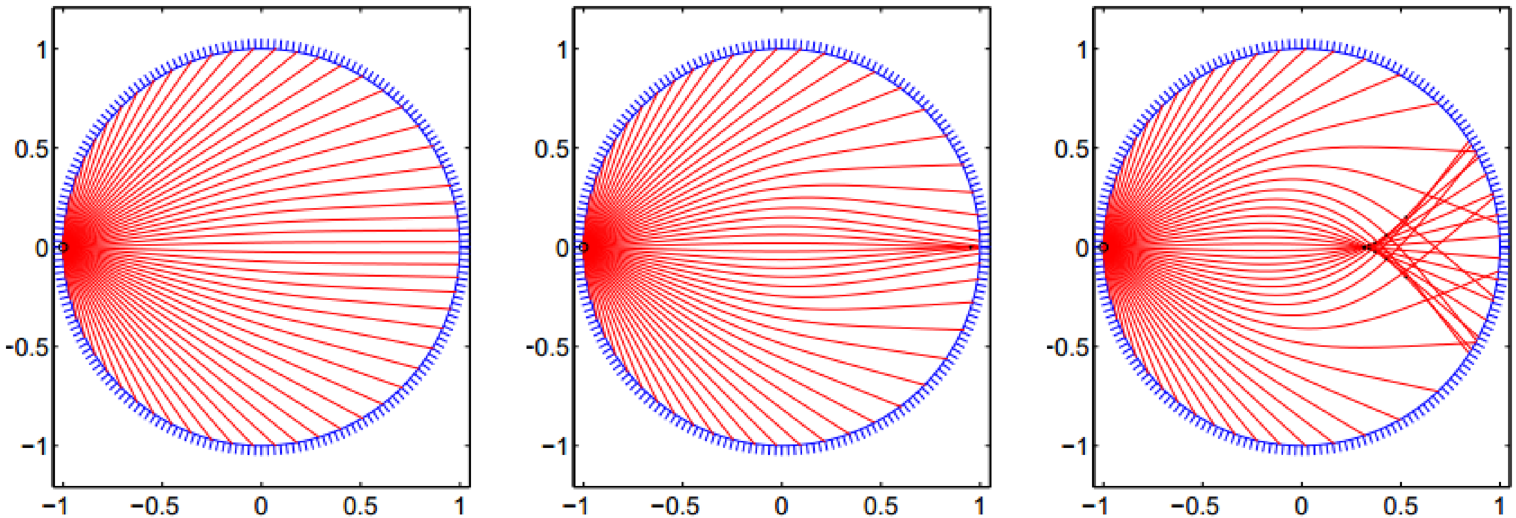
Therefore, $\Lambda \subset \{H_c = 0\}$.

$\Lambda_c = \{(x, d_x\varphi)\}$ locally some φ

Then $c^2(x)|\nabla_x\varphi|^2 - 1 = 0$.

EIKONAL EQUATION

The EE has only local solutions

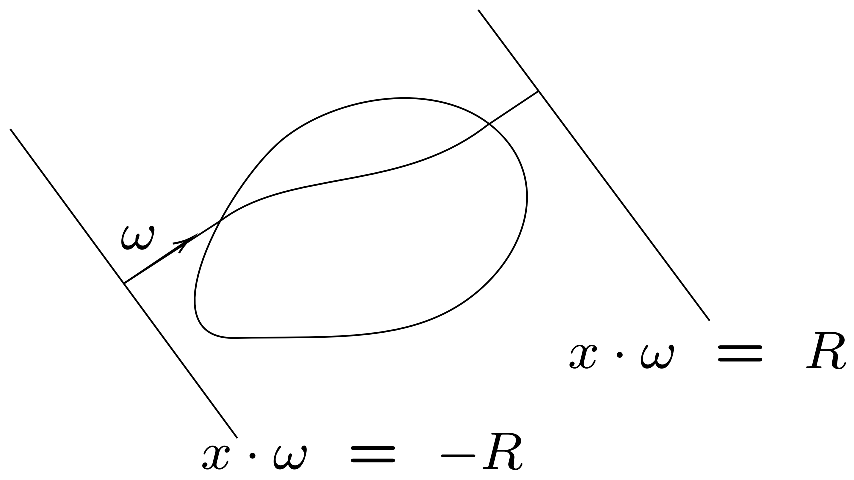


$k = 0.20$ (simple) $k = 0.49$ (non-simple) $k = 1.23$ (non-simple)

$$c_k(r) = exp(k\, exp(-\frac{r^2}{2\sigma^2})),\ 0 \leq \sigma \leq 1,\ \sigma \text{ fixed}$$

Francois Monard: SIAM J. Imaging Sciences (2014)

GEODESIC DISTANCE



$\nabla_x \varphi$ is perpendicular to the distorted plane wave $\{x : \varphi(x, \omega) = t\}$

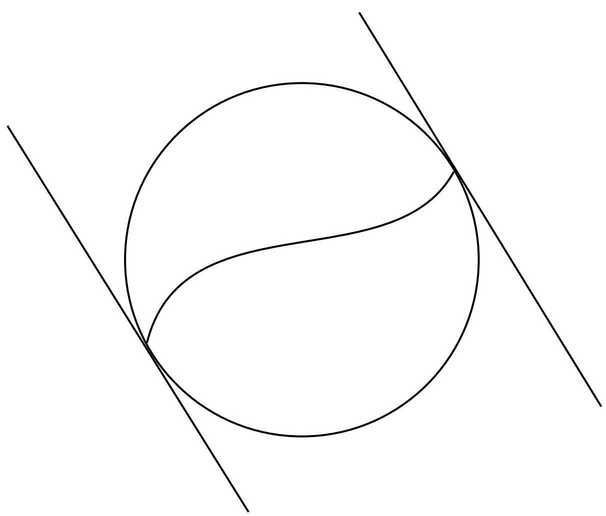
Therefore, $\varphi(x, \omega)$ measures the geodesic distance between $\{x : x \cdot \omega = -R\}$ and $\{x : \varphi(x, \omega) = t\}$ (assume EE can be solved globally)

BOUNDARY RIGIDITY

IP: Suppose we know

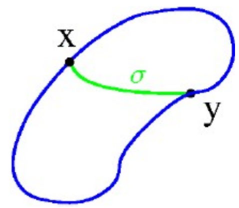
$$\varphi(x, \omega), \qquad |x| = R$$

Can we recover $c(x)$?



BOUNDARY RIGIDITY

Let M be a bounded domain in $\mathbb{R}^n$ with smooth boundary, $c \in C^\infty(M)$, $c > 0$



$x, y \in \partial M$

Boundary distance function

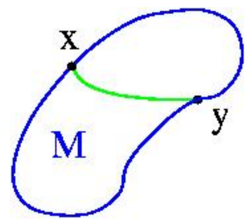
$$d_c(x,y) = \inf_{\substack{\sigma(0)=x \\ \sigma(1)=y}} L(\sigma)$$

$$L(\sigma) = \int_0^1 \frac{1}{c} \left| \frac{d\sigma}{dt} \right| dt$$

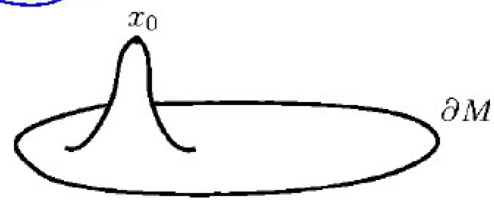
Inverse problem (Boundary Rigidity)

Determine c knowing $d_c(x,y)$ $x, y \in \partial M$

BOUNDARY RIGIDITY



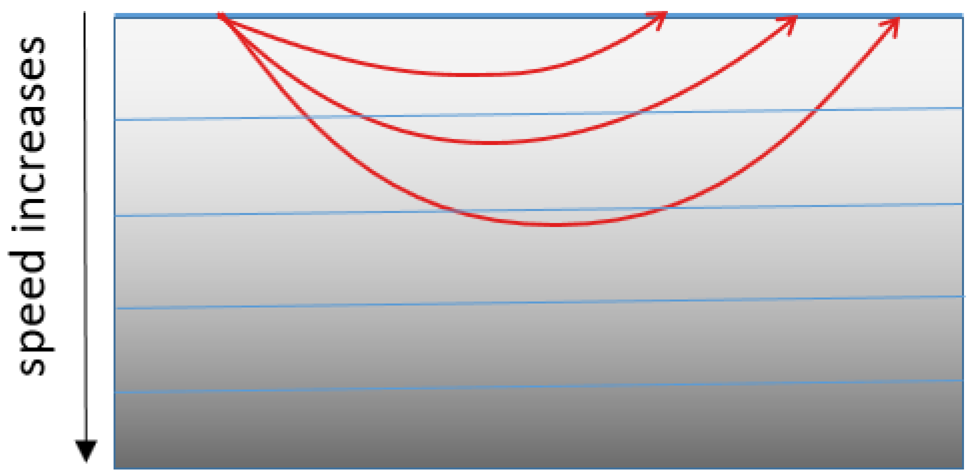
$d_c \Rightarrow c$?



$$d_c(x_0, \partial M) > \mathsf{sup}_{x,y \in \partial M}\, d_c(x,y)$$

Need an a-priori condition to recover c from d_c.

DEF (M, c) is simple if given two points $x, y \in \partial M$, $\exists!$ minimizing geodesic joining x and y and ∂M is strictly convex



strictly convex

THEOREM (Mukhometov [M]) One can determine c uniquely and stably from d_c if (M, c) is simple.

LAGRANGIAN DISTRIBUTIONS

$$CP: \quad u = A_0(x,\omega)\delta(t-\varphi(x,\omega)) + A_1(x,\omega)H(t-\varphi(x,\omega)) + \text{smoother}$$

u is a Lagrangian distribution

Another representation (locally) :

$$u(t,x,\omega) = \int e^{i(t-\varphi(x,\omega))\rho} a(t,x,\omega,\rho)d\rho$$

where $a \in S^0(\mathbb{R}_t \times \mathbb{R}^n_x \times S^{n-1}_\omega \times \mathbb{R}_\rho)$

INVERSE PROBLEM FOR ACOUSTIC EQUATION

Theorem. *If we know $u(t,x,\omega)$, any $t > R$, then we can determine $c(x)$ if $(B(0,R),c)$ is simple.*

Sketch of Proof: We can determine $\varphi(x,\omega)$, $|x| > R$ and therefore d_c and then $c(x)$ using Mukhometov's theorem.

TRANSPORT EQUATION

$$\begin{aligned}(\partial_t^2 - c^2(x)\Delta)u &= 0 \\ u &= \delta(t - x\cdot\omega), \quad t < -R\end{aligned}$$

$$u = A_0(x,\omega)\delta(t-\varphi(x,\omega)) + A_1(x,\omega)H(t-\varphi(x,\omega)) + \text{ smoother}$$

$$\begin{cases} \nabla_x\varphi\cdot\nabla_x A_0 + \frac{1}{2}c^2(x)\Delta A_0 = 0 \\ A_0 = 1, \quad x\cdot\omega < -R \end{cases}$$

Integrating along geodesics,

$$\begin{aligned}\frac{dx}{ds} &= \nabla_x\varphi(x(s),\omega) \\ \varphi(x(s),\omega) &= e^{-\frac{1}{2}\int_{-\infty}^{s} c^2(x(r))dr}\int_0^s A_0(x(r))dr\end{aligned}$$

This leads to the geodesic X-ray transform.

GEODESIC X-RAY TRANSFORM

Let $c \in C^\infty(M)$, $c > 0$. Linearizing $c \mapsto d_c$ leads to the *ray transform*

$$If(x,\xi) = \int_0^{\tau(x,\xi)} f(\gamma(t,x,\xi))\,dt$$

where $x \in \partial M$ and $\xi \in S_x M = \{\xi \in T_x M \,:\, |\xi| = 1\}$.

Here $\gamma(t,x,\xi)$ is the geodesic starting from point x in direction ξ, and $\tau(x,\xi)$ is the time when γ exits M. We assume that (M,c) is nontrapping, i.e. τ is always finite.

GEODESIC X-RAY TRANSFORM

$$\boxed{If(x,\xi) = \int_0^{\tau(x,\xi)} f(\gamma(t,x,\xi))\,dt}$$

Theorem ([M]). *If (M,c) simple, then I_c is injective.*

$$I_c f = 0, \quad f \in C^\infty(M) \Longrightarrow f = 0$$

Moreover, stability estimates are valid.

The geodesic X-ray transform is the linearization of $c \mapsto d_c$.

GEODESIC X-RAY TRANSFORM

Let $X \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $c \in C^\infty(X), c(x) > 0$.

$$I_c f(x,\xi) = \int f(\gamma(x,s,\xi))d\xi \quad x \in X, \xi \in S^*_x X$$

$\gamma(x,s,\xi)$ is the geodesic through (x,ξ)

$$S^*_x X = \{\xi \in T^*_x X : c(x)|\xi| = 1\}$$

Theorem ([G1],[SU1]).

Assume $\left\{ \begin{array}{rl} T^*_x X \to & X \\ v \mapsto & \gamma(x,v) \end{array} \right.$ *is a diffeomorphism.*

Then $I_c^* I_c$ *is an elliptic pesudodifferential operator of order* -1 *with principal symbol* $c(x)|\xi|^{-1}$.

NORMAL OPERATOR

<u>Sketch of Proof</u>: $X \subset \mathbb{R}^n$, X open

$$I_c^* I_c f(x) = \int_{S_x^* X} \int_0^\infty f(\exp_x tv)\, dt\, d\lambda \qquad f \in C_0^\infty(X)$$

$d\lambda$ is the standard measure on $S_x^* X$

$\gamma(x, tv) = \exp_x(tv)$

NORMAL OPERATOR

Transformation $\qquad \exp_x(tv) = y$

$$I_c^* I_c f(x) = \int \frac{f(y)J(x,y)}{d_c(x,y)^{n-1}} dy$$

$J(x,y)$ is the Jacobian of the transformation

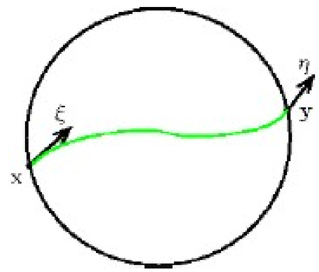
$$J(x,y) = \left| \det \frac{\partial^2 f}{\partial x \partial y} \right|, \qquad f(x,y) = \frac{1}{2} d_c(x,y)^2$$

Exercise [SU1]: $d_c^2(x,y) = G(x,y)|x-y|^2$, G smooth, $G(x,x) = \frac{1}{c^2(x)}$.

SCATTERING RELATION

d_c only measures first arrival times of waves.

We need to look at behavior of all geodesics



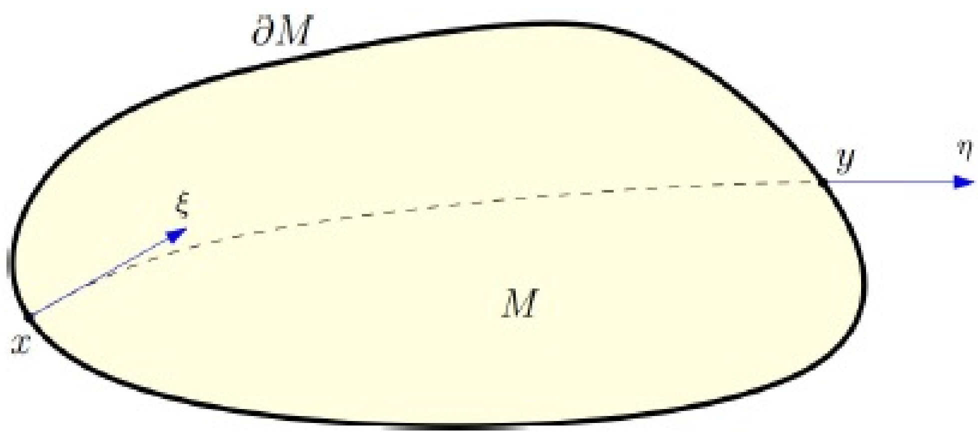
$$\|\xi\|_c = \|\eta\|_c = 1$$

$\alpha_c(x, \xi) = (y, \eta)$, α_c is SCATTERING RELATION ([G2],[U])

If we know direction and point of entrance of geodesic then we know its direction and point of exit.

Travel Time Tomography

Define the scattering relation α_c.



$$\alpha_c : (x, \xi) \to (y, \eta).$$

α_c, d_c follows all geodesics.

Inverse Problem: *Do α_c, d_c determine c?*

NON-SIMPLE SPEEDS

IP: Do α_g, d_c determine c?

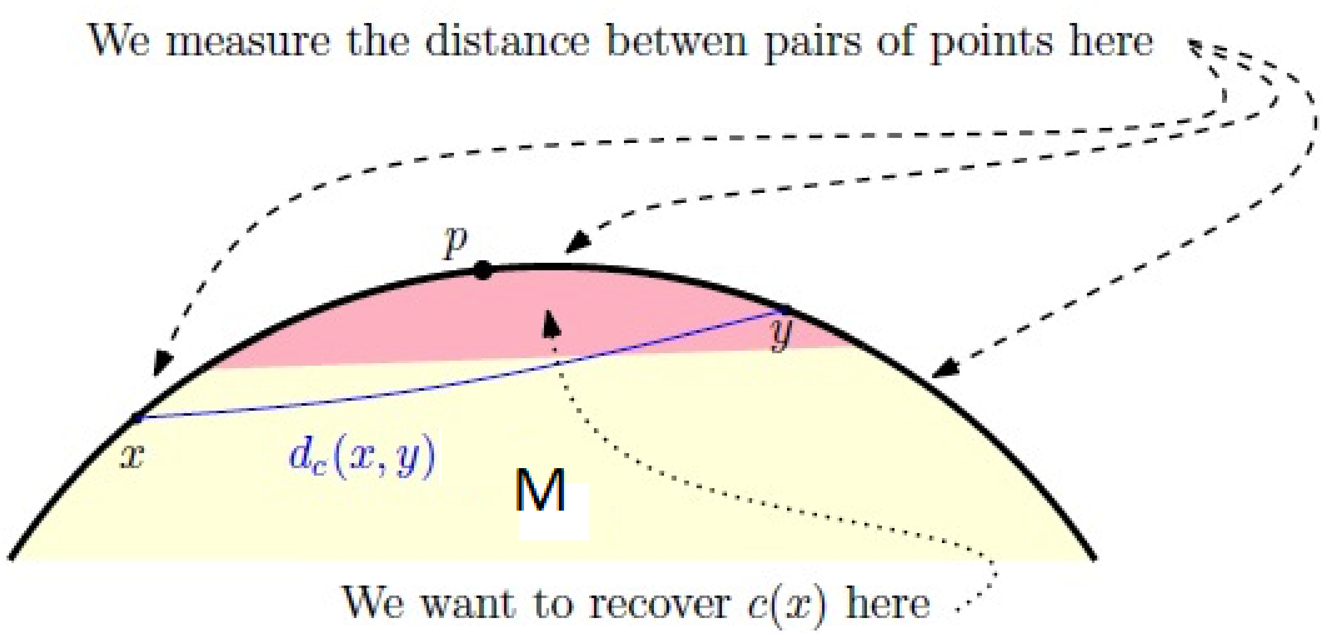
Remark: If (M, c) is simple, α_c is equivalent to d_c.

For non-simple metrics (caustics and/or non-convex boundary), this is the right problem to study.

For some of the results, see the survey [SUVZ].

PARTIAL DATA

Travel time with partial data: Does d_c, known on $\partial M \times \partial M$ near some p, determine c near p uniquely?



PARTIAL DATA

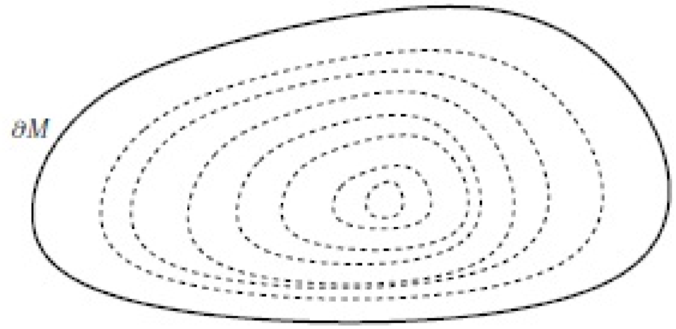
Theorem ([SUV]). *Let* $\dim M \geq 3$. *If* ∂M *is strictly convex near* p *for* c *and* $\widetilde{c}$, *and* $d_c = d_{\widetilde{c}}$ *near* (p, p), *then* $c = \widetilde{c}$ *near* p.

Also stability and reconstruction.

FOLIATION CONDITION

We could use a layer stripping argument to get deeper and deeper in M and prove that one can determine c in the whole M.

Foliation condition: M is foliated by strictly convex hypersurfaces if, up to a nowhere dense set, $M = \cup_{t \in [0,T)} \Sigma_t$, where Σ_t is a smooth family of strictly convex hypersurfaces and $\Sigma_0 = \partial M$.



A more general condition: several families, starting from outside M.

GLOBAL RESULT

Theorem ([SUV]). *Let* $\dim M \geq 3$, *let* c *and* $\widetilde{c}$ *be two smooth sound speeds on* M, *let* ∂M *be strictly convex with respect to both* c *and* $\widetilde{c}$. *Assume that* M *can be foliated by strictly convex hypersurfaces for* c. *Then if* $\alpha_c = \alpha_{\widetilde{c}}, d_c = d_{\widetilde{c}}$ *we have* $c = \widetilde{c}$ *in* M.

Also stability and reconstruction.

Examples: The foliation condition is satisfied for strictly convex domains of non-negative sectional curvature, simply connected domains with non-positive sectional curvature and simply connected domains with no focal points. Also if sound speed increases with depth.

IDEAS OF THE ROOF

The proof is based on two main ideas.

First, we use the approach in a recent paper by U-Vasy (2016) on the linearized problem with partial data.

Second, we convert the non-linear boundary rigidity problem to a "pseudo-linear" one. Straightforward linearization, which works for the problem with full data, fails here.

GEODESIC X-RAY TRANSFORM WITH PARTIAL DATA

U-Vasy result: Consider the inversion of the geodesic ray transform

$$If(\gamma) = \int f(\gamma(s))\,ds$$

known for geodesics intersecting some neighborhood of $p \in \partial M$ (where ∂M is strictly convex) "almost tangentially". It is proven that those integrals determine f near p uniquely. It is a Helgason support type of theorem for non-analytic curves! This was extended recently by H. Zhou for arbitrary curves (∂M must be strictly convex w.r.t. them) and non-vanishing weights.

U-VASY

The main idea in U-Vasy is the following:

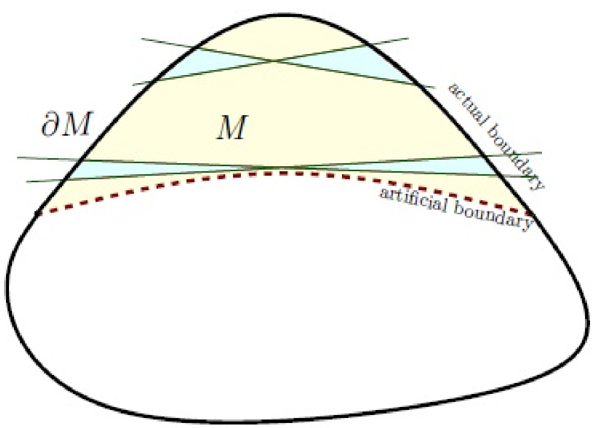
Introduce an artificial, still strictly convex boundary near p which cuts a small subdomain near p. Then use Melrose's scattering calculus to show that the I, composed with a suitable "back-projection" is elliptic in that calculus. Since the subdomain is small, it would be invertible as well.

U-VASY

Consider

$$Pf(z) := I^*\chi If(z) = \int_{S_zM} x^{-2}\chi If(\gamma_{z,v})dv,$$

where χ is a smooth cutoff sketched below (angle $\sim x$), and x is the distance to the artificial boundary.



INVERSION OF LOCAL GEODESIC TRANSFORM

$$Pf(z) := I^*\chi If(z) = \int_{S_zM} x^{-2}\chi If(\gamma_{z,v})dv,$$

Main result: P is an elliptic pseudodifferential operator in Melrose's scattering calculus.

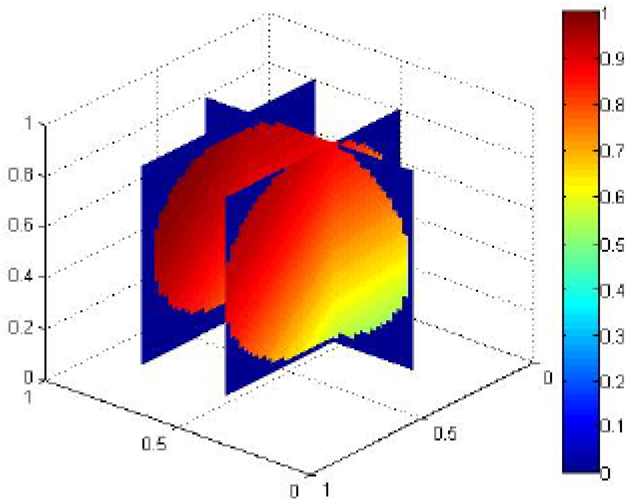
There exists A such that $AP = Identity + R$

This is Fredholm and R has a small norm in a neighborhood of p. Therefore invertible near p using Neumann series.

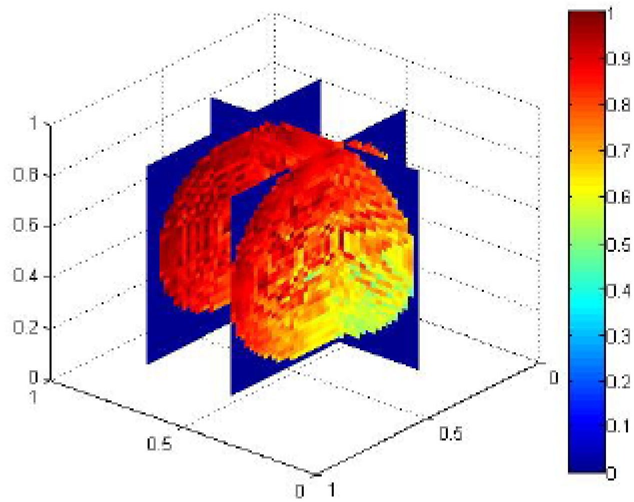
$$(Identity + R)^{-1}APf = f$$

.

SOME NUMERICAL RESULTS FOR INVERSE GEODESIC X-RAY TRANSFORM



(a) exact solution for f_1

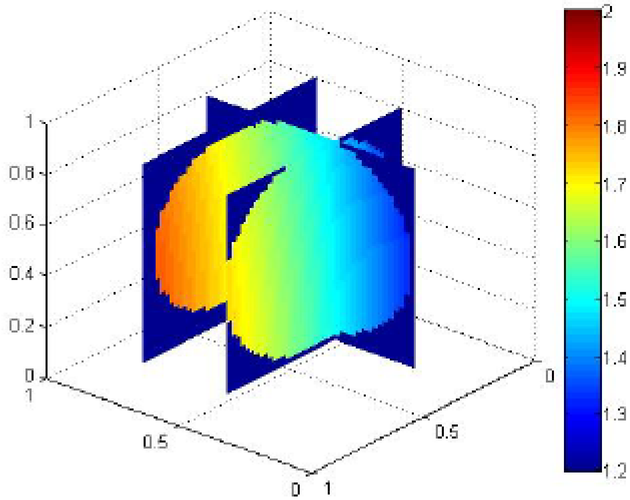


(b) approximate solution for f_1

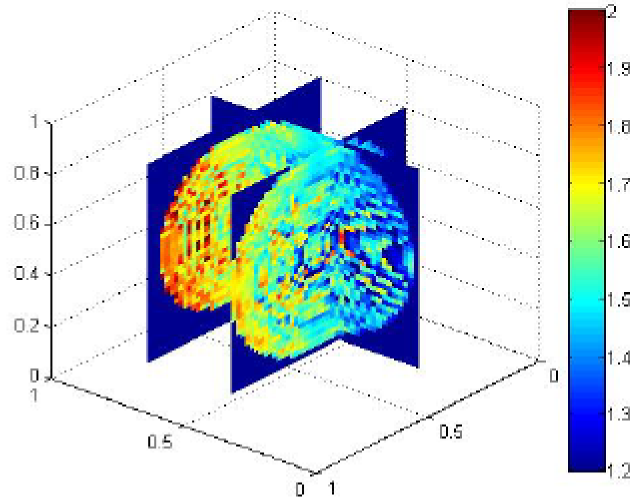
$$f_1 = 0.01 + \sin\left(2\pi(x+y+z)/10\right)$$

[ACU]

SOME NUMERICAL RESULTS FOR INVERSE GEODESIC X-RAY TRANSFORM



(c) exact solution for f_2

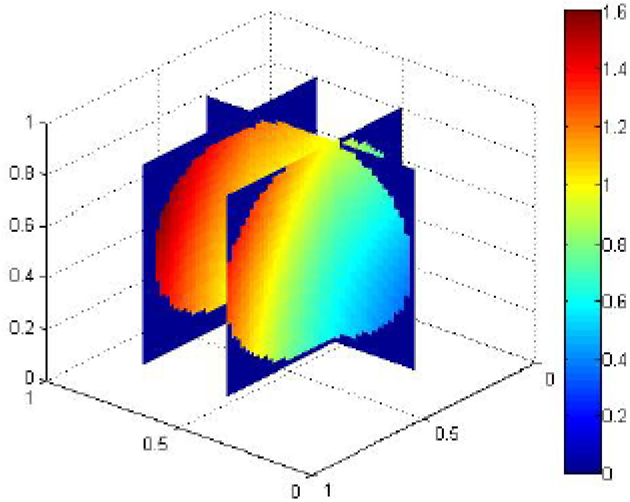


(d) approximate solution for f_2

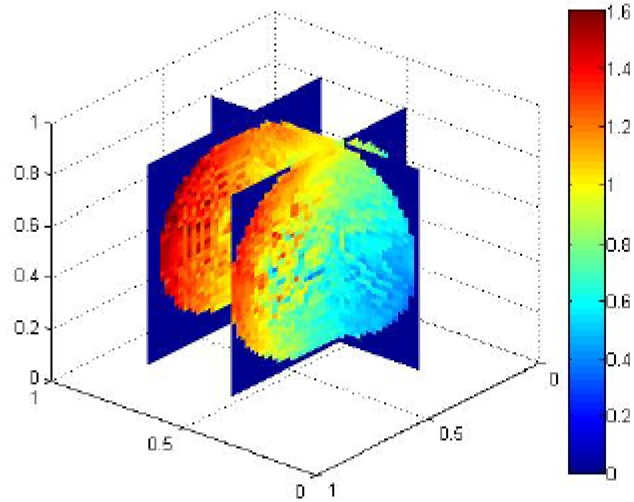
$$f_2 = 0.01 + \sin\left(2\pi(x+y)/10\right) + \cos\left(2\pi z/20\right)$$

[ACU]

SOME NUMERICAL RESULTS FOR INVERSE GEODESIC X-RAY TRANSFORM



(e) exact solution for f_3

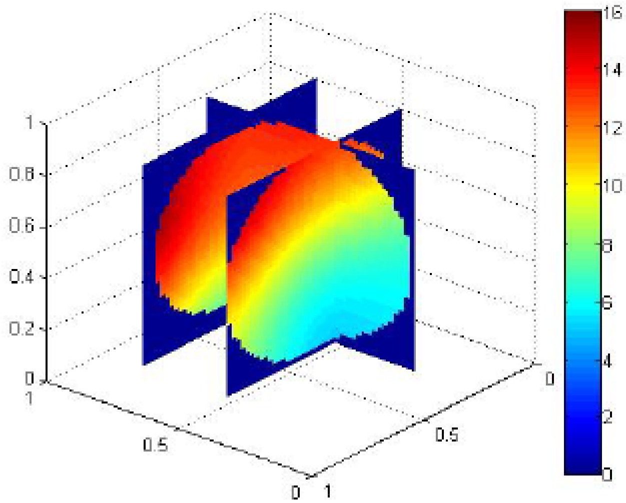


(f) approximate solution for f_3

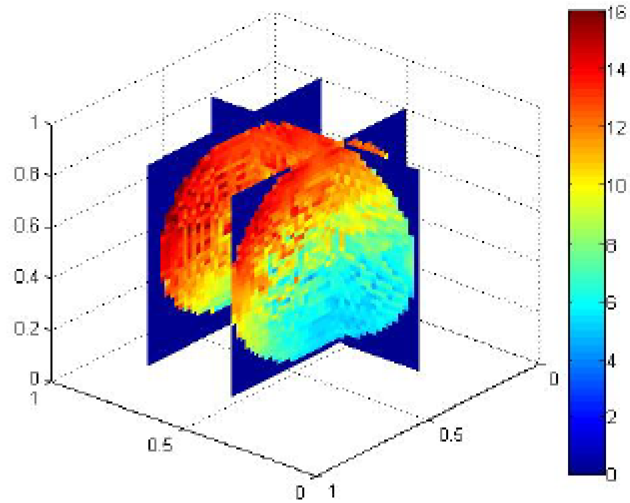
$$f_3 = x + y^2 + z^2/2$$

[ACU]

SOME NUMERICAL RESULTS FOR INVERSE GEODESIC X-RAY TRANSFORM



(a) exact solution for f_4

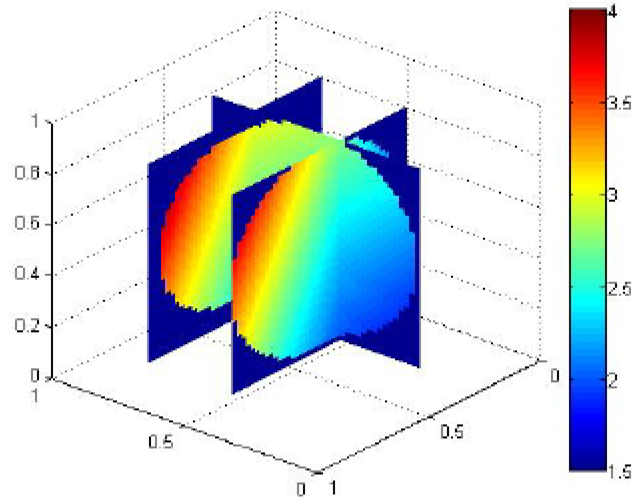


(b) approximate solution for f_4

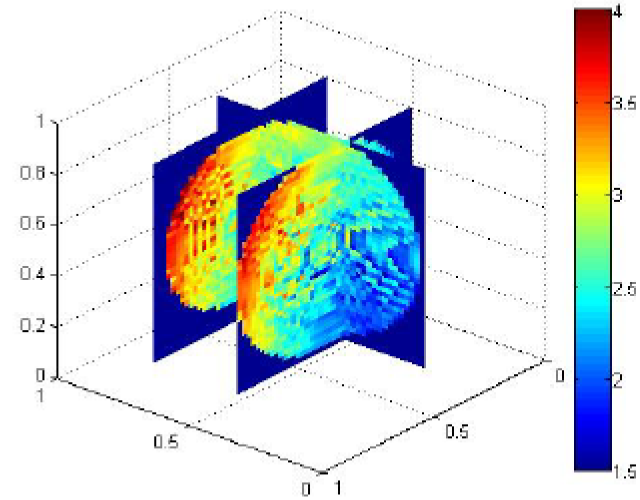
$$f_4 = 1 + 6x + 4y + 9z + \sin\left(2\pi(x+z)\right) + \cos\left(2\pi y\right)$$

[ACU]

SOME NUMERICAL RESULTS FOR INVERSE GEODESIC X-RAY TRANSFORM



(c) exact solution for f_5



(d) approximate solution for f_5

$$f_5 = x + e^{y+z/2}$$

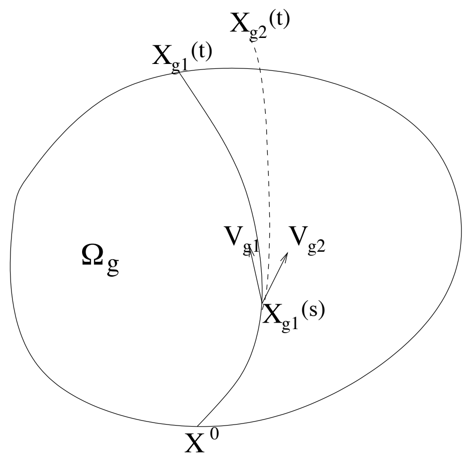
[ACU]

- Relative errors for using up to 4 terms in the Neumann series

relative error	f_1	f_2	f_3	f_4	f_5
n=0	37.1%	37.08%	37.13%	37.27%	37.25%
n=1	15.74 %	15.63%	15.81%	16.2%	16.32 %
n=2	8.92%	8.65%	9.09%	9.98%	10.28%
n=3	6.99%	6.55%	7.26%	8.61%	9.02%

SECOND STEP: REDUCTION TO PSEUDOLINEAR PROBLEM

Identity ([SU2])



$$g_i = \frac{1}{c_i^2} dx^2,$$
$$T = d_{c_1},$$
$$F(s) = X_{c_2}\big(T-s, X_{c_1}(s, X^0)\big),$$
$$F(0) = X_{c_2}(T, X^0), \quad F(T) = X_{c_1}(T, X^0),$$
$$\int_0^T F'(s)ds = X_{c_1}(T, X^0) - X_{c_2}(T, X^0)$$

$$\int_0^T \frac{\partial X_{c_2}}{\partial X^0}\left(T-s, X_{c_1}(s, X^0)\right)\left(V_{c_1} - V_{c_2}\right)\Big|_{X_{c_1}(s,X^0)} dS = X_{c_1}(T, X^0) - X_{c_2}(T, X^0)$$

IDENTITY([SU2])

$$\int_0^T \frac{\partial X_{c_2}}{\partial X^0}\left(T-s, X_{c_1}(s,X^0)\right)(V_{c_1}-V_{c_2})\Big|_{X_{c_1}(s,X^0)} dS = X_{c_1}(T,X^0) - X_{c_2}(T,X^0)$$

$$V_{c_j} := \left(\frac{\partial H_{c_j}}{\partial \xi}, -\frac{\partial H_{c_j}}{\partial x}\right)$$ the Hamiltonian vector field.

$$(g_k) = \frac{1}{c_k^2}\left(\delta_{ij}\right), \quad k = 1,2$$

$$V_{g_k} = (c_k^2 \xi,\ -\frac{1}{2}\nabla(c_k^2)|\xi|^2)$$

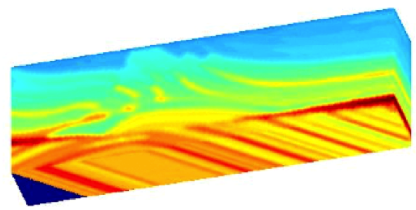
Linear in c_k^2!

RECONSTRUCTION

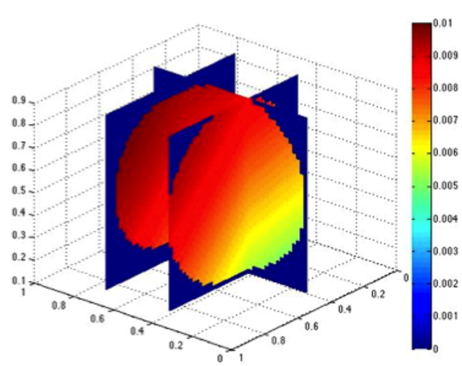
$$\int_0^T \frac{\partial X_{c_1}}{\partial X^0}\Big(T-s, X_{c_2}(s,X^0)\Big) \times \\ \Big((c_1^2-c_2^2)\xi, -\frac{1}{2}\nabla(c_1^2-c_2^2)|\xi|^2\Big)\Big|_{X_{c_2}(s,X^0)} dS \\ = \underbrace{X_{c_1}(T,X^0)}_{\text{data}} - X_{c_2}(T,X^0)$$

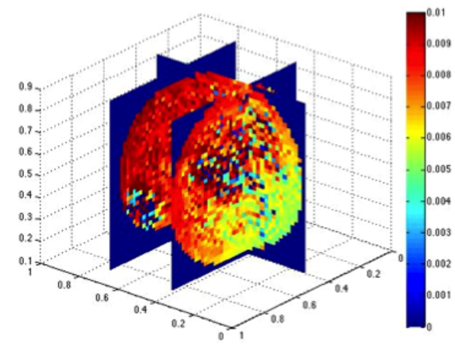
Inversion of weighted geodesic ray transform and use similar methods to U-Vasy.

- We test the method using a spherical section of the Marmousi model



- Results





	$n = 0$	$n = 1$	$n = 2$	$n = 3$
relative error	40.92%	19.89%	14.48%	14.20%
relative error with 5% noisy data	42.15%	22.33%	17.47%	17.12%

SCATTERING CALCULUS

The scattering calculus ([M1],[M2]) is a version of the classical one on $\mathbb{R}^n_x$ with a compactification of $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$. Consider pseudodifferential operators with symbols $a(z, \zeta)$ satisfying symbol-like estimates both w.r.t. z and ζ (Hörmander, Parenti, Shubin)

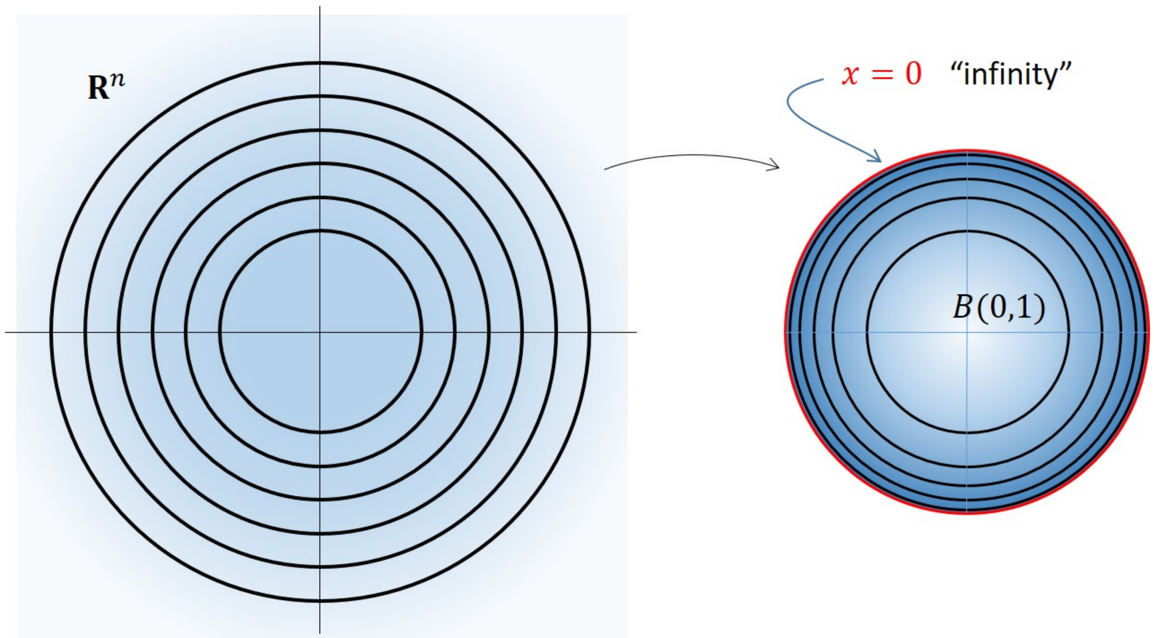
$$|\partial_z^\alpha \partial_\zeta^\beta a(z,\zeta)| \leq C_{\alpha,\beta} \langle z \rangle^{l-|\alpha|} \langle \zeta \rangle^{m-|\beta|}$$

This defines the class $S^{l.m}(\mathbb{R}^n \times \mathbb{R}^n)$. Lower order means both lower order of differentiaion and a slower growth at infinity.

Now compactify both $\mathbb{R}^n_x$ and $\mathbb{R}^n_\xi$ to get the scattering calculus.

SCATTERING CALCULUS

In polar coordinates $r\omega, r > 0, \omega \in S^{n-1}$, perform the change of variables $x = 1/r$ for $r \gg 1$. Then a neighborhood of ∞ becomes a neighborhood of 0, i.e., $0 < x < C$; and $x = 0$ is the "infinite boundary".



If one parametrizes S^{n-1} locally by $y \in \mathbb{R}^{n-1}$, then we have the coordinates

$$(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^{n-1} : \mathbb{R}_+^n$$

wth $x = 0$ defining S^{n-1}, flattened. The standard basic vector fields $\partial/\partial r$, $\partial/\partial(ry^j)$ take the form

$$x^2 \frac{\partial}{\partial x}, \quad x \frac{\partial}{\partial y^j},$$

and they are complete, tangent to $x = 0$ and unit. Those are the fields we use in the quantization and in the Sobolev spaces, as well.

We do that both for z and for its dual ζ. Then the class $\Psi^{l,m}(\mathbb{R}^n)$ becomes the class $\Psi^{l,m}(\mathbb{R}_+^n)$ with symbols in

$S^{l.m}(\mathbb{R}^n_+ \times \mathbb{R}^n_+)$. . This can be done on manifolds with boundary, as well.

There is a Fredholm theory of compact operators on such spaces.

Why the scattering calculus? When we approach the artificial boundary, the "angle of view" becomes smaller and the ellipticity degenerates. The classical calculus would not give us an elliptic operator.

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