$\mathbb{R}_x \times \mathbb{S}^1_\theta$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2}$$

$V \in L^\infty_c(X;\mathbb{C})$

$-\Delta + V$ on $\mathcal{X}$

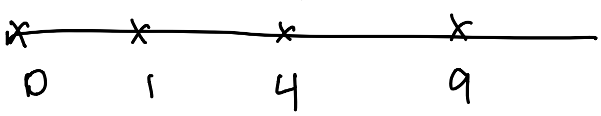
one-dim Schrödinger operator

$$-\frac{d^2}{dx^2} + V_0$$

$$V_0(x) = \frac{1}{2\pi}\int_0^{2\pi} V(x,\theta)\,d\theta$$

Spectrum $-\Delta+V$ on $\mathcal{X}$

continuous spectrum $[0,\infty)$



up to infinitely many eigenvalues.

only accumlation point at infinity

Resolvent: $R_V(\zeta) = (-\Delta + V - \zeta^2)^{-1}$, If $\operatorname{Im} \zeta > 0$

If $\chi \in C_c^\infty(X)$, then $\chi R_V(\zeta)\chi$ has a meromorphic continuation to $\hat{Z}$: smallest Riemann surface on which

$$T_j(\zeta) := (\zeta^2 - j^2)^{\frac{1}{2}}$$

is a single-valued analytic function for all $j \in \mathbb{N}_0$

($\operatorname{Im} T_j(\zeta) > 0$ if ζ in the physical space ($\operatorname{Im}\zeta > 0$)

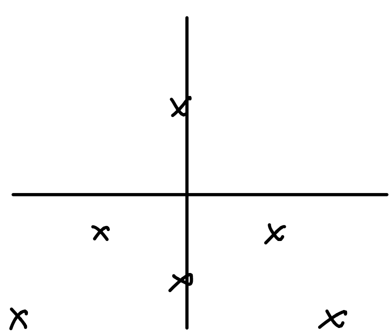
$R_V(\zeta)$ is bounded on $L^2(X)$.

poles of $\chi R_V(\zeta)\chi$ are resonances.

1-D problem: $-\frac{d^2}{dx^2} + W$, $W \in L_c^\infty(\mathbb{R})$

$R_{W,0}(\lambda) = (-\frac{d^2}{dx^2} + W - \lambda^2)^{-1}$, if $\operatorname{Im}\lambda > 0$

meromorphic continuation to $\mathbb{C}$.

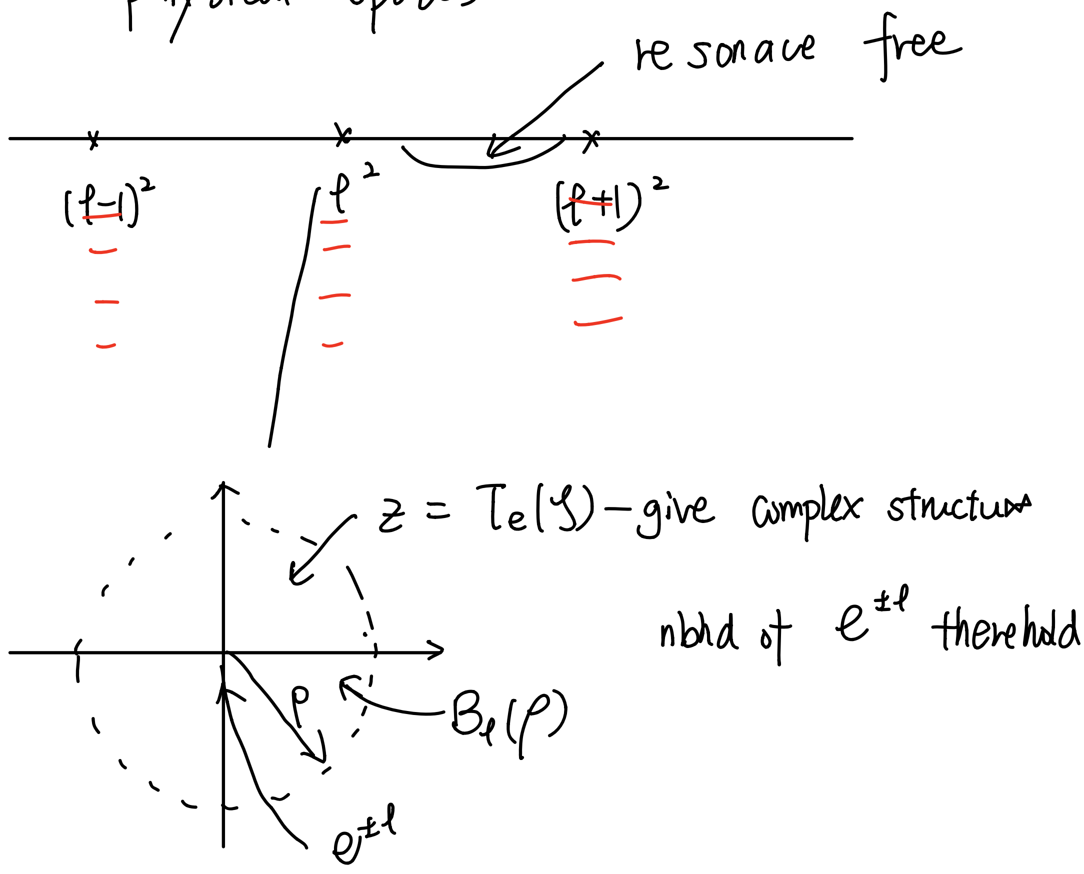


$$R_{V_0}(\varsigma) = \sum_{j=0}^{\infty} R_{V_0,0}\big(T_j(\varsigma)\big) P_j \qquad j>0$$

P_j: projection onto span $e^{ij\theta} \wedge e^{-ij\theta}$

Cartoon of partition of $\hat{\mathbb{Z}}$

physical spaces



Set $V_m(x) = \frac{1}{2\pi}\int V(x,\theta)\, e^{-im\theta}\, d\theta \qquad m \in \mathbb{Z}$

Theorem: Let $V \in L^{\infty}_c(X)$, $\|V_m\|_{L^\infty} = \mathcal{O}(m^{-\frac{1}{2}})$

Suppose $\lambda_0 \in \mathbb{C}$ is a pole of $R_{V_0,0}(\lambda)$ with multiplicity $m_{V_0,0}(\lambda_0)$. Then there is a $C>0$ so that for ℓ suff large there are exactly $2m_{V_0,0}(\lambda_0)$ poles of $R_V(\zeta)$, when counted with multiplicity in $\{\zeta \in B_\ell(|\lambda_0|+\ell) : |\tau_\ell(\zeta)-\lambda_0|<C\}$

Theorem: V be as above. Given $\rho>0$, set

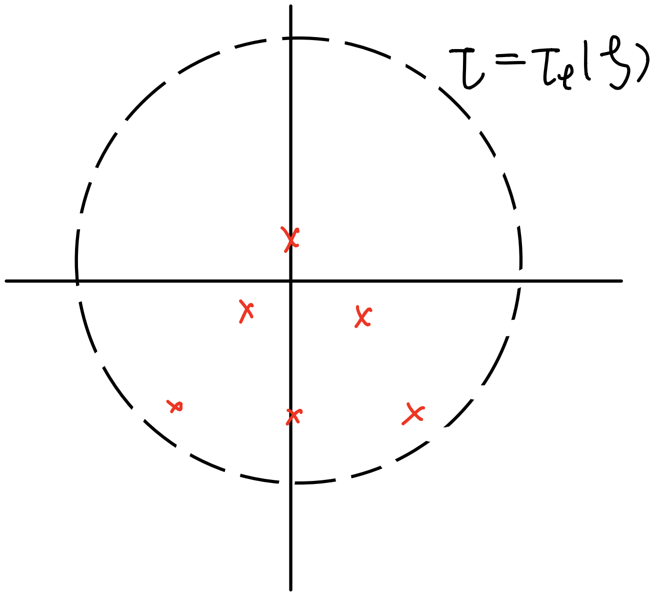
$$\Lambda_\rho = \{\lambda_j \in \mathbb{C} : \lambda_j \text{ is a pole of } R_{V_0,0}(\lambda),\ |\lambda_j| \le \rho+1\}.$$

Then there is a $C>0$ so that for ℓ sufficient large there are no poles of $R_V(\zeta)$ in

$$\{\zeta \in B_\ell(\rho) : |\tau_\ell(\zeta)-\lambda_j| > C\ell^{-1/2m_{V_0,0}(\lambda_j)} \text{ for all } \lambda_j \in \Lambda_\rho\}$$

If $V \in C^\infty_c$, can improve to

$$|\tau_\ell(\zeta) - \lambda_j| > C\ell^{-2/m_{V_0,0}(\lambda_j)}$$



Thm: Let $V \in C_c^\infty(X)$, & let λ_0 is a simple pole of $R_{V_0 0}(\lambda)$, and that

$$R_{V_0 0}(\lambda) - \frac{i}{\lambda - \lambda_0} u \otimes u$$

is analytic near $\lambda - \lambda_0$.

Then for ℓ suff large, $R_V(\zeta)$ has

<u>either</u> two simple poles $\zeta_{\ell\pm}$ in $B_\ell(|\lambda_0|+1)$ satisfying

$$\tau_\ell(\zeta_{\ell\pm}) = \lambda_0 - \frac{i}{4\ell} \sum_{k \neq 0} \frac{1}{k^2} \int (k^2 V_k V_k + V_k' V_{-k}') u^2 \, dx + O(\ell^{-3})$$

<u>or</u> a simple pole of multiplicity 2 with same asympt expansion.

Cor: Suppose $V \in C_c^\infty(X; \mathbb{R})$. Suppose for each $\rho > 0$ there is a sequence $\{\ell_j\} = \{\ell_j(\rho)\} \subset \mathbb{N}$, $\ell_j \to +\infty$ as $j \to +\infty$, so that $-\Delta$ and $-\Delta + V$ have the same resonances in $B_{\ell_j}(\rho)$.

Then $V \equiv 0$.

Steps: ① $V_0 \equiv 0$ by showing $-\frac{d}{dx^2} + V_0$ has reso only at 0.

② correction term, $V_{-k} = \overline{V_k}$

Wave equation:

$$(\partial_t^2 - \Delta + V)u = 0 \quad \text{on} \quad X \times [0,\infty),$$
$$(u, u_t)|_{t=0} \in C_c^\infty(X) \times C_c^\infty(X)$$

Thm: $V \in C_c^\infty(X, \underline{\underline{\mathbb{R}}})$. Suppose $-\frac{d^2}{dx^2} + V_0$ on $\mathbb{R}$ has no negative eigenvalues & does not have a resonance at 0. For $k_0 \in \mathbb{N}$, can write

$$u(t) = u_{ev}(t) + u_{thr}(t) + u_r(t)$$

where u_{ev} contribution of eigenvalues / efcns of $-\frac{d^2}{dx^2} + V_0$

$$u_{thr}(t) = b_{00} + \sum_{k=0}^{k_0-1} t^{-k-\frac{1}{2}} \sum_{j=1}^{\infty} \left(e^{it_j} b_{jk+} + e^{-it_j} b_{jk-}\right)$$

If $\chi \in C_c^\infty(X)$, $m \in \mathbb{N}$,

$$\sum_j \|\chi b_{jk\pm}\|_{H^m} < \infty$$
$$\|\chi u_r\|_{H^m} \leq C t^{-k_0 - \frac{1}{2}}$$

The last thm follows rest + work w/ K. Datchev.

Sources of inspiration :

* paper of Drouot

  resonances of $-\Delta + W_\varepsilon$ on $\mathbb{R}^d$, d odd

  $$W_\varepsilon(x) = V_0(x) + \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} V_k(x) e^{ik \cdot x/\varepsilon} \qquad \varepsilon > 0 \text{ small}$$

  as $\varepsilon \downarrow 0$, resonances of $-\Delta + W_\varepsilon$ well-approximated by resonances of $-\Delta + V_0$.

* paper on eigenvalues of $-\Delta + V$ on $\mathbb{S}^d$

  Weinstein (Guillemin, Widom, Friedlander ⋯)