Dispersive estimates for the semi-classical Schrödinger equation on strictly convex domains

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The Schrödinger equation on domains

Let Ω be a domain of dimension $d \geq 2$, Δ the Laplace operator :

$$i\partial_t u - \Delta u = 0, \text{ in } \Omega, \quad u|_{t=0} = u_0, \quad u|_{\partial\Omega} = 0 \text{ if } \partial\Omega \neq \emptyset$$

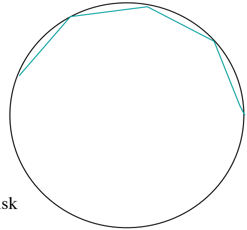
[(Blair-)Smith-Sogge; Seeger; Staffilani-Tataru; Koch-Tataru; Smith-Tataru; Burq-Gérard-Tzvetkov; Burq-Lebeau-Planchon; Planchon-Vega; Anton, etc]

- $\underline{\Omega = \mathbb{R}^d} \Rightarrow \|e^{\pm it\Delta_{\mathbb{R}^d}}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq C(d)t^{-d/2}$.
- On bounded domains : **NO** satisfactory estimates to deal with important applications (cubic NLS on a 3D ball is not well-understood in the natural energy space...)
- Compact manifolds (even when $\partial\Omega = \emptyset$): eventually a loss will occur, due to the volume being finite;
- $\underline{\partial\Omega = \emptyset} \Rightarrow$ replace the exact formula by a local parametrix in semi-classical time; deal with wavelength sized intervals;
- $\underline{\partial\Omega \neq \emptyset} \Rightarrow$ starting point : knowledge of a ”good” parametrix Available parametrices (very efficient for propagation of singularities) : [Melrose-Taylor; Melrose-Sjöstrand; Eskin]

Model for convex boundaries in 2D

Unit disk: $\{r \leq 1\}$
$\Delta_{disk} = \partial_r^2 + \frac{1}{r^2}\partial_\theta^2$

Model domain:
$\Omega_2 = \{(x, y), x > 0, y \in \mathbb{R}\}$
$\Delta_F = \partial_x^2 + (1 + x)\partial_y^2$



Disk

Model domain

$(\Omega_{d=2}, \Delta_F)$ = the Friedlander model domain :

- good model for a strictly convex (as first approximation of the disk : take $r = 1 - x/2$, $\theta = y$)
- Fourier transform ($y \to \theta$) : $-\Delta_F$ becomes $-\partial_x^2 + (1 + x)\theta^2$
- for $\theta \neq 0 \Rightarrow$ positive, self-adjoint, with eigenfunctions $\{e_k(x, \theta)\}_k$ = Hilbert basis on $L^2(\mathbb{R}_+)$

Sharp dispersion for semi-classical Schrödinger on Friedlander domain

Let $h \in (0,1]$, $0 < a \lesssim 1$ and let the semi-classical Schrödinger equation:

$$ih\partial_t^2 u - h^2 \Delta_F u = 0 \text{ on } \Omega_d, \quad u|_{t=0} = \chi(hD_y)\delta_{x=a,y=0}, \quad u|_{\partial\Omega_d} = 0.$$

Theorem: ([I,2019] For all $0 < a \lesssim 1$, $h \in (0,1]$ and $t \in (0,1]$ we have:

$$\|u\|_{L^\infty(\Omega_d)} \lesssim \frac{1}{h^d} \min\{1, (\frac{h}{t})^{\frac{d-1}{2}} \gamma_{h,a}(\frac{t}{h})\},$$

where $\gamma_{h,a}(\frac{t}{h}) = \begin{cases} \sqrt{a}, \text{ if } a \lesssim (th)^{1/2}, \\ (\frac{ht}{a})^{1/2}, \text{ if } (th)^{1/2} \lesssim a \lesssim th^{1/3}, \\ (\frac{ha}{t})^{1/4}, \text{ if } th^{1/3} < a \lesssim 1. \end{cases}$

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- for every $a \lesssim th^{1/3}$ we have $\gamma_{h,a}(\frac{t}{h}) \lesssim (ht)^{1/4}$;
- **for every $t \in (h, 1]$ AND every $th^{1/3} < a \lesssim 1$ the bound is sharp** :

$$\|u(t)\|_{L^\infty(\Omega_d)} \simeq \frac{1}{h^d} (\frac{h}{t})^{\frac{(d-1)}{2}} \left(\frac{ha}{t}\right)^{\frac{1}{4}};$$

- Loss of 1/4 : recall $\sup \left|\chi(hD_t) e^{\pm i \frac{t}{h} \Delta_{\mathbb{R}^d}}\right| \lesssim \frac{1}{h^d} \min(1, (\frac{h}{t})^{\frac{d}{2}})$.

Staring point: construction of a parametrix using spectral decomposition, d=2

- Fourier transform in the tangential variable y :

$$u(t,x,y) = \int_{\mathbb{R}} e^{i(y\theta + ht\theta^2)} \chi(h\theta) v(t,x,\theta)\, d\theta\,,$$

where, for $x \geq 0$,

$$\Big(ih\partial_t + h^2(-\partial_x^2 + x\theta^2)\Big) v(t,x,h) = 0, \quad v_{|x=0} = 0, \quad v_{|t=0} = \delta_{x=a}.$$

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- Recall $(-\partial_x^2 + x\theta^2)$ has eigenfunctions $\{e_k(x,\theta)\}_k$ with eigenvalues $\lambda_k(\theta)$: the Dirac distribution on $\mathbb{R}_+$ reads as

$$\delta_{x=a} = \sum_{k\geq 1} e_k(x,\theta) e_k(a,\theta),$$

which yields

$$v(t,x,\theta) = \sum_{k\geq 1} e^{iht\lambda_k(\theta)} \chi(h\theta) e_k(x,\theta) e_k(a,\theta).$$

- Take $t = \sqrt{a}T$ and $x = aX$, then $T \in [1, 1/\sqrt{a}]$ and $X \in [0,1]$.

Spectral decomposition of $-\partial_x^2 + x\theta^2$, $\theta \neq 0$, $x \geq 0$, with Dirichlet on $\{x = 0\}$

Let $-\partial_x^2 + x\theta^2$, $\theta \neq 0$ with eigenfunctions and eigenvalues:

$$e_k(x,\theta) = \frac{\theta^{1/3}}{\sqrt{L'(\omega_k)}} Ai\Big(\theta^{2/3}x - \omega_k\Big), \quad \text{with} \quad \lambda_k(\theta) = \omega_k \theta^{4/3},$$

WHERE $\theta \simeq \frac{1}{h}$ AND

$$A_\pm(\omega) = e^{\mp i\pi/3} Ai(e^{\mp i\pi/3}\omega), \ Ai(-\omega) = A_+(\omega) + A_-(\omega),$$

$$L(\omega) = \pi + i \log \frac{A_-(\omega)}{A_+(\omega)}, \quad \text{for } \omega \in \mathbb{R}.$$

- L is real analytic and strictly increasing ;
- $L(\omega) = \frac{4}{3}\omega^{\frac{3}{2}} - B(\omega^{\frac{3}{2}})$, for $\omega \geq 1$, $B(\omega^{3/2}) \simeq \sum_{j\geq 1} b_j \omega^{-3j/2}$;
- $Ai(-\omega_k) = 0$ iff $L(\omega_k) = 2\pi k$ and $L'(\omega_k) = \int_0^\infty (Ai)^2(x - \omega_k)\, dx$;
- $\{(-\omega_k)\}_{k\in\mathbb{N}^*}$ are the zeros of Airy, $\omega_k \simeq k^{2/3}(1 + O(k^{-1}))$.

Bouncing ping-pong ball

Consider the semi-classical 1D Schrödinger with potential on the half line :

$$ih\partial_t V - h^2\partial^2_{xx} V + xV = 0, x > 0; \quad V|_{x=0} = 0, \quad V|_{t=0} = \delta_{x=a}.$$

- Let $t = \sqrt{a}T$ and $x = aX$ and set $\lambda := \frac{a^{3/2}}{h}$;
- $e_k(X,\lambda) = \frac{\lambda^{1/3}}{\sqrt{L'(\omega_k)}} Ai(\lambda^{2/3}X - \omega_k)$, $\lambda_k(h) = h^{-4/3}\omega_k$;
- In the new variables

$$V(T,X) = \sum_k e^{i\lambda T(\frac{\omega_k}{\lambda^{2/3}})} e_k(X,\lambda) e_k(1,\lambda).$$

(indeed $ht\lambda_k(\frac{1}{h}) = hth^{-4/3}\omega_k = T\omega_k\sqrt{a}h^{-1/3} = T\lambda^{1/3}\omega_k = \lambda T(\frac{\omega_k}{\lambda^{2/3}}))$

- If $\theta = \frac{\eta}{h}$, $\eta \simeq 1$: eigenvalues $\lambda_k(h) = (h/\eta)^{-4/3}\omega_k$, then change of variables $\omega = (\eta\lambda)^{2/3}\alpha$. Same formula for V with $\eta^2 T$ instead of T.

Airy - Poisson formula (from "gallery modes" to oscillating integrals)

Lemma : In $\mathcal{D}'(\mathbb{R}_\omega)$, one has

$$\sum_{N\in\mathbb{Z}} e^{-iNL(\omega)} = 2\pi \sum_{k\in\mathbb{N}^*} \frac{1}{L'(\omega_k)}\delta(\omega-\omega_k)\,.$$

In other words, for $\phi(\omega) \in C_0^\infty$,

$$\sum_{N\in\mathbb{Z}} \int e^{-iNL(\omega)}\phi(\omega)\,d\omega = 2\pi \sum_{k\in\mathbb{N}^*} \frac{1}{L'(\omega_k)}\phi(\omega_k)\,.$$

Two formulas for V : (using Airy-Poisson)

- in terms of the eigenfunctions $\left\{ e_k(X,\lambda) = \frac{\lambda^{1/3}}{\sqrt{L'(\omega_k)}} Ai(\lambda^{2/3}X - \omega_k) \right\}$

$$V(T,X) = \sum_{k\geq 1} e^{i\lambda T(\frac{\omega_k}{\lambda^{2/3}})} \frac{\lambda^{2/3}}{L'(\omega_k)} Ai(\lambda^{2/3}X - \omega_k) Ai(\lambda^{2/3} - \omega_k).$$

- as a sum over the reflections on the boundary :

$$V(T,X) = \sum_{N\in\mathbb{Z}} \int_{\mathbb{R}} e^{i\lambda T(\frac{\omega}{\lambda^{2/3}}) - iNL(\omega)} \lambda^{2/3} Ai(\lambda^{2/3}X - \omega) Ai(\lambda^{2/3} - \omega) \frac{d\omega}{2\pi}.$$

- Duality between k and N: **same number if** $a \simeq (th)^{1/2}$ (iff $T \simeq \lambda$).
- **Main contribution :** $\omega_k \simeq \lambda^{2/3}$, hence $k \simeq \lambda$; for $\omega_k > 2\lambda^{2/3}$ use the asymptotic of Airy function $Ai(-z) \simeq \sum_{\pm} z^{-1/4} e^{\pm z^{3/2}}$.

Parametrix in terms of reflections $(ht)^{1/2} \lesssim a$

$$V(T,X) = \sum_N V_N(T,X),$$

$$V_N(T,X) = \lambda^{2/3} \int e^{i\lambda(T(\frac{\omega}{\lambda^{2/3}}) - \frac{N}{\lambda}L(\omega))} \chi(\frac{\omega}{\lambda^{2/3}}) Ai(\lambda^{2/3}X - \omega) Ai(\lambda^{2/3} - \omega) d\omega\,.$$

Using $Ai(-z) = \int e^{i(\frac{s^3}{3} - sz)} ds$, $L(\omega) = \frac{4}{3}\omega^{3/2} - B(\omega^{3/2})$, gives

$$V_N(T,X) = \lambda^2 \int e^{i\lambda(T\alpha - \frac{4}{3}N\alpha^{3/2} + \frac{N}{\lambda}B(\lambda\alpha^{3/2}) + \frac{s^3}{3} + s(X-\alpha) + \frac{\sigma^3}{3} + \sigma(1-\alpha))} \chi(\alpha) d\alpha ds d\sigma\,.$$

- Stationary phase in α : $T - (s+\sigma) \simeq 2N\sqrt{\alpha_c}$, $|\partial_\alpha^2 \Phi_N||_{\alpha_c} \simeq \frac{N}{\sqrt{\alpha_c}}$;
- Critical value : for $K := \frac{T}{2N} \simeq 1$ on support χ, $|s|, |\sigma| \lesssim 1$

$$\Phi_N(\alpha_c) = \frac{s^3}{3} + s(X - K^2) + \frac{\sigma^3}{3} + \sigma(1 - K^2) + \frac{K}{2N}(s+\sigma)^2 - \frac{1}{12N^2}(s+\sigma)^3.$$

Proposition 1 For $N \geq \lambda^{1/3}$, we have

$$|V_N(T,X)| \lesssim \frac{\lambda^{2/3}}{((N/\lambda^{1/3})^{1/2} + \lambda^{1/6}|T-2N|^{1/2})}\,.$$

Proposition 2 For $N < \lambda^{1/3}$ and $|T-2N| \gtrsim 1/N$, we have

$$|V_N(T,X)| \lesssim \frac{\lambda^{2/3}}{(1+|N(T-2N)|^{1/4})}\,. \tag{1}$$

Proposition 3 For $N < \lambda^{1/3}$ and $|T-2N| \lesssim 1/N$, we have

$$|V_N(T,X)| \lesssim \frac{\lambda^{2/3}}{(N^{1/4}\lambda^{-1/12} + |N(T-2N)|^{1/6})}\,. \tag{2}$$

- When $T = 2N \leq \lambda^{1/3}$, $X = 1$: $|V_N(T,1)| \simeq \frac{\lambda^{2/3}}{N^{1/4}\lambda^{-1/12}} \simeq \frac{\lambda^{3/4}}{T^{1/4}}$;

- For $T < \lambda^{1/3}$ (any T, even larger than λ):

$$\sup_X |V(T,X)| \le |V(T,1)| \le \sum_{N \simeq T} |V_N(T,1)| \lesssim \lambda^{2/3} \frac{\lambda^{1/12}}{T^{1/4}} + \lambda^{2/3} \simeq \frac{\lambda^{3/4}}{T^{1/4}},$$

$$|V(2N,1)| \simeq \frac{\lambda^{3/4}}{T^{1/4}}\Big|_{T=2N};$$

- For $T \ge \lambda^{1/3}$ (any T, even larger than λ):

$$|V(T,X)| \le |V(T,1)| \le \sum_{N \simeq T} |V_N(T,1)| \lesssim \sqrt{\lambda T}$$

- $T \simeq \lambda^{1/3}$ means $\frac{t}{\sqrt{a}} \simeq (\frac{a^{3/2}}{h})^{1/3} = \frac{a^{1/2}}{h^{1/3}}$, hence $a \simeq th^{1/3}$;
- $T \simeq \lambda$ means $\frac{t}{\sqrt{a}} \ge \frac{a^{3/2}}{h}$, hence $a \simeq (th)^{1/2}$;
- if $T > \lambda \Rightarrow$ more terms in the sum over N than in the sum over k! use e_k to find $|V(T,X)| \lesssim \lambda$ (instead of $\sqrt{\lambda T}$ for $T > \lambda$).

What about u ?

- When $\theta = \frac{\eta}{h}$: integration in η gives $(\frac{h}{t})^{1/2}$ and $y + 2t\eta \simeq 0$;
- After integration in α, replace $K = \frac{T}{2N}$ by $\tilde{K} = \frac{Y}{4N}$, $Y = -\frac{y}{\sqrt{a}}$;
- u becomes a sum over N where T is now replaced by $Y/2$;
- For every $T < \lambda^{1/3}$, if $Y = 4\tilde{N} + O(1)$, then in this sum only $N = \tilde{N}$ can provide the worst contribution and the sum of all the other terms is smaller that the value at $N = \tilde{N}$.

Parametrix in terms of modes e_k : $a \lesssim (ht)^{1/2}$

Recall

$$V(T,X) \;=\; \sum_{k\geq 1} e^{i\lambda T(\frac{\omega_k}{\lambda^{2/3}})} \frac{\lambda^{2/3}}{L'(\omega_k)} Ai(\lambda^{2/3}X \;-\; \omega_k) Ai(\lambda^{2/3} \;-\; \omega_k).$$

Lemma: There exists C_0 such that for $L \geq 1$ the following holds true

$$\sup_{b\in\mathbb{R}} \Big(\sum_{1\leq k\leq L} \frac{1}{L'(\omega_k)} Ai^2(b-\omega_k) \Big) \leq C_0 L^{1/3},$$

where recall $L'(\omega_k) \simeq \sqrt{\omega_k}$.

- Since the main contribution of the sum over k comes from $k \simeq \lambda$, take $L \simeq 2\lambda$ and apply CS to deduce $|V(T,X)| \lesssim \lambda$;
- Notice that λ is (much) better than $\sqrt{\lambda T}$ for $T > \lambda$...
- When $\theta = \eta/h$, integrate over η taking advantage that the Airy terms can be seen as part of the symbol (since they do not oscillate much in this regime);
- For $k > L \simeq 2\lambda$: this corresponds to the "transverse" part in u.