Microlocal Analysis of Anosov geodesic flows

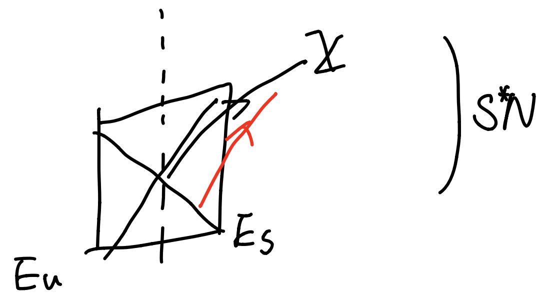
with T. Sujii

① Settings

(N, g) closed manifold, $K_g < 0$, $\dim N = d+1$

X C^∞ geodesic vectorfield on $M = S^*N$

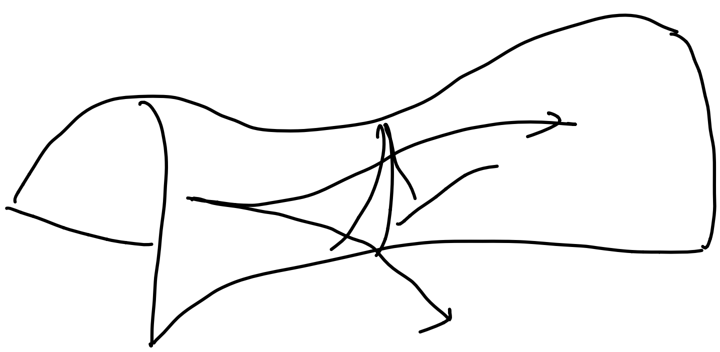
More generally, X contact Anosov flow on M



Vector $F \to M$

X_F : Lie deri

$C^\infty(M, F) \circlearrowleft$



Ⓘ Question : describe long time "effective behaviour of geodesic flow"?

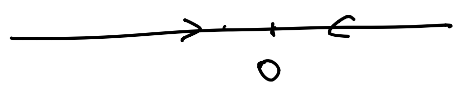
$\Leftrightarrow \quad \forall u, v \in C^\infty(M) \qquad \langle v \mid \underbrace{e^{tX} u}_{u \circ \phi_t} \rangle_{L^2(M)} = \ ? \quad t \gg 1$

$\Leftrightarrow$ "describe the discrete spect" of X in good space $\mathcal{H} \supset C^\infty(M)$

rem : $L^2(M) \quad X^* = -X \Rightarrow$ C^0 spectral on $i\mathbb{R}$

Ⓘ Discrete Ruelle Pollicott spect in simple examples

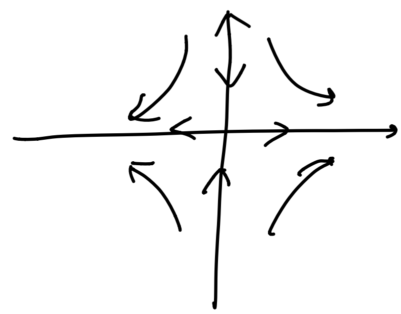
Ex: • $X = -x\partial_x$ on $\mathbb{R}$ flow $x(t) = x(0)e^{-t}$



$x \in \mathbb{R} = E_0$

• on $T^*\mathbb{R} = T^*E_0$

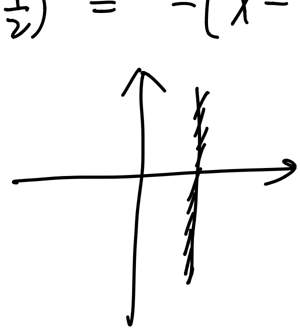
$\tilde{\phi}^t(x, \xi) = (e^t x, e^{-t}\xi)$



Observe: $\forall k \in \mathbb{N} \quad X x^k = -k x^k \quad \Pi_k = x^k \langle \frac{\delta}{k!} | \cdot \rangle$

in L^2, $X^* = -X + 1 \Leftrightarrow (X - \frac{1}{2})^* = -(X - \frac{1}{2})$

$\Leftrightarrow \quad \mathrm{spa}(X) = \frac{1}{2} + i\mathbb{R}$

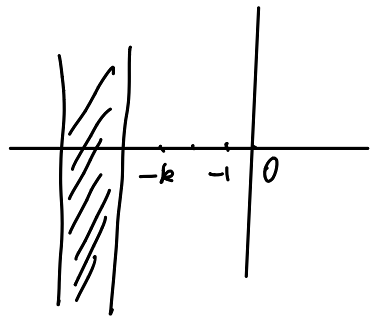


- Let $0 < h \ll 1$, $m \geq 0$ "escape fcn ϕ^{-t}"

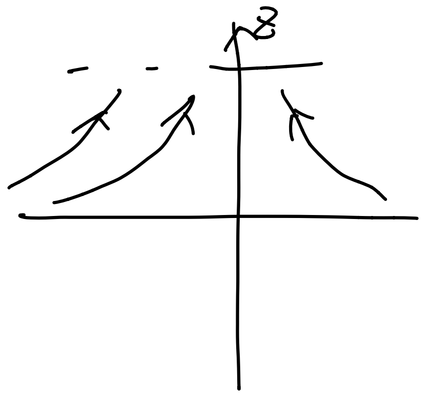
$$W(x,\xi) = \frac{\langle \sqrt{h}\cdot\xi\rangle^m}{\langle \sqrt{h}\cdot x\rangle^m}, \quad \frac{W \circ \Phi^t}{W} \leq \begin{cases} C \\ e^{-mt} \text{ far from } (0,0) \end{cases}$$

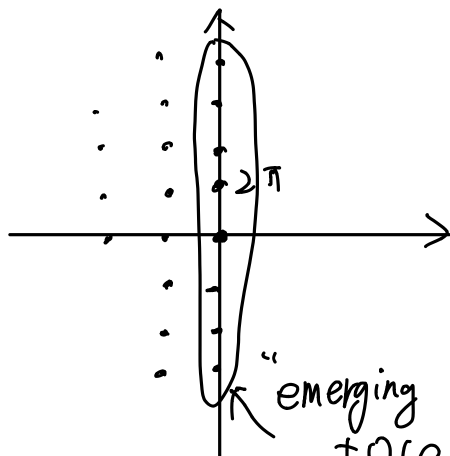
- in $\mathcal{H}_W(\mathbb{R}) = Op(W^{-1})L^2(\mathbb{R})$,

$x^k \in \mathcal{H}_W$, $\|\Pi_k\|_{\mathcal{H}_W} \leq C$



Ex: $X = -x\partial_x + \partial_z$ on $\mathbb{R}_x \times S^1_z$





"emerging operation" $+ O(e^{-t})$ ∂_z

IV Some results (1.2.3)

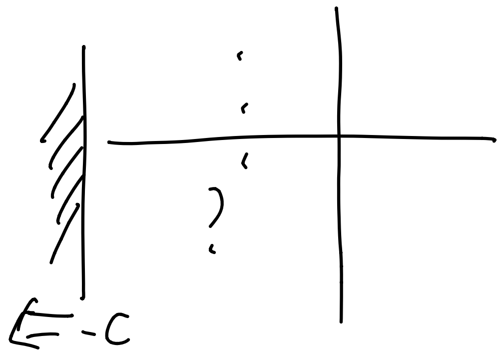
1) Existence of discrete spectrum "Pollicott Ruelle resonance".

Thm 1 (Butterley Liverane 07'. F. Roy Sjöstrand 11'.

F – Tsujii (8)

$\forall c > 0, \; \exists \; C^\infty(M) \subset H_W \subset \mathcal{D}'(M)$

X is generator of a strongly C^0 group in H_W and X has intrinstic discrete sp on $\mathrm{Re}(z) \geq -c$



$H_W = Op(W^{-1}) L^2(M)$

$W(x, \xi) =$ " $\dfrac{\langle \xi_s \rangle^m}{\langle \xi_u \rangle^m}$ "

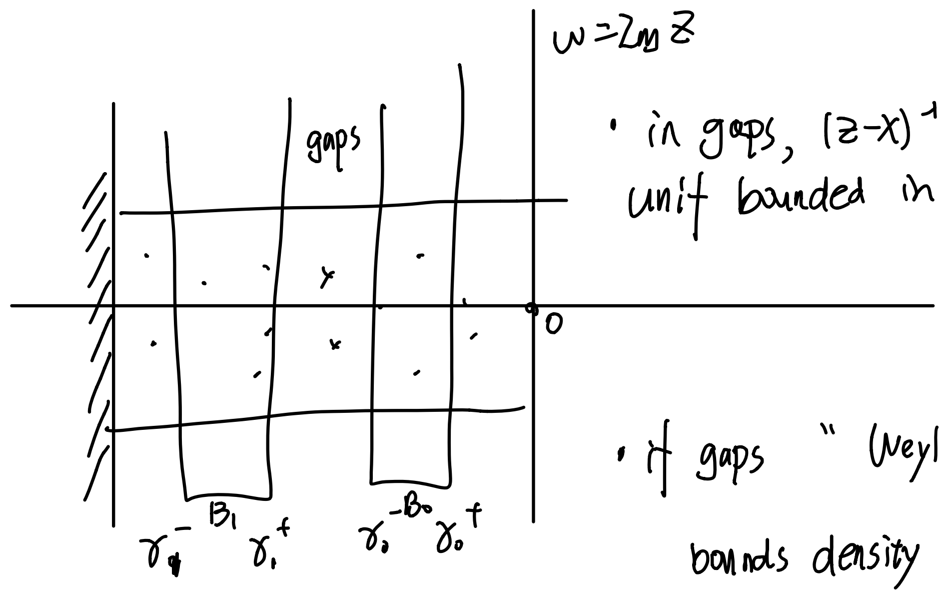
2) "RP speitrum is contained in vertical bands"

- P. Leboeuf 2004. for $k = -1$
- F 2006 contact $U(1)$ extension of cat map
- F Tsujii 2013 " " Anosov flow
- Dyatlov 2013 bands for wave around black holes.

Thm F. Tsujii 2015 $\quad X$ contact Anosov

$\forall \varepsilon>0, \ \exists \omega_\varepsilon > 0$

$$\left(\sigma_{RP}(X) \cap \{Re(z) > -c\}\right) \subset \bigcup_{k\in\mathbb{N}} \{Re z \subset [\gamma_k^- - \varepsilon, \gamma_k^+ + \varepsilon]\}$$
$$\cup \{|Im z| < \omega_\varepsilon\}$$



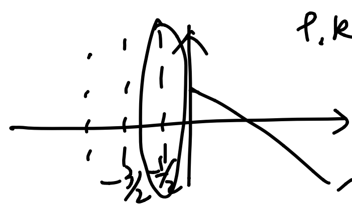
- in gaps, $(z-X)^{-1}$ is unif bounded in H_ω
- if gaps "Weyl wave" in each bands density $\sim \operatorname{rank}(F_k) \dfrac{Vol(M)\,\omega^d}{(2\pi)^{d+1}}$

Ex: Flaminu Forne 2002 $\quad$ Leboeuf 04 $\quad$ Dyatlov–F–Guell 2013

if N surface $d+1=2$, $X=-1$, $N = \Gamma \backslash SL_2\mathbb{R} / SO_2$

$$\sigma_{RP}(X) = \{z_{k,l} = -\tfrac{1}{2} - k \pm i\sqrt{\mu_l - \tfrac{1}{4}}\} \cup \{-\mathbb{N}\}$$
$l, k \in \mathbb{N}$

$\Delta \psi_l = \mu_l \psi_l$



$\gamma_k^\pm = -\tfrac{1}{2} - k$

on T^*N

$$\sqrt{\Delta} \sim Op(|\xi|) \sim Op_{geom}(-i\mathcal{X})$$

emerging wave operator

3) " Emergence of a quantum operators "

Thm3: (F- Tsujii 19 in progress)

in $H_W(M)$ for any $K \geq 0$

$$e^{tX} = \bigoplus_{k=0}^{K} e^{t\, Op_{geom}(\mathcal{X}_{F_k})} + \Psi^{-\infty}(M) + O(e^{t\gamma_{k+1}^{+}}) + O_t(|w|^{-\beta/2})$$

operator value symbol on some symplect E_0^*

Hölder exp of E_u, E_s

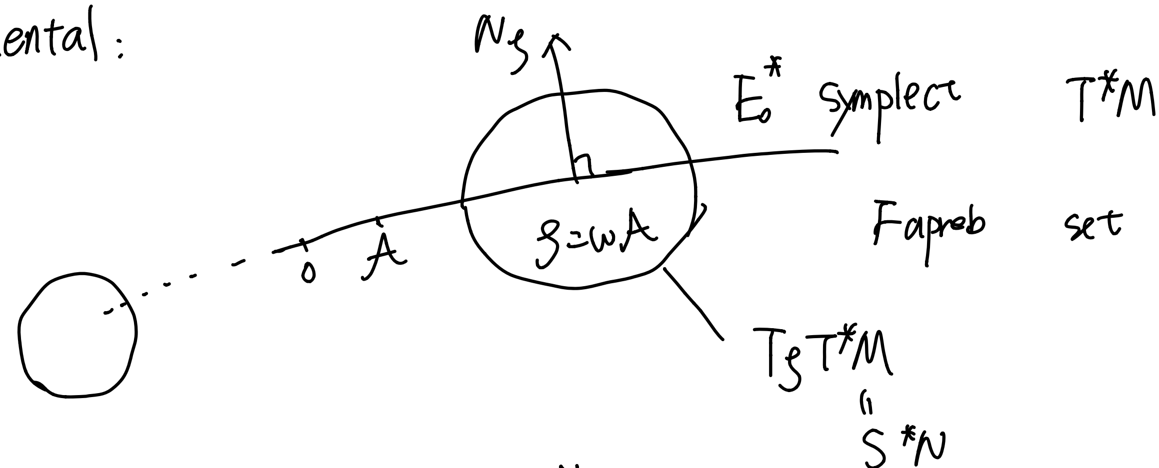
Let $E_0^* = \mathbb{R}_\omega A \subset T^*M$. $\dim E_0^* = 2d+2$

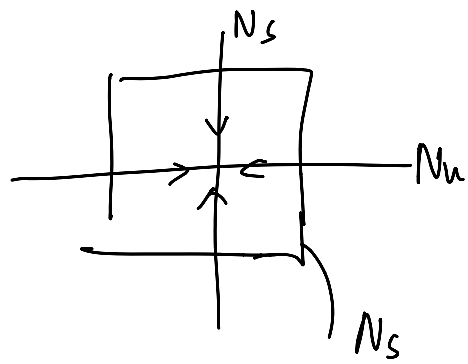
Liouville one form

Lemma : $E_0^* \setminus \{0\}$ is simplect

rem: $E_{0,+}^* = \mathbb{R}_{\omega>0} A \underset{\text{symple}}{\cong} T^*M \setminus \{0\}$

Fundamental :





For $k \in \mathbb{N}$

vector bundle

$$\begin{tikzcd} F_k := |E_s|^{1/2} \otimes \mathrm{Pol}_k(E_s) \arrow[r] \arrow[dr] & E_0^* \arrow[d] \\ & M \end{tikzcd}$$

remerber: $X = -x\partial_x$

$\leftarrow$ Homog of degree k

$x^k |dx|^{\frac{1}{2}} \in Pol_k(E_s) \otimes |E_s|^{\frac{1}{2}}$

X_{F_k}: $C(E_0^*/F_k) \circlearrowleft$ "Lie deri induced by X"

$Op_{geom}(X_{F_k}) = \mathcal{T}_m^* \, X_{F_k} \, \mathcal{T}_m$ $\quad \omega >> 1$

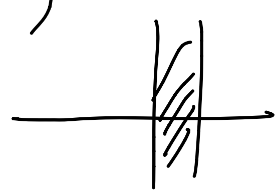
$\uparrow$

wave packet transform

$\mathcal{T}_m : C^\infty(M) \to \mathcal{S}_w(E_0^*; F_k)$

Lemma: $\Phi^t_{F_k} = e^{tX_{F_k}}$ $L^2(E_0^*/F_k) \circlearrowleft$ linear boundle map

$\gamma_k^\pm = \lim_{t\to\pm\infty} \log\left(\max_{E_0^*} \| \Phi^t_{F_k}(m) \|^{\frac{1}{t}}\right)$



then

$X_{F_k}: L^2 \circlearrowleft$ has C^∞ spect contained in $[\gamma_k^-, \gamma_k^+]$

$F_{k=0} = |E_0|^{\frac{1}{2}} \otimes Pol_{k=0}(E_0) \otimes F$

$\searrow |E_u|^{1/2} \equiv |E_s|^{-\frac{1}{2}}$

$= \mathbb{C}$ trivial

$\Rightarrow \quad \gamma_0^{\pm} = 0 \qquad B_0 \equiv i\mathbb{R}$

zeta function $\equiv$ Gutzwiller of Voros

quantum mechanics