MSRI LECTURE ON MICROLOCAL ANALYSIS AND INVERSE PROBLEMS

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ABSTRACT. Rough notes for the lecture on microlocal analysis and inverse problems at the MSRI introductory workshop in Fall 2019.

- Question: Can we determine internal properties of a medium by measurements on the boundary?
- Two types of scattering:
  - Linear: propagation independent of the medium
  - Nonlinear: propagation along curves determined by medium
- Hybrid methods: superposition of two images obtained by a single wave
- Model:
  - Consider $u_{tt} - c^2(x)\Delta u = 0$
  - Solve the inverse problem $u|_{t=0} = \beta H(x)$ and $\partial_t u|_{t=0} = 0$
  - Reconstruct H
- Simple case:
  - Consider for $q \in C_c^\infty(\mathbb{R}^n)$ with support contained in $B_R(0)$,
$$\begin{cases} u_{tt} - \Delta u + qu = 0 \\ u = (t - x \cdot w), \ t < -R \end{cases} \tag{1}$$
  - Inverse problem: Assuming we know $u(t, x \cdot w)$ for $t \gg 1$, can we recover q?
    * $\Box\delta(t - x \cdot w) = 0$ and $(\Box + q)\delta(t - x \cdot w) = q\delta(t - x \cdot w)$
    * Next try: $u_1(t, x \cdot w) = \delta(t - x \cdot w) + a_1(x, w)H(t - x \cdot w)$ where H is Heaviside function
    * Obtain $(\Box + q)u_1 = (q + 2\nabla a_1 \cdot w)\delta(t - x \cdot w) + (qa_1 - \Delta a_1)H(t - x \cdot w)$
    * First term being zero tells us $a_1(x, w) = -\frac{1}{2}\int_{-\infty}^{x\cdot w} q(x + (s - x \cdot w)w)ds$
    * If $x \cdot w > R$, then a_1 is the X-ray transform of $-\frac{q}{2}$
      - X-ray transform:
$$If(x, w) = \int f(x + sw)ds$$
for $f \in C_0^\infty(\mathbb{R}^n)$
    * Next try: $u_2 = \delta(t - x \cdot w) + a_1(x, w)H(t - x \cdot w) + a_2(x, w)(t - x \cdot w)$
    * Yields $\nabla a_2 \cdot w = -\frac{1}{2}(q(x)a_1 - \Delta a_1)$

    - This yields overall

$$u = \delta(t - x\cdot w) + \sum_{j=0}^{N} a_{j+1}(x,w)(t - x\cdot w) + C^{N-2}(\mathbb{R}^n_x \times \mathbb{R}_t)$$

      which is a conormal distrobution on $\{t = x\cdot w\}$.
    - Principal symbol determines X-ray transform of q
    - Transpose of X-ray:

$$I^*f(x) = \int_{S^{n-1}} f(x - (x\cdot w)w, w)dw$$

    - $I^*I = (-\Delta)^{1/2}$ which implies $(-\Delta)^{-1/2}I^*If = f$ for $f \in \mathcal{E}'(\mathbb{R}^n)$
      - Non-local inversion formula
    - Thus $WF((-\Delta)^{1/2}f) = WF(f)$
  - Example: $f = \sum_{i=1}^{2} a_i(x)\chi_{\Omega_i}$, Ω_i disjoint bounded domains.
    - Where are singularities of f?
    - Look at the symbol
- Acoustic Wave Equation:
  - Ω a bounded domain contained in a medium with $c(x) = 1$ for $x \notin \Omega$
  - Consider

$$\begin{cases} u_{tt} - c(x)\Delta u + qu = 0 \\ u = (t - x\cdot w) \end{cases} \tag{2}$$

  - Inverse problem: If we know $u(t, x\cdot w)$ for $t \gg 1$, can we recover c?
    - We get the expansion

$$u(t,x,w) = A_0(x,w)\delta(t-\phi(x,w)) + A_1(x,w)H(t-\phi(x,w)) + \sum_{j=1}^{\infty} A_{j+1}(x,w)(t-\phi(x,w))^j + C^\infty$$

  - Eikonal equation:
    - $(1 - c^2(x)|\nabla_x\phi|^2)\delta''(t - x\cdot w)$ yields

$$\begin{cases} |\nabla_x\phi|^2 = \frac{1}{c^2(x)} \\ \phi(x,w) = x\cdot w,\ x\cdot w < -R \end{cases} \tag{3}$$

    - Solve using Hamilton-Jacobi theory
- Transport equation:
  - Eliminate $\delta'(t - \phi(x,w))$ via

$$\begin{cases} 2c(x)\nabla_x\phi\cdot\nabla A_0 - c^2(x)\Delta\phi A_0 = 0 \\ A_0 = 1,\ x\cdot w < -R \end{cases} \tag{4}$$

- Boundary Rigidity:
  - Suppose we know $\phi(x,w)$, the geodesic distance.
  - Can we recover c?

– M bounded, $c \in C^\infty(M)$, smooth boundary
– $d_c(x, y) = \inf L(\sigma)$ where $L(\sigma) = \int_0^1 \frac{1}{c} |\frac{d\sigma}{dt}| dt$ σ a curve
– Determine c knowing d_c

Definition 0.1. *(M, c) is simple if for every $x, y \in \partial M$, there exists a unique minimal geodesic joining x to y and ∂M is strictly convex.*

Theorem 1. *One can determine c uniquely and stably from d_c if (M, c) is simple.*

Theorem 2. *Know u for $t > R$. Can determine c if $(B(0, R), c)$ is simple.*

Theorem 3. *Assume the map $T_x^* X \to X$ given by $v \mapsto \gamma(x, v)$, where $\gamma(x, y)$ is the geodesic starting at $x \in X$ with tangent v is a diffeomorphism. Then $I_c^* I_c$, the geodesic X-ray transform, is an elliptic ΨDO.*