① Setting

(M, g) compact

$$(-\Delta + V)\Psi_\lambda = \lambda \Psi_\lambda$$

$$\|\Psi_\lambda\|_{L^2(M)} = 1$$

How is c typical Ψ_λ, $\lambda \gg 1$.

$W_{\Psi_\lambda} \in D'(T^*M)$

$$\langle W_{\Psi_\lambda}, a \rangle \underset{h=\frac{1}{\sqrt{\lambda}}}{=} (Op_h(a)\Psi_\lambda, \Psi_\lambda)$$

(Ψ_λ) $\exists \lambda_n \uparrow \infty$ (m ∞ subsequences)

$$W_{\Psi_\lambda} \longmapsto \mu \gg 0 \quad (\text{Gårding})$$

semiclassical meas

$\mathcal{M}_V(0) = \{\mu :$ obtained in this way $\}$

- QUE Cangectone $\Leftrightarrow \mathcal{M}_0(0) = \{\alpha\}$ Liouvi meas
- Here (M, g) has completely integrable geodesic flow.

$\mathcal{M}_V(0) \ni \mu \Rightarrow \mu \in \mathbb{P}(S^*M)$ & $(\phi_t)_* \mu = \mu_t$ if

$\phi_t : T^*M \to T^*M$ geodesic flow

Do not confuse: ϕ_t is not the Hamiltonian flow of $\|\xi\|_x^2 + V(x)$

② The Problem

$$h = \frac{1}{\sqrt{\lambda}} \quad (-h^2\Delta + h^2 V)\Psi_h = \Psi_h + R_h$$

$$\uparrow$$

$0 \leftarrow \varepsilon_h^2$ (interested case $\varepsilon_h >> h$)

"weak" KAM approach.

$\|R_h\|_{L^2(M)} = \sqrt{h}$ "Actual" KAM lezu Hein, Ropin.

$\mathcal{M}_V(\sqrt{h})$ Gunes Hessell, Armaiz

③ The Sphere $(\mathbb{S}^d$ can$)$

Thm: (Kkobson, Zelditch)

$$\mathcal{M}_0(0) = \mathcal{M}_{inv} = \{\mu \in \mathcal{P}(S^*\mathbb{S}^d),\ \mu \text{ invariant}\}$$

Proof: $\varphi_k(x) = C_k(x_1 + i x_2)^k$ $\quad \mathbb{S}^d \subset \mathbb{R}^{d+1}$

$\|\varphi_k\| = 1$

$|\varphi_k|^2(x) = C_k^2(|x_1|^2 + |x_2|^2)^k$

concentrating $x_3 = \cdots = x_{d+1} = 0$

$\Rightarrow \mu = \delta_\gamma, \quad \gamma \circlearrowleft$

- $\{\delta_\gamma : \gamma \text{ geodesic}\} \subseteq \mathcal{M}_0(0)$
- Krein Milmam thm: $\forall$ invariant meas is a weak $*$ limit of convex combination of δ_γ's
- Diagonal extraction

$$I(V)(x,\xi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T V \circ \pi_x \circ \phi_t(x,\xi)\, dt$$

If $V \in C^\infty$

$I(V) \in C^\infty(T^*\mathbb{S}^d \setminus \{0\})$

$I(V) \in C^\infty(G(\mathbb{S}^d))$

Thm: (M. GRivière) $\mu \in \mathcal{M}_V(h\varepsilon_h^2) \supsetneq \mathcal{M}_V(0)$,

then if $\mu(\gamma) > 0$ on some γ geodesic

$\Rightarrow$ $dI(V)_\gamma = 0$

Thm' $\forall \mu \in \mathcal{M}_V(O(h\varepsilon_h^2))$

$(\phi_V^t)_* \mu = \mu$ $\quad$ ϕ_V^t is the Hamiltonian flow of $I(V)$

$-h^2\Delta + \varepsilon_h^2 V$

Consequences

① $d=2$, $\varepsilon_h = h$ then $\exists V$ s.t.

$\forall \mu \in \mathcal{M}_V(h^2)$ is of the form

$\mu = \rho\, dx + \alpha_1 \delta_{\gamma_1} + \alpha_2 \delta_{\gamma_2} + \alpha_3 \delta_{\gamma_3}$

$\rho \in L^1_+(\mathbb{S}^2)$ $\quad$ $V = I^{-1}(a x^2 + b\eta^2 + c\xi^2)$

② $\exists (M, g)$ Zoll surface s.t.

$\mathcal{M}_0(0) \neq \mathcal{M}_{inv}$

(Weinstein, Duistermaat-Guillemin, Colin de Verdière)

$-\Delta = A + Q \leftarrow \Psi^0(M)$

$\uparrow$ $sp \subseteq \mathbb{N}$ Zelditch.

Proof: (Weinstein, Guillhem, Uribe, etc.)

$$Op_h(a) \longmapsto I(Op_h(a)) = \int_0^{2\pi} e^{-isA} Op_h(a)\, e^{isA}\, ds$$

$$A^2 - \tfrac{d-1}{4} = -\Delta \Rightarrow e^{i2\pi A} = e^{i2\pi} Id$$

$\uparrow$ has spectral on $\mathbb{N} + \frac{1}{2}$.

$$[I(Op_h(a)), -\Delta] = 0$$

$$I(Op_h(a)) = Op_h(I(a))$$

$$\Rightarrow ([I(Op_h(a)), -h^2\Delta + \varepsilon U]\Psi_n \mid \Psi_n) = O(h\varepsilon_h^2)$$

$$\Rightarrow \int \{ I(a), v \}\, d\mu = 0$$

④ $M = \mathbb{T}^d$ [$\mathcal{M}_0(0)^{d=2}$ Jakobson]

- $\varepsilon_h = h$ Anantharaman M'

$\mathcal{M}_V(o(h^2))$ are absolutely continuous

Change every open set $\omega \times \mathbb{R}^d$

$(d=2)$ Burq Zworski ;

Thm (M. G. Rivière) $\varepsilon_h >> h$, $d=2$ $\mu \in \mathcal{M}_V(o(h\varepsilon_h))$

$$\mu = \rho\, dx + \sum \alpha_j \delta_{\gamma_j}$$

γ_j are periodic geodesic that are critical points of $I(V)(\cdot, \xi)$

Averaging Method on $\mathbb{T}^2$

$\xi_0 \in \mathbb{R}^2$ $\xi_0^1 / \xi_0^2 \in \mathbb{Q}$

$$|\xi|^2 = H_{\xi_0}(\xi)^2 + H_{\xi_0^\perp}(\xi)^2$$

$$-h^2\Delta + \varepsilon_h^2 V =$$

$$= Op_h(H_\xi^2) + \varepsilon_h^2\left(\left(\frac{H^2_{\xi_0^\perp}}{\varepsilon_h}\right) + V\right)$$

$$\eta = \frac{H^{1}_{g_0}}{\varepsilon_h} \in \widehat{\mathbb{R}}$$