C. Sun (Cergy)

$$\Omega = \mathbb{T}_y \times (-1, 1)_x$$

$$\begin{cases} (i\partial_t + \underbrace{\partial_x^2 + x^2\partial_y^2}_{H})\, u = 0 \\ u|_{\partial\Omega} = 0 \\ u|_{t=0} = u_0 \end{cases}$$

- concentration
- propogation

Fourier transform on y

$$(\partial_x^2 - x^2 n^2)\, e = \lambda^2 e$$

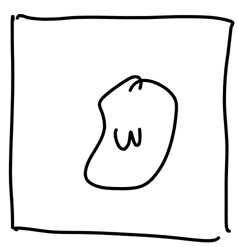
← Not what we are interested in

Def: exact controllability iff

$\forall u_0, u_1 \in L^2(\Omega)$, $\exists f \in L^2_{t,x,y}$

$$(i\partial_t + H)\, u = f\, \mathbb{1}_{\omega \times (0,1)} + \text{bc}$$

s.t. $u|_{t=0} = u_0$, $u|_{t=T} = u_1$



Def: observable (T, ω) iff $\exists C, \forall u_0$

$$\|u_0\|_{L^2}^2 \leq C \int_0^T \int_\omega |e^{itH} u_0|^2 \, dx\, dy\, dt$$

Thm (HUM)

E.C. TW $\Leftrightarrow$ Obs T, ω

$$\partial_x^2 + x^2 \partial_y^2 \to \partial_x^2 + \partial_y^2$$

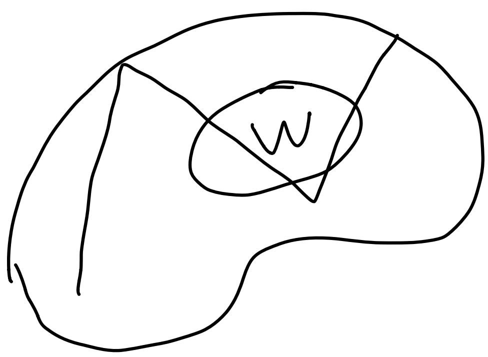
Thm: For heat equation $\forall \omega, \forall T > 0$, obs

$$\|e^{T\Delta} u_0\| \leq C \|e^{t\Delta} u_0\|_{L^2(0,T) \times \omega}$$

For wave equation

$$\|u_0\|_{L^2} \leq C \|e^{it\sqrt{-\Delta}} u_0\|_{L^2((0,T)\times\omega)}$$

$\Leftrightarrow$ geometric control condition



For Schrödinger equation

$$\|u_0\|_{L^2} \le C\|e^{it\Delta}u_0\|_{L^2((0,T)\times\omega)}$$

$\Leftarrow$ Geometric control condition (T_0)

Thm: M^2 ($\uparrow$ compact), $K<0$, $\forall \omega$ open, $\forall T>0$

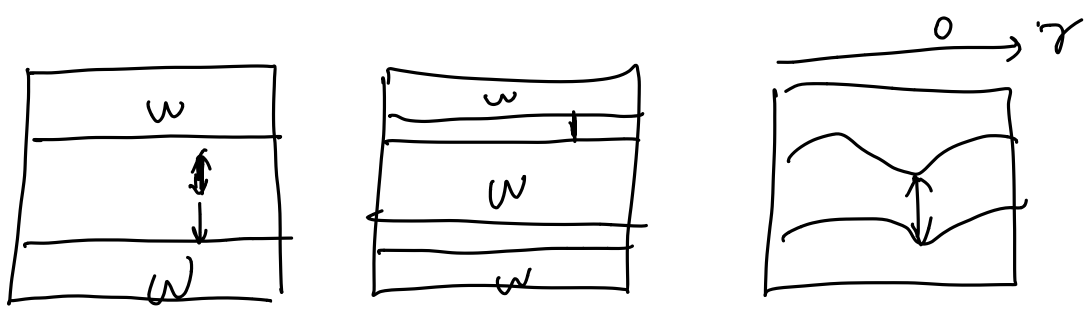
$$\|u_0\|_{L^2} \le C\|e^{it\Delta}u_0\|_{L^2((0,T)\times\omega)}$$

$M=\mathbb{T}^d$, OK $\forall\omega$, $\forall T>0$ 90'

$M=T^2$, ω $|\omega|>0$

Def: $\omega = \bigcup$ horizontal bounds

$\mathcal{L}(\omega)$ = the size of largest bound in ω



Thm (N.B, C. Sun 19) $\partial_x^2 + x^2\partial_y^2$

Obs $\forall T > \mathcal{L}(\omega)$

obs false $\forall T \leq \mathcal{L}(\omega)$

$-\partial_x^2 - x^2\partial_y^2$

Fourier transform in $y \Rightarrow -\partial_x^2 + x^2\eta^2 = \eta^2(-\eta^{-2}\partial_x^2 + x^2)$

$e_{\eta,1}(x) \sim e^{-\eta\frac{x^2}{2}}$, $\lambda_{\eta,1} \sim |\eta|$

$u_0 = \sum_\eta u_\eta e_{\eta,1} e^{i\eta y}$

$i\partial_t u + Hu = 0$

$(i\partial_t - |D_y|)u \sim 0$

1) Reduce Prob to spectrally local data

$$u_0 \to \chi(h^2 H) u_0$$

2) Understand regimes (semi-classical)

1) Wave $\quad \xi = 0,\ x = 0,\ \eta = +\infty$

2) Semiclassical propogation $\begin{cases} 0 < \eta < +\infty \\ \xi, x < +\infty \end{cases}$

3) Transversal propogation $\quad \eta = 0$

$$u_0 = \chi(h^2 H)\, \chi(\varepsilon |D_y|)\, u_0 \qquad \chi \in C^\infty(\tfrac{1}{2}, 2)$$

$$h^2 \leq \varepsilon \leq 1 \qquad H \gg \varepsilon^{-1} \gg h^{-2}$$

Semi-Classical

$$h^2 D_t \sim \tau \qquad \tau \sim \xi^2 + x^2 \eta^2$$

$$h D_x \sim \xi$$

$$h D_y \sim \eta$$

$u = e^{itH} u_0$

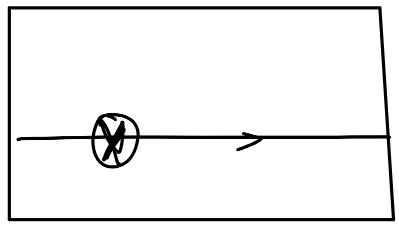
$\chi(h^2 H) u = \chi(h^2 D_t) u$

$u = e^{itH} u_0$

$v(s,x) = u(hs, x)$

$= e^{\frac{ish^2 H}{h}} u_0$

$(ih\partial_s + h^2 H) v = 0$



$GC \Rightarrow \quad \|v_0\|_{L^2} \leq \int_0^{\alpha} \int_\omega |v|^2(s,x)\, ds\, dx$

$\|u_0\|_{L^2} \leq \frac{1}{h} \int_0^{\alpha h} \int_\omega |u|^2(s,x)\, dt\, dx$

$\leq \frac{1}{h} \int_{k\alpha h}^{(k+1)\alpha h}$ ———

$\frac{C}{h} \|u_0\|_{L^2} \leq \frac{1}{h} \int_0^T$ ———

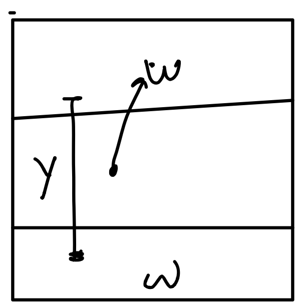
$\xi^2 + x^2 \eta^2$

$\dot{x} = 2\xi$

$\dot{\xi} = -2x\eta^2$

$\dot{y} = 2x^2 \eta$

$\dot{\eta} = 0$



Wave regime:

$v \quad s \sim \frac{\mathcal{L}(w)}{h}$

$$[\overbrace{i\partial_t + H}^{P}, x\partial_x + 2y\partial_y] = 2H$$

$$([P, x\partial_x + y\partial_y]u, u)_{L^2_{t,x,y}} = 2(-Hu, u)$$

$$\| $$

$$2\|u\|^2_{H^1_G}$$

$$(-Hu, u) = \int |\partial_x u|^2 + |x\partial_y u|^2$$

$$= ((PQ - QP)u, u)$$

$$= -(Q, P^* u) = 0$$

$$u \mapsto \varphi\left(\frac{t}{T}\right)\chi(x)\zeta(y)u$$

* Lots of terms including φ', χ', ζ'

* Hypoellipticity

$$\|\langle\partial_y\rangle^{\frac{1}{2}} u\|_{L^2} \leq C\|u\|^2_{H^1_G}$$

$$2t\|u\|^2_{H^1_G} \leq \text{Obser} + 2\mathcal{L}(w)\|u\|^2_{H^1_G} + \text{l.o.t.}$$

Observability and control for Grushin Schrödinger equations

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Joint work with Chenmin Sun (Université Cergy Pontoise)

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Laboratoire de Mathématiques d'Orsay,
CNRS UMR 8628
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MSRI 14-18 oct. 2019

Outline

1. Grushin-Schrödinger equation
   - Controllability and observability
   - The geometric control condition
   - Main results and motivations
2. The different regimes
   - The half wave regime
   - The different regimes
     - Half-wave
     - Semi-classical propagation
     - Transversal propagation
3. Proofs
   - Semi-classical propagation
   - The half wave regime
   - Transversal propagation

2D Grushin-Schrödinger equation

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the 1D torus, and $\Omega = (-1,1)_x \times \mathbb{T}_y$. Let $\Delta_G = \partial_x^2 + x^2\partial_y^2$ be the Grushin operator with domain

$$D(\Delta_G) = \{f \in L^2(\Omega) : \Delta_G f \in L^2(\Omega), f|_{\partial\Omega} = 0\}.$$

$$\Delta_G = X_1^2 + X_2^2\text{: type I}$$

and

$$X_1 = \partial_x, X_2 = x\partial_y; [X_1, X_2] = \partial_y, \mathrm{span}\{X_1, X_2, [X_1, X_2]\} = \mathbb{R}^2 = T_x\mathbb{R}^2.$$

Objective: Observability and Controlability for Grushin-Schrödinger equation with Dirichlet boundary condition:

$$\begin{cases} i\partial_t u - \Delta_G u = 0, \quad (t,x,y) \in \mathbb{R} \times \Omega \\ u\big|_{t=0} = u_0 \\ u\big|_{\mathbb{R}\times\partial\Omega} = 0 \end{cases}$$

2D Grushin-Schrödinger equation

- Conservation of mass:

$$M(u) = \int_\Omega |u(t,x,y)|^2 dxdy.$$

- Conservation of energy:

$$\|u\|^2_{H^1_G} = \int_\Omega (|\partial_x u(t,x,y)|^2 + |x\partial_y u(t,x,y)|^2) dxdy.$$

- Subellipticity: Energy controls $H^{1/2}$ norm

$$[\partial_x, x\partial_y] = \partial_y \Rightarrow \exists C > 0; \|u\|^2_{H^{1/2}_{x,y}} \leq C|u\|^2_{H^1_G}$$

Controllability

Let $\omega \subset \Omega$ be a non-empty open set, and $T > 0$.

Definition (Exact-controllability on ω at time T)

$\forall u_0, u_1 \in L^2(\Omega)$,$\exists f \in L^2([0,T] \times \omega)$ such that the solution u of

$$(i\partial_t - \Delta_G)u = f, \quad u|_{\partial\Omega} = 0, u|_{t=0} = u_0$$

satisfies $u|_{t=T} = u_1$

Definition (Observability on ω at time T)

$\forall u_0 \in L^2(\Omega)$, $\exists f \in L^2([0,T] \times \omega)$ such that the solution u of

$$(i\partial_t - \Delta_G)u = f, \quad u|_{t=0} = u_0, \quad u = e^{it\Delta_G} u_0,$$

$$\|(e^{iT\Delta} u_0)\|^2_{L^2(\Omega)} \leq C_T \int_0^T \|(e^{it\Delta_G} u_0)\|^2_{L^2(\omega)} dt$$

Observability results for three typical PDE's

- Heat: (Lebeau-Robbiano, Fursikov-Imanuvilov): $\omega \neq \emptyset$, $T > 0$,

$$\|e^{T\Delta} u_0\|^2_{L^2(\Omega)} \leq C_T \int_0^T \|e^{t\Delta} u_0\|^2_{L^2(\omega)} dt.$$

  Spectral inequality(Carleman) + very fast dissipation of HF.
- Wave: (80' Rauch-Taylor, Bardos-Lebeau-Rauch, Burq-Gérard) ω satisfies the geometric control condition (GCC), $T > T_{GCC}$,

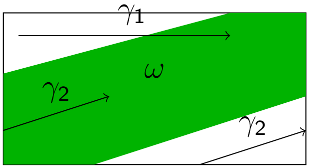
$$\|u_0\|^2_{L^2(\Omega)} \leq C_T \int_0^T \|e^{it\sqrt{-\Delta}} u_0\|^2_{L^2(\omega)} dt.$$

  Melrose-Sjöstrand propagation of singularities. Finite propagation speed.
- Schrödinger: Infinite propagation speed
  - (Lebeau 90') ω satisfies (GCC), $T > 0$, then

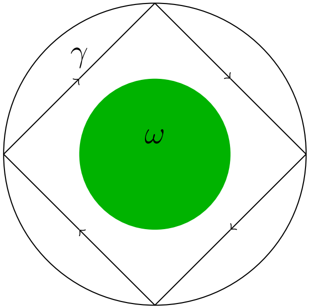
$$\|u_0\|^2_{L^2(\Omega)} \leq C_T \int_0^T \|e^{it\Delta} u_0\|^2_{L^2(\omega)} dt.$$

GCC: pictures

ω satisfies (GCC) if there exists $T_{GCC} > 0$, such that all generalized geodesics of length $T > T_{GCC}$ intersects with ω. .



GCC is satisfied



GCC is not satisfied

Observability for Schrödinger equation beyond GCC

Schrödinger: GCC not necessary (stability/instability) of geodesic flow

Theorem (Jaffard 90, N.B.-Zworski 03, 12, Anantharaman-Macià 14, Bourgain-N.B.-Zworski 13, N.B.-Zworski 17)

$\Omega = \mathbb{T}^2$, $\forall E, |E| > 0$ *and* $T > 0$+ *stable perturbations* $V \in L^2(\mathbb{T}^2)$ *and let* $P = -\Delta + V(x)$. $\exists C(T, \omega) > 0$, $\forall u_0 \in L^2(\mathbb{T}^2)$,

$$\|u_0\|^2_{L^2(\mathbb{T}^2)} \leq C(T, E) \int_0^T \|e^{itP} u_0\|^2_{L^2(E)} dt.$$

Theorem (Schenck 10, Anantharaman-Rivière 10, Bourgain-Dyatlov 16, Jin 17, Dyatlov-Jin-Nonnenmacher 19)

Ω *surface neg curvature,* $\forall \omega$, *open,*$\forall T > 0$. $\exists C > 0$, $\forall u_0 \in L^2(\mathbb{T}^2)$,

$$\|u_0\|^2_{L^2(\Omega)} \leq C(T, \omega) \int_0^T \|e^{it\Delta} u_0\|^2_{L^2(\omega)} dt.$$

Bibliographic for the results of parabolic-Grushin:

Question: Hypoelliptic geometry?

- Heat type equation:(Alabau, Beauchard, Cannarsa, Duprez, Guglielmi, Koenig, Pravda-Starov, ...) for different operators A and control domains ω, new phenomena happen: observability false for some $T > 0$ or even for all finite $T > 0$!

In particular, for Grushin (heat) equation

$$\partial_t u - \Delta_G u = 0$$

on $\Omega = (-1, 1)_x \times \mathbb{T}_y$, the following striking result holds:

Theorem (A. Koenig '17)

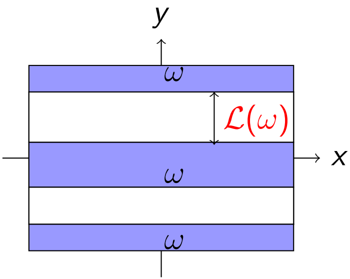
Assume that there exists a horizontal strip $(-1, 1)_x \times (a, b)_y$ which does not encounter ω. Then for any $T > 0$, the heat-observability is untrue (as well as null-controllability).

Schrödinger: No $T \leq \mathcal{L}(\omega)$ observability!

Let ω be of the form $(-1,1)_x \times I$, where $I \subset \mathbb{T}$ is a finite union of intervals. For such ω, we define $\mathcal{L}(\omega)$:

$$\mathcal{L}(\omega) := \sup\{s : \exists y_1, y_2 \in \mathbb{T}, \mathrm{dist}_{\mathbb{T}}(y_1, y_2) = s, [(0,y_1),(0,y_2)] \cap \omega = \emptyset\}$$

the lenght of largest interval in $\Omega \setminus \omega \cap \{x = 0\}$.



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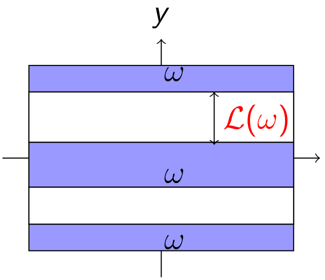

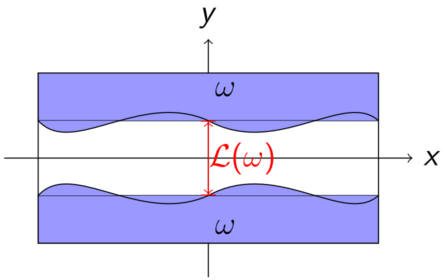

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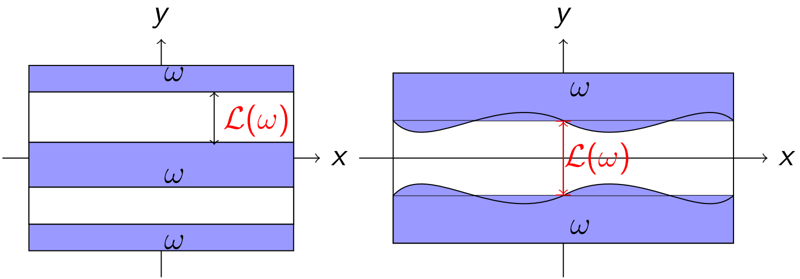

Theorem (N. B, Chenmin Sun. '19)

If $T \leq \mathcal{L}(\omega)$, the observability by (T, ω) is false.

Observability $T > \mathcal{L}(\omega)$

Theorem (N. B, Chenmin Sun '19)

Assume that $T > \mathcal{L}(\omega)$. There exists $C_T > 0$, such that for all $u_0 \in L^2(\Omega)$,

$$\|u_0\|^2_{L^2(\Omega)} \leq C_T \int_0^T \|e^{it\Delta_G} u_0\|^2_{L^2(\omega)} dt.$$

Corollary (Exact-controllability)

Assume that $T > \mathcal{L}(\omega)$. For any $u_0, v_0 \in L^2(\Omega)$, there exists $f \in L^2([0, T] \times \omega)$, such that the solution of

$$i\partial_t u + \Delta_G u = \mathbf{1}_\omega f, \qquad u \mid_{\partial\Omega} = 0, \qquad u|_{t=0} = u_0$$

satisfies that $u|_{t=T} = v_0$.

A harmonic oscillator

Take fourier transform w.r.t. y

$$(i\partial_t u + \partial_x^2 u - x^2\eta^2)\widehat{u}(t,x,\eta) = 0,$$

The harmonic oscillator

$$-\partial_x^2 u + x^2\eta^2 = \eta^2(-\eta^{-2}\partial_x^2 u + x^2)$$

has a fondamental (ground) state, with eigenvalue $\lambda_{\eta,1}$

$$e_{\eta,1}(x) \sim e^{-\frac{|\eta|x^2}{2}}, \quad \lambda_{\eta,1} \sim |\eta|,$$

Take initial data distributed on these ground state

$$u_0 = \sum_{\eta\in\mathbb{Z}} \alpha_n e^{i\eta y} e_{\eta,1}(x),$$

Then solution of Grushin-Schrödinger with initial data u_0 is.a solution of (half) wave equation

$$(i\partial_t + |D_y|)u \sim 0.$$

Harmonic oscillator: consequences

- Finite speed of propagation for Half wave equation responsible for the requirement $T \geq \mathcal{L}(\omega)$.
- If $|D_y| \sim \epsilon^{-1}$, then $-H \geq \epsilon^{-1}$.

Semi-classical reduction and prove observability only for initial data

$$u_0 = \chi(-h^2 H)u_0, \chi \in C_0^\infty(\frac{1}{2}, 2), \chi\mid_{(\frac{1}{\sqrt{2}}, \sqrt{2})} = 1$$

Work on Characteristic manifold after (anisotropic) semi-classical scaling, $\tau \sim h^{-2}, -\Delta_G \sim h^{-2}$

$$\{(t, \tau, x, \xi, y, \eta); \tau = \xi^2 + x^2\eta^2, \tau \in (\frac{1}{2}, 2)\}$$

Second microlocalize w.r.t. η variable and assume

$$u_0 = \chi(-h^2 H)\chi(\epsilon|D_y|)u_0, \chi \in C_0^\infty(\frac{1}{2}, 2), \chi\mid_{(\frac{1}{\sqrt{2}}, \sqrt{2})} = 1$$

We can assume

$$h^2 \leq \epsilon \leq 1$$

The semi-classical regimes ($\xi = hD_x, \eta = hD_y$)

- The Half wave regime $\xi = 0, x = 0, \eta = +\infty$,

$$|D_x| \ll h^{-1}, \quad |x| \ll 1, \quad |D_y| \gg h^{-1} \text{ (but } |D_y| \leq h^{-2})$$

  $\eta \sim h^{-2}$ responsible for finite time observation. Careful positive commutator estimates

- The Semi-classical propagation regime $0 < \eta < +\infty$

$$(x, hD_x) \text{ bounded }, ch^{-1} < |D_y| < Ch^{-1}$$

  semi-classical propagation $\Rightarrow$ arbitrary small time observation

- The Transversal propagation regime $\eta = 0$
  - Rapid propagation regime $h^{-\delta} \leq |D_y| \ll h^{-1}$, $0 < \delta < 1/4$: semi-classical propagation + positive commutator
  - Normal form regime: $|D_y| \leq h^{-\delta}$: normal form + positive commutator

Semi-classical propagation: Lebeau's method I

Theorem (Lebeau '92)

Non degenerate Laplace. Assume geometric control condition.

$$\forall T > 0, \exists C_T > 0, \quad \|u_0\|^2_{L^2(\Omega)} \leq C_T \int_0^T \|e^{it\Delta}(t,\cdot)\|^2_{L^2(\omega)} dt,$$

- Unique continuation + semi-classical observation $0 < h \ll 1$ ($\psi \in C_c^\infty(1/2 \leq |r| \leq 2)$.)

$$\|\psi(h^2\Delta)u_0\|_{L^2(\Omega)} \leq C_T \int_0^T \|\psi(h^2\Delta)u(t,\cdot)\|^2_{L^2(\omega)} d,$$

- Rescaling in time $v(s,x) = u(hs,x)$: semi-classical Schrödinger equation:

$$ih\partial_s v_h + h^2\Delta v_h = 0, \text{ where } v_h(s,x) = \psi(h^2\Delta)u(hs,x).$$

Semi-classical propagation: Lebeau's method I

- Propagation of singularities: for semi-classical Schrödinger $ih\partial_s v_h + h^2 \Delta v_h = 0$,

$$\mathrm{WF}_h(v_h) \subset \mathrm{Char}(P_h) = \{(s, x; \tau, \xi) \in T^*(\mathbb{R}_s \times \Omega_x) : \tau - |\xi|_g^2 = 0\}.$$

$\mathrm{WF}_h(v_h)$ invariant under Hamiltonian (geodesic) flow of $p = \tau - |\xi|_g^2$.

- (GCC) assumption

$$(\star) \qquad \exists \alpha > 0, \quad \| v_h \mid_{t=0} \|^2_{L^2(\Omega_x)} \leq C \int_0^\alpha \| v_h(s,x)\|^2_{L^2(\omega)} ds.$$

- Back to the classical time scale t: From $(\star)$

$$\| u_h \mid_{t=0} \|^2_{L^2\Omega_x)} \leq \frac{C}{h} \int_0^{\alpha h} \| u_h(t,x)\|^2_{L^2(\omega)} dt.$$

Write for $t = 0, t = \alpha h, \cdots, t = T$, combine with conservation L^2 norm, $\exists 0 < h_0 = h_0(T, a) \ll 1, \forall 0 < h < h_0$,

$$\| u_h(0)\|^2_{L^2(\Omega)} \leq C_T \int_0^T \| u_h(t,x)\|^2_{L^2(\omega)} dt.$$

The half wave regime $x = 0, \xi - 0, \eta = +\infty$

- Key point

$$(\star) \qquad [i\partial_t - \Delta_G, x\partial x + y\partial_y] = -2\Delta_G$$

- Idea: compute

$$([i\partial_t - \Delta_G, x\partial x + y\partial_y]u, u)_{L^2}$$

integrate by parts and use coercivity to control H^1_H norm of u from$(\star)$

- Problems: boundary terms in x, t, and $y\partial_y$ is *not* y periodic
- Solution introduce cut of in x, t, y (and deal with the additional terms)

A priori estimates

- Hypoelliptic estimate

$$\|f\|^2_{L^2(\Omega)} + \||D_y|^{1/2} f\|^2_{L^2(\Omega))} \leq \|f\|^2_{H^1_G} = ((-\Delta_G)f, f)_{L^2(\Omega)}.$$

- Elliptic estimate: Characteristic manifold

$$\text{Char} = \{\tau = \xi^2 + x^2\eta^2 \in (\frac{1}{2}, 2)\}$$

We deduce that if $|D_y| \geq Ch^{-1}$ then $|x| \leq \frac{\sqrt{2}}{C}$. i.e. for any $\chi \in C^\infty_0(\mathbb{R})$ equal to 1 on $(-1, 1)$,

$$\|(1 - \chi(\frac{Cx}{\sqrt{2}})u\|_{H^1} = \mathcal{O}(h^\infty).$$

Positive commutator

Let $\varphi_T \in C_0^\infty(\mathbb{R}_t)$ with support in $(-0, T)$ equal to 1 0n $\varepsilon, T-\varepsilon)$ and $\zeta \in C_0^\infty(\mathbb{R}_y)$. Compute

$$(\star) = \Big([i\partial_t - \Delta_G, x\partial x + y\partial_y]\varphi_T(t)\chi(x)\zeta(y)v, v\Big)_{L^2_{t,x,y}}$$

$$= \Big(-2\Delta_G\varphi_T(t)\chi(x)\zeta(y)u, u\Big)_{L^2_{t,x,y}}$$

$$= 2\int_{t,x,y}\varphi_T(t)\chi(x)\zeta(y)\big(|\partial_x u|^2 + x^2|\partial_y u|^2\big)dxdydt + \mathcal{O}(\|u\|_{H^1_G}\|u\|_{L^2})$$

$$(\star) = ((x\partial_x + y\partial_y)[i\partial_t - \partial_x^2 - x^2\partial_y^2, \varphi_T(t)\chi(x)\zeta(y)]u, u)_{L^2_{t,x,y}}$$

$$= \int_{t,x,y}\varphi_T'(t)\chi(x)\zeta(y)uy\partial_y\overline{u}dxdydt$$

$$+ \mathcal{O}(\|u\|_{H^1_G}\|u\|_{L^2}) + O(h^\infty)\|u\|^2_{L^2} + \mathcal{O}(\|u\|^2_{H^1_G(\omega)}$$

$$2T\|u\|^2_{H^1_G} \leq \text{Observation} + 2\mathcal{L}(\omega)\|u\|^2_{H^1_G} + l.o.t.$$

Transversal propagation regime $\eta = 0$

The method in this regime is inspired from control for Schrödinger on $\mathbb{T}^2$ (Burq-Zworski 03)

- Semi-classical propagation does not give result because geodesic are horizontal !
- Step 1: apply semi-classical propagation to escape set $\{x = 0\}$
- Step 2 apply positive commutator $[-\Delta_G, y\partial_y] = -x^2\partial_y^2 \geq -c\partial_y^2$ (away from $\{x = 0\}$)
- Since u microlocalized $|D_y| \sim \epsilon^{-1}$, $u_0 = \chi(-h^2 H)\chi(\epsilon|D_y|)u_0$

$$\forall T > 0; \exists C > 0, h_0 > 0, \epsilon_0 > 0, \forall 0 < h \leq h_0, \forall h^2 < \epsilon \leq \epsilon_0,$$

$$\|u_0\|^2_{L^2(\Omega)} \leq C_T \int_0^T \|u(t,x)\|^2_{L^2(\omega)} dt$$

Problem: How to glue the second semi-classical estimates? Defect of compactness! (i..e. errors $O(|D_y|^{-\infty}$ are not necessarily compact)!
Solution: stop at $\epsilon < h^\delta$

Transversal propagation: The normal form regime $|D_y| \lesssim h^{-\delta}$ (inspired from Burq-Zworski 12)

$$i\partial_t u + \partial_x^2 u + x^2\partial_y^2 u = 0$$

$$v = (\mathrm{Id} + hQ(x, hD_x)\partial_y^2)u.$$

$$(i\partial_t u + \partial_x^2 u + x^2\partial_y^2 - h[\Delta_G, Q]\partial_y^2)v = 0$$

$$\begin{aligned} h[\Delta_G, Q] &= 2(i\xi\partial_x q)(x, hD_x)\partial_y^2 + h(\partial_x^2 q)(x, hD_x)\partial_y^2 + h[x^2, Q]\partial_y^4 \\ &= 2(i\xi\partial_x q)(x, hD_x) + \mathcal{O}_{\mathcal{L}(L^2)}(h^{1-2\delta} + h^{1-4\delta}). \end{aligned}$$

choose

$$q(x,\xi) = \frac{1}{2i\xi}\int_{-1}^{x}(z^2 - M)dx \Leftrightarrow x^2 - 2i\xi\partial_x q(x,\xi) = M$$

$$i\partial_t v + \partial_x^2 v + \underbrace{M}_{\text{average of } x^2 \text{ along } x = \text{const.}} \cdot\partial_y^2 v = O_{L^2}(h^\theta)$$

Conclude rom the observability for Schrödinger on $\mathbb{T}^2$.