Inverse boundary problems for semilinear elliptic PDE

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MSRI, October 17, 2019

Non-linearity helps in solving inverse problems!

- Hyperbolic case: Kurylev–Lassas–Uhlmann, 2018, Lassas–Uhlmann–Wang, 2019, Sá Barreto–Wang, 2018, Chen–Lassas–Oksanen–Paternain, 2019, Hintz–Uhlmann, 2018, ...,
- Elliptic case (full data problem): Feizmohammadi–Oksanen; Lassas–Liimatainen–Lin–Salo, 2019

A common feature of these works is that the presence of a nonlinearity allows one to solve inverse problems for non-linear equations in cases where the corresponding inverse problem in the linear setting is open.

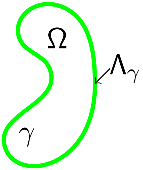
The purpose of the talk is to point out that the same phenomenon remains valid for partial data inverse boundary problems for a class of semilinear elliptic PDE.

The Calderón problem, 1980

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set with smooth boundary. Consider the boundary value problem,

$$\begin{cases} L_\gamma u = \nabla \cdot (\gamma \nabla u) = 0 \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = f. \end{cases}$$

Here $\gamma = \gamma(x)$ is the electrical conductivity, $\gamma > 0$ on $\overline{\Omega}$, f represents the imposed voltage potential at the boundary.



The Dirichlet–to–Neumann map: $\Lambda_\gamma(f) = (\gamma \partial_\nu u)|_{\partial\Omega}$, where ν is the unit outer normal to $\partial\Omega$.

We apply the voltage potential on $\partial\Omega$, measure the resulting current flux at $\partial\Omega$, and encode this information into the Dirichlet–to–Neumann map.

The Calderón problem: Does Λ_γ determine γ in Ω?

Applications:

- Medical imaging (electrical impedance tomography)

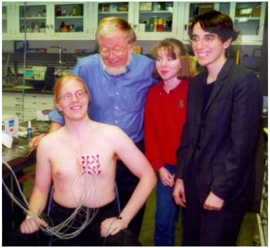


Figure: Source: https://matematiikkalehtisolmu.fi/2000/1/naatanen/

- Non-destructive testing (corrosion, cracks)

  ⋮

- Geophysical exploration (oil prospecting)

First global uniqueness result in dimension $n \geq 3$:

Theorem (Sylvester–Uhlmann, 1987)

Let $0 < \gamma_j \in C^{1,1}(\overline{\Omega})$, $j = 1, 2$. *If* $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ *then* $\gamma_1 = \gamma_2$ *in* Ω.

Idea of proof

Step 1. Reduction of the conductivity equation to the Schrödinger equation,

$$\gamma_j^{-1/2} \circ L_{\gamma_j} \circ \gamma_j^{-1/2} = \Delta - q_j, \quad q_j = \frac{\Delta \gamma_j^{1/2}}{\gamma_j^{1/2}} \in L^\infty(\Omega), \quad j = 1, 2.$$

Step 2. $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Longrightarrow$ the integral identity:

$$\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0,$$

for all u_1, u_2 solutions to $(\Delta - q_j)u_j = 0$ in Ω.
Step 3. Show that

$$\mathrm{Span}\{u_1 u_2 : (\Delta - q_j)u_j = 0 \text{ in } \Omega\}$$

is dense in $L^1(\Omega)$.

Construct enough special solutions to $(\Delta - q_j)u_j = 0$, called complex geometric optics (CGO) solutions:

$$u_j(x; h) = e^{\frac{x \cdot \zeta_j}{h}}(1 + r_j(x; h)), \quad j = 1, 2,$$

for all $h > 0$ small enough.

Here $\zeta_j \in \mathbb{C}^n$, $\zeta_j \cdot \zeta_j = 0$, $|\zeta_j| \sim 1 \Longrightarrow -\Delta(e^{\frac{x \cdot \zeta_j}{h}}) = 0$, r_j: remainder which tends to zero in a suitable sense, as $h \to 0$.

The issue of regularity of conductivity

- Sylvester–Uhlmann, 1987: $\gamma \in C^{1,1}(\overline{\Omega})$;
- Brown, 1996: $\gamma \in C^{1,\frac{1}{2}+\delta}(\overline{\Omega})$, $\delta > 0$;
- Päivärinta–Panchenko–Uhlmann, 2003: $\gamma \in W^{\frac{3}{2},\infty}(\Omega)$;
- Brown–Torres, 2003: $\gamma \in W^{\frac{3}{2},p}(\Omega)$, $p > 2n$;
- Haberman-Tataru, 2013: $\gamma \in C^1(\overline{\Omega})$ and $\gamma \in W^{1,\infty}(\Omega)$ with $\|\nabla \log \gamma\|_{L^\infty}$ small;
- Caro-Rogers, 2016: $\gamma \in W^{1,\infty}(\Omega)$;

Conjecture: global uniqueness holds for $\gamma \in W^{1,n}(\Omega)$.

- Haberman, 2016: proved the conjecture for $n = 3, 4$.
- Ham–Kwon–Lee, 2019, Ponce-Vanegas, 2019: further progress towards the conjecture for $n \geq 5$.

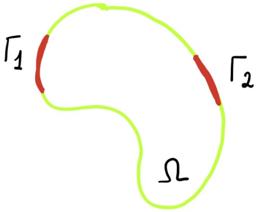
The Calderón problem with partial data

In practice impedance tomography measurements cannot be taken on the entire boundary due to limitations in resources or obstructions from natural obstacles.

This leads us to consider the Calderón problem with partial data.

Let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be arbitrary open non-empty. The partial Dirichlet–to–Neumann map,

$$\Lambda_\gamma^{\Gamma_1,\Gamma_2}(f) = (\gamma \partial_\nu u)|_{\Gamma_2}, \quad \text{supp}\,(f) \subset \Gamma_1.$$



The Calderón problem with partial data: Does $\Lambda_\gamma^{\Gamma_1,\Gamma_2}$ determine γ in Ω? Open in general.

Results for $C^{1,1}$ conductivities

Let $\gamma_1, \gamma_2 \in C^{1,1}(\overline{\Omega})$. If $\Lambda_{\gamma_1}^{\Gamma_1,\Gamma_2} = \Lambda_{\gamma_2}^{\Gamma_1,\Gamma_2}$ then $\gamma_1 = \gamma_2$ in Ω.

- Ammari–Uhlmann, 2004: $\gamma_1 = \gamma_2$ near $\partial\Omega$, $\Gamma_1 = \Gamma_2 \subset \partial\Omega$ arbitrary
- Isakov, 2007: $\Gamma_1 = \Gamma_2 = \Gamma$ and $\partial\Omega \setminus \Gamma$ is either a part of a hyperplane or a sphere

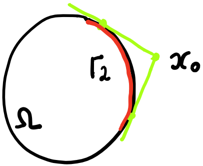
A characteristic feature of these results: reduction of the partial data problem to a full data one using unique continuation and symmetry type arguments.

- Bukhgeim–Uhlmann, 2002:

  $$\Gamma_1 = \partial\Omega, \quad \Gamma_2 = \{x \in \partial\Omega : \xi \cdot \nu(x) < \varepsilon\}, \quad \xi \in \mathbb{S}^{n-1}, \quad \varepsilon > 0.$$

  Note: Γ_2 is slightly more than a half of the boundary
- Kenig–Sjöstrand–Uhlmann, 2007:

  $$\Gamma_1 = \partial\Omega, \quad \Gamma_2 = \{x \in \partial\Omega : \frac{(x - x_0)}{|x - x_0|} \cdot \nu(x) < \varepsilon\},$$
  $$x_0 \notin \overline{ch(\Omega)}, \quad \varepsilon > 0.$$



  Note: when Ω is strictly convex, Γ_2 could be arbitrarily small

- Kenig–Salo, 2014: unifies approaches of Kenig–Sjöstrand–Uhlmann and Isakov and extends both of them.

The issue of regularity of conductivity in the partial data problem

- In the Bukhgeim–Uhlmann result:

$$\Gamma_1 = \partial\Omega, \quad \Gamma_2 = \{x \in \partial\Omega : \xi \cdot \nu(x) < \varepsilon\}, \quad \xi \in \mathbb{S}^{n-1}, \quad \varepsilon > 0.$$

  - Knudsen, 2006: $\gamma \in W^{\frac{3}{2}+\delta,2n}(\Omega)$, $\delta > 0$
  - Zhang, 2012: $\gamma \in C^{1,\delta}(\overline{\Omega}) \cap H^{\frac{3}{2}}(\Omega)$, $\delta > 0$

- In the Kenig–Sjöstrand–Uhlmann result:

$$\Gamma_1 = \partial\Omega, \quad \Gamma_2 = \{x \in \partial\Omega : \frac{(x - x_0)}{|x - x_0|} \cdot \nu(x) < \varepsilon\},$$
$$x_0 \notin \overline{ch(\Omega)}, \quad \varepsilon > 0.$$

Note: when Ω is strictly convex, Γ_2 could be arbitrarily small

Theorem (K.–Uhlmann, 2016)
Let $\gamma_1, \gamma_2 \in C^{1,\delta}(\overline{\Omega}) \cap H^{\frac{3}{2}}(\Omega)$, $\delta > 0$ arbitrarily small. Assume that $\gamma_1, \gamma_2 > 0$ in $\overline{\Omega}$, $\gamma_1 = \gamma_2$ and $\partial_\nu\gamma_1 = \partial_\nu\gamma_2$ on $\partial\Omega \setminus \Gamma_2$. If $\Lambda_{\gamma_1}^{\Gamma_1,\Gamma_2} = \Lambda_{\gamma_2}^{\Gamma_1,\Gamma_2}$ then $\gamma_1 = \gamma_2$ in Ω.

Remark. K.–Uhlmann, 2016: the result holds also for $\gamma_1, \gamma_2 \in W^{1,\infty}(\Omega) \cap H^{\frac{3}{2}+\delta}(\Omega)$, $\delta > 0$.

Outline of the proof

Step 1. Complex geometric optics solutions for Lipschitz continuous conductivities

Let $0 < \gamma \in W^{1,\infty}(\Omega)$ and extend it to a function on $\mathbb{R}^n$ so that the extension $0 < \gamma \in W^{1,\infty}(\mathbb{R}^n)$ and $\gamma = 1$ near infinity.

We reduce the conductivity equation to the Schrödinger equation:

$$\gamma^{-1/2} \circ L_\gamma \circ \gamma^{-1/2} = \Delta - q,$$

$$q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} = -\nabla \gamma^{1/2} \cdot \nabla \gamma^{-1/2} + \frac{1}{2} \Delta \log \gamma \in (H^{-1} \cap \mathcal{E}')(\mathbb{R}^n).$$

Define the "multiplication by q" map

$$m_q : H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)$$

by

$$\langle m_q(u), v \rangle_{\mathbb{R}^n} = -\int_{\mathbb{R}^n} (\nabla \gamma^{1/2} \cdot \nabla \gamma^{-1/2}) uv dx - \frac{1}{2} \int_{\mathbb{R}^n} \nabla \log \gamma \cdot \nabla (uv) dx,$$

for $u, v \in H^1(\mathbb{R}^n)$. Here $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ is the distribution duality on $\mathbb{R}^n$.

Construct complex geometric optics solutions for the Schrödinger equation with a singular potential,

$$-\Delta u + m_q(u) = 0 \quad \text{in} \quad \Omega.$$

Following Kenig–Sjöstrand–Uhlmann, 2007, we rely on Carleman estimates for the semiclassical Laplace operator $-h^2\Delta$, $0 < h \to 0$.

Carleman estimates with limiting Carleman weights

Let $\varphi \in C^\infty(\overline{\Omega}, \mathbb{R})$. Consider the conjugated operator

$$P_\varphi = e^{\frac{\varphi}{h}}(-h^2\Delta)e^{-\frac{\varphi}{h}},$$

with the semiclassical principal symbol

$$p_\varphi(x,\xi) = \xi^2 + 2i\nabla\varphi\cdot\xi - |\nabla\varphi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n.$$

Definition (Kenig–Sjöstrand–Uhlmann, 2007)

$\varphi \in C^\infty(\overline{\Omega}, \mathbb{R})$ is a limiting Carleman weight for $-h^2\Delta$ if $\nabla\varphi \neq 0$ and the Poisson bracket of $\operatorname{Re} p_\varphi$ and $\operatorname{Im} p_\varphi$ satisfies,

$$\{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\}(x,\xi) = 0 \quad \text{when} \quad p_\varphi(x,\xi) = 0, \quad (x,\xi) \in \Omega \times \mathbb{R}^n.$$

Note: if φ is a LCW then so is $-\varphi$.

Example:

- linear weights $\varphi(x) = \alpha \cdot x$, $\alpha \in \mathbb{R}^n$, $|\alpha| = 1$,
- logarithmic weights $\varphi(x) = \log|x - x_0|$, with $x_0 \notin \overline{\Omega}$.

Dos Santos Ferreira–Kenig–Salo–Uhlmann, 2009: Complete local classification of limiting Carleman weights on $\mathbb{R}^n$.

Proposition (Salo–Tzou, 2009, Kenig–Sjöstrand–Uhlmann, 2007)

Let φ be a limiting Carleman weight for $-h^2\Delta$, and let $\tilde{\varphi} = \varphi + \frac{h}{2\varepsilon}\varphi^2$. Then for $0 < h \ll \varepsilon \ll 1$ and $s \in \mathbb{R}$, we have the following Carleman estimate with a gain of 2 derivatives,

$$\frac{h}{\sqrt{\varepsilon}}\|u\|_{H^{s+2}_{scl}(\mathbb{R}^n)} \leq C\|e^{\tilde{\varphi}/h}(-h^2\Delta)e^{-\tilde{\varphi}/h}u\|_{H^s_{scl}(\mathbb{R}^n)}, \quad C > 0,$$

for all $u \in C_0^\infty(\Omega)$.

Here

$$\|u\|_{H^s_{\mathrm{scl}}(\mathbb{R}^n)} = \|\langle hD\rangle^s u\|_{L^2(\mathbb{R}^n)}, \quad \langle\xi\rangle = (1 + |\xi|^2)^{1/2}.$$

Recalling that

$$m_q : H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n),$$

we should use the Carleman estimate with $s = -1$ and $\varepsilon > 0$ sufficiently small but fixed.

Proposition

For all $h > 0$ sufficiently small, we have

$$h\|u\|_{H^1_{scl}(\mathbb{R}^n)} \leq C\|e^{\varphi/h}(-h^2\Delta + h^2 m_q)e^{-\varphi/h}u\|_{H^{-1}_{scl}(\mathbb{R}^n)},$$

for all $u \in C_0^\infty(\Omega)$.

The formal $L^2(\Omega)$ adjoint of the operator

$$e^{\varphi/h}(-h^2\Delta + h^2 m_q)e^{-\varphi/h}$$

is of the form

$$e^{-\varphi/h}(-h^2\Delta + h^2 m_q)e^{\varphi/h}$$

and therefore, the same Carleman estimate holds for the adjoint.

The Carleman estimate for the adjoint implies the following solvability result.

Proposition

If $h > 0$ is small enough, then for any $v \in H^{-1}(\Omega)$, there is a solution $u \in H^1(\Omega)$ of the equation

$$e^{\varphi/h}(-h^2\Delta + h^2 m_q)e^{-\varphi/h}u = v \quad in \quad \Omega,$$

which satisfies

$$\|u\|_{H^1_{scl}(\Omega)} \leq \frac{C}{h}\|v\|_{H^{-1}_{scl}(\Omega)}.$$

Complex WKB method

Fix a point $x_0 \notin \overline{\text{ch}(\Omega)}$ and let $\varphi(x) = \log|x - x_0|$.
We wish to construct complex geometric optics solutions to $-\Delta u + m_q(u) = 0$, which are of the form

$$u(x;h) = e^{\frac{\varphi + i\psi}{h}}(a(x) + r(x;h)).$$

Here $\psi \in C^\infty(\overline{\Omega}, \mathbb{R})$ should solve the eikonal equation:

$$|\nabla\psi|^2 = |\nabla\varphi|^2, \quad \nabla\varphi \cdot \nabla\psi = 0,$$

and the amplitude $a \in C^\infty(\overline{\Omega})$ should satisfy the transport equation:

$$2(\nabla\varphi + i\nabla\psi)\cdot\nabla a + (\Delta\varphi + i\Delta\psi)a = 0.$$

The eikonal and transport equations have global smooth solutions.

The remainder r should satisfy

$$e^{-\frac{(\varphi+i\psi)}{h}}(-h^2\Delta + h^2 m_q)(e^{\frac{\varphi+i\psi}{h}} r) = h^2\Delta a - h^2 m_q(a).$$

The solvability result together with standard L^2 smoothing estimates for $\nabla \log \gamma \in (L^\infty \cap \mathcal{E}')(\mathbb{R}^n) \Longrightarrow \exists r \in H^1(\Omega)$ such that

$$\|r\|_{H^1_{\mathrm{scl}}(\Omega)} = o(1), \quad h \to 0.$$

Such remainder estimates are not strong enough to solve the inverse problem, even in the full data case.

Improvement for $\gamma \in W^{1,\infty}(\Omega) \cap H^{\frac{3}{2}}(\Omega)$

In this case,

$$\|r\|_{H^1_{\mathrm{scl}}(\Omega)} = o(h^{1/2}), \quad h \to 0.$$

Where does this improvement come from?

- $\gamma \in W^{1,\infty} \implies A = \nabla \log \gamma \in L^\infty \cap \mathcal{E}'$ and therefore,

  $$\|A - A * \Psi_\tau\|_{L^2} = o(1), \quad \tau \to 0.$$

  Here Ψ_τ is the standard mollifier.
- $\gamma \in W^{1,\infty} \cap H^{\frac{3}{2}} \implies A = \nabla \log \gamma \in H^{\frac{1}{2}}$ and therefore,

  $$\|A - A * \Psi_\tau\|_{L^2} = o(\tau^{1/2}), \quad \tau \to 0.$$

Step 2. Converting $\Lambda_{\gamma_1}^{\Gamma_1,\Gamma_2} = \Lambda_{\gamma_2}^{\Gamma_1,\Gamma_2}$ into an integral identity

Let $u_j \in H^1(\Omega)$ satisfy $L_{\gamma_j} u_j = 0$ in Ω, $j = 1, 2$, and let $\widetilde{u}_1 \in H^1(\Omega)$ be an auxiliary solution such that $L_{\gamma_1} \widetilde{u}_1 = 0$ with $\widetilde{u}_1 = u_2$ on $\partial\Omega$. Then

$$\int_\Omega \Big(-\nabla \gamma_1^{1/2} \cdot \nabla(\gamma_2^{1/2} u_1 u_2) + \nabla \gamma_2^{1/2} \cdot \nabla(\gamma_1^{1/2} u_1 u_2) \Big)\, dx = \int_{\partial\Omega \setminus \Gamma_2} (\Lambda_{\gamma_1} \widetilde{u}_1 - \Lambda_{\gamma_2} u_2) u_1 \, dS.$$

Long tradition in inverse boundary problems ... Brown, 1996

Step 3. Testing the integral identity against complex geometric optics solutions

We can extend γ_j to all of $\mathbb{R}^n$ so that $\gamma_j - 1 \in C^{1,\delta}(\mathbb{R}^n) \cap H^{3/2}(\mathbb{R}^n)$, with $\gamma_1 = \gamma_2$ on $\mathbb{R}^n \setminus \Omega$ (thanks to the boundary determination).

Let us substitute the complex geometric optics solutions

$$u_1(x;h) = \gamma_1^{-1/2} e^{-\frac{(\varphi + i\psi)}{h}} \left(a_1(x) + r_1(x;h) \right),$$

$$u_2(x;h) = \gamma_2^{-1/2} e^{\frac{\varphi + i\psi}{h}} \left(a_2(x) + r_2(x;h) \right),$$

into the integral identity and pass to the limit $h \to 0$.

Main Lemma: We have

$$RHS = \int_{\partial\Omega \setminus \Gamma_2} (\Lambda_{\gamma_1} \widetilde{u}_1 - \Lambda_{\gamma_2} u_2) u_1 \, dS \to 0, \quad h \to 0.$$

To show this, we use boundary Carleman estimates.

Boundary Carleman estimates with limiting Carleman weights

Long tradition in PDE: Lebeau–Robbiano, 1994, Burq, 2002, Fursikov–Imanuvilov, 1996, Koch–Tataru, 2001, ..., Dos Santos Ferreira–Kenig–Sjöstrand–Uhlmann, 2007.

We want to apply boundary Carleman estimates to the function

$$\widetilde{u}_1 - u_2$$

and the conductivity operator

$$-\Delta - A_1 \cdot \nabla,$$

where $A_1 = \nabla \log \gamma_1 \in L^\infty \cap H^{\frac{1}{2}}$ (m_q is too singular, so cannot work with $-\Delta + m_q$).

Boundary Carleman estimates

For all $u \in H^2(\Omega)$, $u|_{\partial\Omega} = 0$, and all $h > 0$ small enough,

$$\mathcal{O}(h^3) \int_{\partial\Omega_-} (-\partial_\nu \varphi) e^{-\frac{2\varphi}{h}} |\partial_\nu u|^2 dS$$
$$+ \mathcal{O}(1) \| e^{-\varphi/h} (-h^2 \Delta - hA_1 \cdot h\nabla) u \|^2_{L^2(\Omega)}$$
$$\geq h^2 (\| e^{-\frac{\varphi}{h}} u \|^2_{L^2(\Omega)} + \| e^{-\frac{\varphi}{h}} h\nabla u \|^2_{L^2(\Omega)})$$
$$+ h^3 \int_{\partial\Omega_+} (\partial_\nu \varphi) e^{-\frac{2\varphi}{h}} |\partial_\nu u|^2 dS$$

Here

$$\partial\Omega_\pm = \{x \in \partial\Omega : \pm \partial_\nu \varphi(x) \geq 0\},$$

Recall that

$$\Gamma_2 = \{x \in \partial\Omega : \partial_\nu \varphi(x) < \varepsilon\}.$$

Thus, we can control the integral over the inaccessible boundary portion $\partial\Omega \setminus \Gamma_2 \subset \partial\Omega_+$.

Recall the integral identity:

$$\int_\Omega \left(-\nabla\gamma_1^{1/2} \cdot \nabla(\gamma_2^{1/2} u_1 u_2) + \nabla\gamma_2^{1/2} \cdot \nabla(\gamma_1^{1/2} u_1 u_2) \right) dx$$
$$= \int_{\partial\Omega\setminus\Gamma_2} (\Lambda_{\gamma_1} \widetilde{u}_1 - \Lambda_{\gamma_2} u_2) u_1 \, dS.$$

Hence,

$$RHS = \int_{\partial\Omega\setminus\Gamma_2} (\Lambda_{\gamma_1} \widetilde{u}_1 - \Lambda_{\gamma_2} u_2) u_1 \, dS \to 0, \quad h \to 0.$$

As $h \to 0$, the LHS of the integral identity $\to$

$$\int_\Omega \left(-\nabla\gamma_1^{1/2} \cdot \nabla(\gamma_1^{-1/2} a_1 a_2) + \nabla\gamma_2^{1/2} \cdot \nabla(\gamma_2^{-1/2} a_1 a_2) \right) dx.$$

Here the improved remainder estimates, available for $\gamma_j \in W^{1,\infty} \cap H^{3/2}$,

$$\|r_j\|_{H^1_{\mathrm{scl}}(\Omega)} = o(h^{1/2})$$

are vital.

Step 4. Recovering the conductivity

We have

$$\int_{\Omega} \left(-\nabla \gamma_1^{1/2} \cdot \nabla (\gamma_1^{-1/2} a_1 a_2) + \nabla \gamma_2^{1/2} \cdot \nabla (\gamma_2^{-1/2} a_1 a_2) \right) dx = 0,$$

for all $a_j \in C^\infty(\text{neigh}(\overline{\Omega}), \mathbb{R}^n)$ solving the transport equation

$$(\nabla \varphi + i \nabla \psi) \cdot \nabla a_j + \frac{1}{2} (\Delta \varphi + i \Delta \psi) a_j = 0.$$

Using analytic microlocal arguments, based on the microlocal Helgason and microlocal Holmgren theorems, we conclude that $\gamma_1 = \gamma_2$.

Remark. Can we go below $3/2$ derivatives in the partial data Calderón problem?

- Haberman–Tataru, 2013, Caro–Rogers, 2016: to go below $3/2$ derivatives in the full data case, exploit averaging arguments depending crucially on the linear nature of the limiting Carleman weights.
- Kenig–Sjöstrand–Uhlmann, 2007: a key point in the partial data problem is to use non-linear weights

To reach lower regularity in the partial data problem, it seems therefore that a new approach would be needed.

Partial data inverse boundary problems for semilinear elliptic PDE

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected bounded open set with C^∞ boundary.

Consider the Dirichlet problem for the following semilinear elliptic equation,

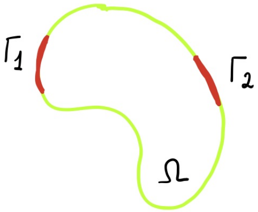
$$\begin{cases} -\Delta u + q(x)u^m = 0 \quad \text{in} \quad \Omega, \\ u = f \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $m \geq 2$, $q \in C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$ (the Hölder space).

There exist $\delta > 0$ and $C > 0$ such that when $f \in B_\delta(\partial\Omega) := \{f \in C^{2,\alpha}(\partial\Omega) : \|f\|_{C^{2,\alpha}(\partial\Omega)} < \delta\}$, the problem (1) has a unique solution $u = u_f \in C^{2,\alpha}(\overline{\Omega})$ satisfying $\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\delta$.

Let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be arbitrary open non-empty. Define the partial Dirichlet–to–Neumann map,

$$\Lambda_q^{\Gamma_1,\Gamma_2}(f) = \partial_\nu u|_{\Gamma_2}, \quad \text{supp}\,(f) \subset \Gamma_1.$$



Theorem (K.–Uhlmann; Lassas–Liimatainen–Lin–Salo, 2019)
$\Lambda_{q_1}^{\Gamma_1,\Gamma_2} = \Lambda_{q_2}^{\Gamma_1,\Gamma_2} \implies q_1 = q_2$ *in* Ω.

Remark. In the linear setting, i.e. $m = 1$,

$$-\Delta u + q(x)u = 0 \quad \text{in} \quad \Omega,$$

the following is known:

- $n \geq 3$: the problem is open in general,
- $n = 2$: open when $\Gamma_1 \cap \Gamma_2 = \emptyset$. The problem is solved
  - when $\Gamma_1 = \Gamma_2$ is an arbitrary open non-empty portion of $\partial\Omega$ and $q \in C^{1,\alpha}(\overline{\Omega})$ (Imanuvilov–Uhlmann–Yamamoto, 2010, Guillarmou–Tzou, 2011 (in the case of Riemann surfaces)),
  - when $\Gamma_1 \cap \Gamma_2 = \emptyset$, provided that some additional geometric assumptions are satisfied, and $q \in C^{2,\alpha}(\overline{\Omega})$ (Imanuvilov–Uhlmann–Yamamoto, 2011).

Remark. We can also consider more general semilinear elliptic equations,

$$-\Delta u + V(x, u) = 0 \quad \text{in} \quad \Omega,$$

where the function $V : \overline{\Omega} \times \mathbb{C} \to \mathbb{C}$ satisfies the following conditions:

(i) the map $\mathbb{C} \ni z \mapsto V(\cdot, z)$ is holomorphic with values in $C^{\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$,

(ii) $V(x, 0) = \partial_z V(x, 0) = 0$, for all $x \in \overline{\Omega}$.

Theorem (K.–Uhlmann; Lassas–Liimatainen–Lin–Salo, 2019)

$\Lambda_{V_1}^{\Gamma_1,\Gamma_2} = \Lambda_{V_2}^{\Gamma_1,\Gamma_2} \implies V_1 = V_2$ *in* $\Omega \times \mathbb{C}$.

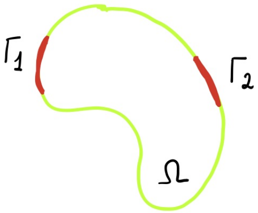
Consider next the following Dirichlet problem,

$$\begin{cases} -\Delta u + q(x)(\nabla u)^2 = 0 \quad \text{in} \quad \Omega, \\ u = f \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

Here $q \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha < 1$, $(\nabla u)^2 = \nabla u \cdot \nabla u$.

For any $f \in C^{2,\alpha}(\partial\Omega)$ small, there exists a unique small solution $u \in C^{2,\alpha}(\overline{\Omega})$. Define the partial Dirichlet–to–Neumann map,

$$\Lambda_q^{\Gamma_1,\Gamma_2}(f) = \partial_\nu u|_{\Gamma_2}, \quad \text{supp } (f) \subset \Gamma_1.$$



Theorem (K.–Uhlmann, 2019)

$\Lambda_{q_1}^{\Gamma_1,\Gamma_2} = \Lambda_{q_2}^{\Gamma_1,\Gamma_2} \implies q_1 = q_2$ *in* Ω.

Remark. Slightly more general nonlinearities can also be treated.

Remark. To best of our knowledge, this result is new even in the full data case $\Gamma_1 = \Gamma_2 = \partial\Omega$.

Let us mention that inverse boundary problems for nonlinear elliptic PDE have been studied quite extensively, both in the semilinear setting, as well as the quasilinear one:

Isakov–Nachman, 1995, Isakov–Sylvester, 1994, Sun, 1996, Sun–Uhlmann, 1997, Sun, 2010, Muñoz–Uhlmann, 2018, ...

A classical method for attacking inverse boundary problems for nonlinear elliptic PDE, going back to Isakov, 1993, (in the case of semilinear parabolic PDE), consists of performing a first order linearization of the given nonlinear Dirichlet-to-Neumann map, allowing one to reduce the inverse problem to an inverse boundary problem for a linear elliptic equation, and to employ the available results in this case.

The recent works by Feizmohammadi–Oksanen; Lassas–Liimatainen–Lin–Salo, 2019 have introduced the natural and powerful method of higher order linearizations of the nonlinear Dirichlet-to-Neumann map for inverse boundary problems for elliptic PDE, in the full data case, allowing one to solve such problems for nonlinear equations in situations where the corresponding inverse problems in the linear setting are open.

Previously, a second order linearization of the nonlinear Dirichlet–to–Neumann map has also been successfully exploited in the works by Assylbekov–Zhou, 2017, Kang–Nakamura, 2002, Sun, 1996, Sun–Uhlmann, 1997, ...

Idea of the proof of partial data results for semilinear PDE

Consider first, for $j = 1, 2$,

$$\begin{cases} -\Delta u_j + q_j(x) u_j^m = 0 & \text{in} \quad \Omega, \\ u_j = f & \text{on} \quad \partial\Omega, \end{cases}$$

Let $m = 2$ and let us perform a second order linearization of this problem.

To that end, let $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{C}^2$, and let $f_j \in C^\infty(\partial\Omega)$, supp $(f_j) \subset \Gamma_1$, $j = 1, 2$. The problem

$$\begin{cases} -\Delta u_j + q(x) u_j^2 = 0 & \text{in} \quad \Omega, \\ u_j = \varepsilon_1 f_1 + \varepsilon_2 f_2 & \text{on} \quad \partial\Omega, \end{cases}$$

has a unique small solution $u_j = u_j(\cdot, \varepsilon) \in C^{2,\alpha}(\overline{\Omega})$, which depends holomorphically on $\varepsilon \in \text{neigh}(0, \mathbb{C}^2)$ with values in $C^{2,\alpha}(\overline{\Omega})$.

Differentiating with respect to ε_l, $l = 1, 2$, taking $\varepsilon = 0$, and using that $u_j(x, 0) = 0$, we get

$$\begin{cases} \Delta v_j^{(l)} = 0 & \text{in} \quad \Omega, \\ v_j^{(l)} = f_l & \text{on} \quad \partial\Omega, \end{cases}$$

where $v_j^{(l)} = \partial_{\varepsilon_l} u_j|_{\varepsilon=0}$, $l = 1, 2$. By the uniqueness and the elliptic regularity for the Dirichlet problem, we see that $v^{(l)} := v_1^{(l)} = v_2^{(l)} \in C^\infty(\overline{\Omega})$, $l = 1, 2$.

Applying $\partial_{\varepsilon_1}\partial_{\varepsilon_2}|_{\varepsilon=0}$, we get next

$$\begin{cases} -\Delta w_j + 2q_j(x)v^{(1)}v^{(2)} = 0 & \text{in} \quad \Omega, \\ w_j = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

where $w_j = \partial_{\varepsilon_1}\partial_{\varepsilon_2} u_j|_{\varepsilon=0}$. The fact that

$$\Lambda_{q_1}^{\Gamma_1,\Gamma_2}(\varepsilon_1 f_1 + \varepsilon_2 f_2) = \Lambda_{q_2}^{\Gamma_1,\Gamma_2}(\varepsilon_1 f_1 + \varepsilon_2 f_2)$$

for all small $\varepsilon_1, \varepsilon_2$ and all $f_1, f_2 \in C^\infty(\partial\Omega)$ with supp (f_1), supp $(f_2) \subset \Gamma_1$ implies that $\partial_\nu w_1|_{\Gamma_2} = \partial_\nu w_2|_{\Gamma_2}$.

Multiplying the last equation by $v^{(3)} \in C^\infty(\overline{\Omega})$ harmonic in Ω and applying Green's formula, we get

$$2\int_\Omega (q_1 - q_2) v^{(1)} v^{(2)} v^{(3)} dx = \int_{\partial\Omega\setminus\Gamma_2} (\partial_\nu w_1 - \partial_\nu w_2) v^{(3)} dS = 0,$$

provided that supp $(v^{(3)}|_{\partial\Omega}) \subset \Gamma_2$. Hence, we obtain that

$$\int_\Omega (q_1 - q_2) v^{(1)} v^{(2)} v^{(3)} dx = 0,$$

for any $v^{(l)} \in C^\infty(\overline{\Omega})$ harmonic in Ω, $l = 1, 2, 3$, such that supp $(v^{(l)}|_{\partial\Omega}) \subset \Gamma_1$, $l = 1, 2$, and supp $(v^{(3)}|_{\partial\Omega}) \subset \Gamma_2$.
Take $v^{(3)} \not\equiv 0$.

Theorem (Dos Santos Ferreira–Kenig–Sjöstrand–Uhlmann, 2009)

Span$\{v^{(1)} v^{(2)} : v^{(l)} \in C^\infty(\overline{\Omega})$ *harmonic*$, v^{(l)}|_{\partial\Omega\setminus\Gamma_1} = 0, l = 1, 2\}$ *is dense in* $L^1(\Omega)$.

Using this result, we conclude that $q_1 = q_2$.

Let us now consider

$$\begin{cases} -\Delta u_j + q_j(x)(\nabla u_j)^2 = 0 \quad \text{in} \quad \Omega, \\ u_j = \varepsilon_1 f_1 + \varepsilon_2 f_2 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Similarly, performing a second order linearization, we get

$$\int_\Omega (q_1 - q_2)(\nabla v^{(1)} \cdot \nabla v^{(2)}) v^{(3)} dx = 0,$$

for any $v^{(l)} \in C^\infty(\overline{\Omega})$ harmonic in Ω, $l = 1, 2, 3$, such that supp $(v^{(l)}|_{\partial\Omega}) \subset \Gamma_1$, $l = 1, 2$, and supp $(v^{(3)}|_{\partial\Omega}) \subset \Gamma_2$. Our inverse theorem follows therefore from the following density result.

Theorem (K.–Uhlmann, 2019)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a connected bounded open set with C^∞ boundary, let $\Gamma \subset \partial\Omega$ be an open nonempty subset of $\partial\Omega$, and let $\tilde{\Gamma} = \partial\Omega \setminus \Gamma$. Then

$$Span\{\nabla u \cdot \nabla v : u, v \in C^\infty(\overline{\Omega}) \text{ harmonic}, u|_{\tilde{\Gamma}} = v|_{\tilde{\Gamma}} = 0\}$$

is dense in $L^1(\Omega)$.

Idea of the proof

We follow the strategy of Dos Santos Ferreira–Kenig–Sjöstrand–Uhlmann, 2009.

The global statement will be obtained as a corollary of the following local result.

Proposition (Local result)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with C^∞ boundary, let $x_0 \in \partial\Omega$, and let $\tilde{\Gamma} \subset \partial\Omega$ be the complement of an open boundary neighborhood of x_0. Then there exists $\delta > 0$ such that if we have

$$\int_\Omega f \nabla u \cdot \nabla v dx = 0,$$

for any harmonic functions $u, v \in C^\infty(\overline{\Omega})$ satisfying $u|_{\tilde{\Gamma}} = v|_{\tilde{\Gamma}} = 0$, then $f = 0$ in $B(x_0, \delta) \cap \Omega$.

From local to global

We use a continuity argument. We know that f vanishes near $x_0 \in \partial\Omega$. Let $x_1 \in \Omega$. Want: $x_1 \notin \text{supp}\,(f)$.

Let $\theta : [0,1] \to \overline{\Omega}$ be a C^1 curve joining x_0 and x_1, such that $\theta'(0)$ is the interior normal to $\partial\Omega$ at x_0, and $\theta(t) \in \Omega$ for all $t \in (0,1]$. We introduce the small neighborhood,

$$\Theta_\varepsilon(t) = \{x \in \overline{\Omega} : d(x, \theta([0,t])) \le \varepsilon\}, \quad 0 < \varepsilon \ll 1,$$

of the curve, ending at $\theta(t)$, and the set

$$I = \{t \in [0,1] : f \text{ vanishes a.e. on } \Theta_\varepsilon(t) \cap \Omega\}.$$

Now $I \neq \emptyset$ as $0 \in I$ if $\varepsilon > 0$ is small enough, by our local result. One can easily see that I is a closed subset of $[0,1]$. To show that I is open, we argue making crucial use of the following Runge type approximation result.

To state it, let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$ be two bounded open sets with smooth boundaries such that $\Omega_2 \setminus \overline{\Omega_1} \neq \emptyset$.

Associated to Ω_2, we let $\mathcal{G} : C^\infty(\overline{\Omega_2}) \to C^\infty(\overline{\Omega_2})$, $a \mapsto w$, be the Green solution operator to the Dirichlet problem,

$$\begin{cases} -\Delta w = a \quad \text{in} \quad \Omega_2, \\ w|_{\partial\Omega_2} = 0. \end{cases}$$

Lemma (Runge type approximation result)

The space

$$\{\mathcal{G}a|_{\Omega_1} : a \in C^\infty(\overline{\Omega_2}),\ supp\ (a) \subset \Omega_2 \setminus \overline{\Omega_1}\}$$

is dense in the space

$$\{u \in C^\infty(\overline{\Omega_1}) : -\Delta u = 0 \ in\ \Omega_1,\ u|_{\partial\Omega_1 \cap \partial\Omega_2} = 0\},$$

with respect to the $H^1(\Omega_1)$–topology.

Note that this result is a little bit more delicate than the corresponding L^2–approximation result, since the dual space of $H^1(\Omega_1)$,

$$\widetilde{H}^{-1}(\Omega_1) := \{v \in H^{-1}(\mathbb{R}^n) : \text{supp } (v) \subset \overline{\Omega_1}\},$$

is not a subspace of $\mathcal{D}'(\Omega_1)$ and special care is needed to handle the case when supp $(v) \cap \partial\Omega_1 \neq \emptyset$.

Local result

Proposition (Local result)

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with C^∞ boundary, let $x_0 \in \partial\Omega$, and let $\tilde{\Gamma} \subset \partial\Omega$ be the complement of an open boundary neighborhood of x_0. Then there exists $\delta > 0$ such that if we have

$$\int_\Omega f \nabla u \cdot \nabla v dx = 0,$$

for any harmonic functions $u, v \in C^\infty(\overline{\Omega})$ satisfying $u|_{\tilde{\Gamma}} = v|_{\tilde{\Gamma}} = 0$, then $f = 0$ in $B(x_0, \delta) \cap \Omega$.

We follow Dos Santos Ferreira–Kenig–Sjöstrand–Uhlmann, 2009. First using a conformal transformation, we reduce to the following setting: $x_0 = 0$, the tangent plane to Ω at x_0 is given by $x_1 = 0$, where $x = (x_1, x')$,

$$\Omega \subset \{x \in \mathbb{R}^n : |x + e_1| < 1\}, \quad \tilde{\Gamma} = \{x \in \partial\Omega : x_1 \leq -2c\}$$

for some $c > 0$. Here $e_1 = (1, 0, \ldots, 0)$ is the first basis vector.

We would like to construct enough harmonic functions vanishing on $\tilde{\Gamma}$, to conclude from the cancellation property that $f = 0$ near x_0.

We work with the corrected harmonic exponentials,

$$u(x, \zeta) = e^{-\frac{i}{h} x \cdot \zeta} + w(x, \zeta),$$

where $\zeta \in \mathbb{C}^n$, $\zeta \cdot \zeta = 0$, w is the solution to the Dirichlet problem,

$$\begin{cases} -\Delta w = 0 \quad \text{in} \quad \Omega, \\ w|_{\partial\Omega} = -(e^{-\frac{i}{h} x \cdot \zeta} \chi)|_{\partial\Omega}, \end{cases}$$

$\chi \in C_0^\infty(\mathbb{R}^n)$ a cutoff function such that $\chi = 1$ on $\tilde{\Gamma}$.

Thus, $u \in C^\infty(\overline{\Omega})$, u is harmonic in Ω, and $u|_{\tilde{\Gamma}} = 0$.

Cancellation property for $\zeta, \eta \in \mathbb{C}^n$, $\zeta \cdot \zeta = \eta \cdot \eta = 0$,

$$\int_\Omega f(x) hDu(x,\zeta) \cdot hDu(x,\eta) dx = 0.$$

Let $\gamma = ie_1 + e_2$. There exists $\varepsilon > 0$ small such that any $z \in \mathbb{C}^n$ such that $|z - 2ie_1| < 2\varepsilon$ can be decomposed as follows,

$$z = \zeta + \eta,$$

where $|\zeta - \gamma| < C_1\varepsilon$, $|\eta + \overline{\gamma}| < C_1\varepsilon$, $\zeta \cdot \zeta = \eta \cdot \eta = 0$.
Note that here $|\zeta \cdot \eta|$ is bounded from below. The cancellation property leads to the following bound for the h–Fourier transform of f,

$$\left| \int_\Omega f(x) e^{-\frac{ix \cdot z}{h}} dx \right| \leq C e^{-\frac{c}{2h}} e^{\frac{2C_1\varepsilon}{h}},$$

for all $z \in \mathbb{C}^n$ such that $|z - 2ie_1| < 2\varepsilon$.

Recall that we want to show that $f(x) = 0$ near $x = 0$. To that end, we pass from the Fourier transform of f to the FBI transform of f,

$$Tf(z) = \int_{\mathbb{R}^n} e^{-\frac{(z-y)^2}{2h}} f(y) dy, \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

Using the formula

$$Tf(z) = (2\pi h)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2h}(z^2+t^2)} e^{-\frac{1}{h} y\cdot(t+iz)} f(y) dtdy,$$

we get the bound,

$$e^{-\frac{\Phi(z_1)}{2h}} |Tf(z_1, x')| \leq Ch^{-1} \begin{cases} 1, & z_1 \in \mathbb{C}, \\ e^{-\frac{ca}{4h}}, & |z_1 - 2a| \leq \frac{\varepsilon a}{2}, |x'| < \frac{\varepsilon a}{2}, \end{cases} x' \in \mathbb{R}^{n-1},$$

where $a > 0$ is large but fixed and the weight Φ is given by

$$\Phi(z_1) = \begin{cases} (\operatorname{Im} z_1)^2, & \operatorname{Re} z_1 \leq 0, \\ (\operatorname{Im} z_1)^2 - (\operatorname{Re} z_1)^2, & \operatorname{Re} z_1 \geq 0. \end{cases}$$

Finally, one has to propagate the exponential decay of $Tf(z_1, x')$ from a neighborhood of $2a$ in z_1 to a neighborhood of 0. This is done by Dos Santos Ferreira–Kenig–Sjöstrand–Uhlmann, 2009, using complex analysis methods. Such ideas of propagating exponential decay estimates for FBI transforms by the use of maximum principle has a long tradition in Analytic Microlocal Analysis in connection with Kashiwara's watermelon theorem.

We obtain that

$$Tf(x) = \mathcal{O}(e^{-\frac{c'}{h}}), \quad c' > 0,$$

as $h \to 0$. Here $x \in \Omega$ near 0. As

$$Tf(x) = \int_{\mathbb{R}^n} e^{-\frac{(x-y)^2}{2h}} f(y) dy,$$

we see that $(2\pi h)^{-n/2} Tf(x) \to f(x)$, as $h \to 0$, a.e. Hence, we get $f = 0$ in a neighborhood of 0. This completes the proof of the local result.

Inverse boundary problems for semilinear elliptic equations with quadratic gradient terms, in the presence of an unknown obstacle

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with a connected C^∞ boundary, and let $D \subset\subset \Omega$ be such that $\Omega \setminus \overline{D}$ is connected and $\partial D \in C^\infty$. Let us consider the following boundary problem,

$$\begin{cases} -\Delta u + q(x)(\nabla u)^2 = 0 & \text{in} \quad \Omega \setminus \overline{D}, \\ u = 0 & \text{on} \quad \partial D, \\ u = f & \text{on} \quad \partial \Omega, \end{cases}$$

where $q \in C^\alpha(\overline{\Omega} \setminus D)$. For any $f \in C^{2,\alpha}(\partial\Omega)$ small, there is a unique small solution $u \in C^{2,\alpha}(\overline{\Omega} \setminus D)$. Let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be non-empty open subsets of the boundary $\partial\Omega$. Define the partial Dirichlet–to–Neumann map by

$$\Lambda_q^{D,\Gamma_1,\Gamma_2}(f) = \partial_\nu u|_{\Gamma_2}, \quad f \in C^{2,\alpha}(\partial\Omega) \quad \text{small}, \quad \text{supp}\,(f) \subset \Gamma_1.$$

Inverse problem: Does $\Lambda_q^{D,\Gamma_1,\Gamma_2}$ determine the unknown obstacle D and the potential q?

Theorem (K.–Uhlmann, 2019)

Let $D_1, D_2 \subset\subset \Omega$ be non-empty open subsets with C^∞ boundaries such that $\Omega \setminus \overline{D_j}$ are connected, $j = 1, 2$. Let $q_j \in C^\alpha(\overline{\Omega} \setminus D_j)$, $j = 1, 2$. Assume that $\Lambda_{q_1}^{D_1,\Gamma_1,\Gamma_2} = \Lambda_{q_2}^{D_2,\Gamma_1,\Gamma_2}$. Then $D := D_1 = D_2$, $q_1 = q_2$ in $\Omega \setminus \overline{D}$.

Remark. This is a generalization of a result by Lassas–Liimatainen–Lin–Salo, 2019 where semilinear elliptic equations without gradient terms are considered.

Remark. The problem of determining an unknown obstacle is of central significance in inverse scattering. The first uniqueness result for this problem goes back to Schiffer and Lax and Phillips, 1967. There have been numerous further contributions to this problem: Isakov, 1990, Kirsch–Kress, 1993, ...

However, the simultaneous recovery of an obstacle and surrounding potentials in the linear setting, say in the case of the linear Schrödinger equation, constitutes an open problem.

A word about the proof

It is an immediate consequence of our partial data result, once the obstacle has been recovered. To determine the obstacle, performing the first order linearization of the problem and the partial Dirichlet–to–Neumann map, we get

$$\begin{cases} -\Delta v_j = 0 & \text{in} \quad \Omega \setminus \overline{D_j}, \\ v_j = 0 & \text{on} \quad \partial D_j, \\ v_j = f & \text{on} \quad \partial\Omega, \end{cases}$$

where $f \in C^\infty(\partial\Omega)$, supp $(f) \subset \Gamma_1$, and $\partial_\nu v_1|_{\Gamma_2} = \partial_\nu v_2|_{\Gamma_2}$.

A standard contradiction argument implies that $D_1 = D_2$.

THANK YOU VERY MUCH FOR YOUR ATTENTION!