Semi-classical analysis of Schrödinger equation on H-type groups

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Motivation

- [Folland & Stein 1974], [Rotschild & Stein 1976], [Folland, 1977]: Analysis of sublaplacians can be performed using Lie groups theory, lifting procedure.

  Example: Heisenberg group and the operator L on $\mathbb{R}^3$:

$$L = X^2 + Y^2, \;\; X = \partial_x - \frac{y}{2}\partial_z, \;\; Y = \partial_y + \frac{x}{2}\partial_z, \;\; [X, Y] = \partial_z$$

- What would be a semiclassical/microlocal approach on graded Lie groups using theory of representations ?
  Which class of pseudodifferential operators ?

  [Taylor 1984], [Beals & Greiner 1988], [Christ, Geller, Glowasky & Polin 1992] [Geller 1990]
  [Bahouri, FK & Gallagher 2012], [Fischer & Ruzhansky 2016], [FK & Fischer, 2018 & 2019]

The question: Large time evolution of energy density of families of solutions of a Schrödinger equation

- Let V_j be $2d$ vector fields of $\mathbb{R}^{2d+p}$ and

$$-\Delta_G = \sum_{1\leq j\leq 2d} V_j^2.$$

- Assume $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ is a Lie algebra of Heisenberg type with centre $\mathfrak{z}$,

$$\mathfrak{v} = \mathrm{Vect}(V_i,\ 1 \leq j \leq 2d\},\ \ \mathfrak{z} = \mathrm{Vect}([V_i, V_j],\ 1 \leq i,j \leq 2d),$$
$$2d = \dim\mathfrak{v},\ \ \ p = \dim\mathfrak{z}.$$

- Let $(\psi_0^\varepsilon)_{\varepsilon>0}$ be bounded in $L^2(\mathbb{R}^d)$ such that $\exists s, C_s > 0$, $\forall\varepsilon > 0$,

$$\varepsilon^s \|(-\Delta_G)^{\frac{s}{2}}\psi_0^\varepsilon\|_{L^2(G)} + \varepsilon^{-s}\|(-\Delta_G)^{-\frac{s}{2}}\psi_0^\varepsilon\|_{L^2(G)} \leq C_s.$$

Describe for $\phi \in \mathcal{C}_c^\infty(G)$, $T \in \mathbb{R}$, $\tau > 0$

$$\lim_{\varepsilon\to 0} \frac{1}{T}\int_0^T \int_G \phi(x) |\mathrm{e}^{i\frac{t}{2\varepsilon^\tau}\varepsilon^2\Delta_G}\psi_0^\varepsilon(x)|^2 dx\, dt.$$

The same question in the Euclidean case

Let (ψ_0^ε) be a bounded family in $L^2(\mathbb{R}^d)$ satisfying

$$\exists s, C_s > 0,\ \ \forall \varepsilon > 0,\ \ \varepsilon^s \|(-\Delta)^{\frac{s}{2}}\psi_0^\varepsilon\|_{L^2(\mathbb{R}^d)} + \varepsilon^{-s}\|(-\Delta)^{-\frac{s}{2}}\psi_0^\varepsilon\|_{L^2(\mathbb{R}^d)} \leq C_c.$$

Then any limit point of the measure

$$\left|\mathrm{e}^{-i\frac{t}{2\varepsilon^\tau}\varepsilon^2\Delta}\psi_0^\varepsilon\right|^2 dxdt$$

is of the form $\varrho_t(x)dt$ where $\varrho_t \in \mathcal{M}^+(\mathbb{R}^d)$ and

1. If $\tau \in (0,1)$, then $\varrho_t = \varrho_0$.
2. If $\tau = 1$ then $\varrho_t(x) = \int_{\mathbb{R}^d} \mu_0(x - t\xi, d\xi)$.
3. If $\tau > 1$ then $\varrho_t = 0$.

The result

Theorem (FK & Fischer 2019)

Any weak limit of $|\mathrm{e}^{i\frac{t}{2\varepsilon^\tau}\varepsilon^2\Delta_G}\psi_0^\varepsilon(x)|^2 dx\, dt$ *writes* $\varrho_t dt = \left(\varrho_t^{\mathfrak{v}^*} + \varrho_t^{\mathfrak{z}^*}\right) dt$ *with*

- *If* $\tau \in (0,1)$, *for all* $t \in \mathbb{R}$, $\varrho_t = \varrho_0$
- *If* $\tau = 1$, *then* $\varrho_t^{\mathfrak{z}^*} = \varrho_0^{\mathfrak{z}^*}$ *and* $\varrho_t^{\mathfrak{v}^*}(x) = \int_{\mathfrak{v}^*} \varsigma_0\left(\mathrm{Exp}(t\,\omega \cdot V)x, d\omega\right).$
- *If* $\tau \in (1,2)$, *then* $\varrho_t^{\mathfrak{v}^*} = 0$ *and* $\partial_t \varrho_t^{\mathfrak{z}^*} = 0$ *holds in* $\mathcal{D}'(\mathbb{R} \times G)$.
- *If* $\tau = 2$, *then* $\varrho_t^{\mathfrak{v}^*} = 0$ *and* $\varrho_t^{\mathfrak{z}^*} = \sum_{n\in\mathbb{N}} \int_{\mathfrak{z}^*\setminus\{0\}} \gamma_{n,t}(x, d\lambda)$ *where*

$$\left(\partial_t - \frac{2n+d}{2|\lambda|}\mathcal{Z}^{(\lambda)}\right)\gamma_{n,t} = 0,$$

  where $\mathcal{Z}(\lambda)$ *is the left invariant vector field corresponding to* $\lambda \in \mathfrak{z}^*$.
- *If* $\tau > 2$, *then* $\varrho_t = 0$ *for all* $t \in \mathbb{R}$.

Remarks

- Quantum limits split into two parts, with different threshold indexes.
- Dispersion takes longer than in the Euclidean case [Bahouri, Gérard & Xu 2000], [Del Hierro 2005], [Bahouri, FK, Gallagher 2016]
- Splitting and invariance properties already noticed ($p = 1$ - contact manifolds, Grauert tubes) in [Zelditch 1997], [Colin de Verdière, Hillairet & Trélat 2018], [Burq & Sun 2019]

Schedule of the talk

1. Graded, stratified Lie groups, H-type groups
2. Analysis on graded Lie groups and Fourier transform
3. Pseudodifferential operators, Egorov theorem
4. Semi-classical measures and Schrödinger equation.

Graded Lie groups

Graded Lie groups - Definition

Definition

A simply connected Lie group G is graded if the Lie algebra $\mathfrak{g}$ of its left-invariant vector fields is graded :

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n, \quad [\mathfrak{g}_\ell, \mathfrak{g}_{\ell'}] \subset \mathfrak{g}_{\ell+\ell'}.$$

The group G is stratified if

$$\forall \ell \in \{1, \cdots n-1\}, \quad [\mathfrak{g}_\ell, \mathfrak{g}_1] = \mathfrak{g}_{\ell+1}.$$

- G identifies to $\mathfrak{g}$ via the exponential map:

$$\begin{array}{cccc} \exp & : & \mathfrak{g} & \to & G \\ & & X & \mapsto & \exp(X) e_G \end{array}$$

- The law group on G is a polynomial map (Campbell-Baker-Hausdorff).

Graded Lie groups - Heisenberg group

- The Heisenberg group:
  - The Lie algebra $\mathfrak{h} = \mathfrak{v} \oplus \mathfrak{z}$ is generated by

$$X = \partial_x - \frac{y}{2}\partial_z, \quad Y = \partial_y + \frac{x}{2}\partial_z, \quad Z = \partial_z = [X, Y].$$

$\Longrightarrow$ The group $\mathbb{H}$ is stratified and has two steps.

  - The points of G are the elements

$$w = \mathrm{Exp}(xX + yY + sS), \ \ (x, y, s) \in \mathbb{R}^3.$$

  - $\mathbb{H} = \mathbb{R}^3$ with the product law

$$(x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' + \frac{1}{2}(x \cdot y' - x' \cdot y)).$$

  - The center of $\mathbb{H}$ is the set of points of the form $(0, 0, s)$.

Graded Lie groups - H-type groups

- H-type groups (multidimensional versions of $\mathbb{H}$ [Kaplan 80])

  1. The Lie algebra of G is stratified with two steps $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$.
  2. If $B(\lambda)$ is the skew symmetric form defined on $\mathfrak{v} \times \mathfrak{v}$ by

$$\forall \lambda \in \mathfrak{z}^*, \ \ \forall U, V \in \mathfrak{v}, \ \ B(\lambda)(U, V) = \lambda([U, V]),$$

  then $B(\lambda)^2 = -|\lambda|^2 \mathrm{Id}.$

- Notation (1): Let $(V_1, \ldots, V_{2d})$ be an ONB of $\mathfrak{v}$ and $(Z_1, \cdots, Z_p)$ of $\mathfrak{z}$, then $x \in G$ or $X \in \mathfrak{g}$ writes

$$x = \mathrm{Exp}(X), \qquad X = v_1 V_1 + \ldots + v_{2d} V_{2d} \ + \ z_1 Z_1 + \ldots + z_p Z_p.$$

- Notation (2): $\mathfrak{v}$ decomposes in a λ-depending way as $\mathfrak{v} = \mathfrak{p}_\lambda \oplus \mathfrak{q}_\lambda$ with

$$\mathfrak{p} := \mathfrak{p}_\lambda := \mathrm{Span}\left(P_1, \dots, P_d\right), \quad \mathfrak{q} := \mathfrak{q}_\lambda := \mathrm{Span}\left(Q_1, \dots, Q_d\right).$$

where $(P_1, \dots, P_d, Q_1, \dots, Q_d)$ be an ONB of $\mathfrak{v}$ such that

$$B(\lambda)(U, V) = |\lambda| U^t J V, \;\; J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

Then $x \in G$ or $X \in \mathfrak{g}$ write

$$x = \mathrm{Exp}(X), \;\; X = p_1 P_1 + \ldots + p_d P_d + q_1 Q_1 + \ldots + q_d Q_d + z_1 Z_1 + \ldots + z_p Z_p.$$

Besides

$$[P_j, Q_j] = \mathcal{Z}^\lambda, \;\; \forall j \in \{1, \cdots, d\}$$

where $\mathcal{Z}^\lambda$ corresponds to the vector λ in the identification $\mathfrak{z} \sim \mathfrak{z}^*$.

Analysis on H-type groups

Analysis on H-type groups

- The Haar measure, dx is deduced on G from the Lebesgue meas. on $\mathfrak{g}$. Associated Lebesgue spaces on G:

$$\|f\|_{L^p(G)} := \left(\int_G |f(x)|^p \, dx \right)^{\frac{1}{p}}, \ \ p \in [1, \infty[.$$

- The dilatation. For $r > 0$ and $X = V + Z \in \mathfrak{g}$, one sets $\delta_r X = rV + r^2 Z$.

$$\begin{array}{ccccc} & \mathfrak{g} & \overset{\delta_r}{\longrightarrow} & \mathfrak{g} & \\ \exp & \downarrow & & \downarrow & \exp \\ & G & \underset{\exp \circ \delta_r \circ \exp^{-1}}{\longrightarrow} & G & \end{array}$$

$$r \cdot \mathrm{Exp}(V + Z) := \mathrm{Exp}(r\,V + r^2 Z), \ r > 0.$$

$\Longrightarrow$ Homogeneous functions and operators

- Homogeneous dimension. The Haar measure is Q-homogeneous:

$$Q := \dim \mathfrak{v} + 2 \dim \mathfrak{z} = 2d + 2p.$$

The dual set of H-type groups

Let $\hat{G}$ be the set of irreducible representations of G:

$$\widehat{G} = \{\text{class of } \pi^{\lambda} \ : \ \lambda \in \mathfrak{z}^* \setminus \{0\}\} \sqcup \{\text{class of } \pi^{0,\omega} \ : \ \omega \in \mathfrak{v}^*\}.$$

- Infinite dimensional representations are parametrized by $\mathfrak{z}^* \setminus \{0\}$: for $\lambda \in \mathfrak{z}^* \setminus \{0\}$, then $\mathcal{H}_\lambda = L^2(\mathfrak{p}_\lambda)$ with for $x = \mathrm{Exp}(P + Q + Z) \in G$,

$$\pi_x^{\lambda}\Phi(\xi) = \exp\left[i\lambda(Z) + \frac{i}{2}|\lambda|\, P \cdot Q + i\sqrt{|\lambda|}\,\xi \cdot Q\right] \Phi\left(\xi + \sqrt{|\lambda|}P\right).$$

- Finite dimensional representations are parametrized by $\mathfrak{v}^*$: for $\omega \in \mathfrak{v}^*$,

$$\pi_x^{0,\omega} = \mathrm{e}^{i\omega(V)}, \quad x = \mathrm{Exp}(V + Z) \in G.$$

The dual set of H-type groups

- Plancherel measure: $d\mu(\pi^\lambda) = |\lambda|^d d\lambda$.

$\Longrightarrow$ The set of finite dimensional representations is of 0 Plancherel measure.

- Dilations on $\hat{G}$: $r \cdot \pi_x = \pi_{r.x}$
  - for $r \in \mathbb{R}_+^*$, $\lambda \in \mathfrak{z}^* \setminus \{0\}$, $x \in G$,

$$\pi_{r\cdot x}^\lambda = T_r \pi_x^{r^2\lambda} T_r^*, \quad T_r f(\xi) = r^{1/2} f(r\xi),$$

$\Longrightarrow \quad r \cdot \lambda = r^2 \lambda$

  - for $r \in \mathbb{R}_+^*$, $\omega \in \mathfrak{v}^*$, $x \in G$, $\pi_{r\cdot x}^{0,\omega} = \mathrm{e}^{ir\omega\cdot x}$

$\Longrightarrow \quad r \cdot (0, \omega) = (0, r\omega)$

Fourier transform

Definition

Let $f \in L^1(G)$, the Fourier transform of f is the operator of $\mathcal{L}(\mathcal{H}_\lambda)$,

$$\widehat{f}(\lambda) = \int_G f(x)\pi^\lambda(x)^* dx.$$

- Notation: $\widehat{f}(\lambda) = \widehat{f}(\pi^\lambda)$, $\widehat{f}(0,\omega) = \widehat{f}(\pi^{0,\omega})$.
- Extension to $L^2(G)$: if $f \in L^2(G)$, $\widehat{f}(\lambda) \in HS(\mathcal{H}_\lambda)$ and Plancherel formula

$$\int_G |f(x)|^2 dx = c_0 \int_{\hat{G}} \|\widehat{f}(\lambda)\|^2_{HS(\mathcal{H}_\lambda)} |\lambda|^d d\lambda.$$

- Inversion formula : with ad-hoc assumptions

$$f(x) = c_0 \int_{\hat{G}} \mathrm{tr}(\pi^\lambda_x \widehat{f}(\lambda))|\lambda|^d d\lambda.$$

The sublalacian

The Sublaplacian: $-\Delta_G = \sum_{1\leq j\leq 2d} V_j^2$,

$$\widehat{\Delta_G f}(\lambda) = H(\lambda)\widehat{f}(\lambda), \;\; \lambda \in \widehat{G}, \;\; f \in \mathcal{S}(G)$$

with

$$H(\lambda) := |\lambda| \sum_{1\leq j\leq d} (-\partial_{\xi_j}^2 + \xi_j^2) \;\; \textit{if} \;\; \lambda \in \mathfrak{z}^* \setminus \{0\},$$
$$H((0,\omega)) = |\omega|^2, \;\; \textit{if} \;\; \omega \in \mathfrak{v}^*.$$

- Spectrum of $H(\lambda)$: $|\lambda|(2|\alpha| + d)$, $\alpha \in \mathbb{N}^d$
- Eigenprojectors of $H(\lambda)$:

$$\Pi_n = \sum_{|\alpha|=n} |h_\alpha\rangle\langle h_\alpha|,$$

with h_α Hermite functions $h_\alpha(\xi) = \Pi_{1\leq j\leq d} h_{\alpha_j}(\xi_j)$.

Pseudodifferential operators and Egorov Theorem

Pseudodifferential operators on H-type groups

- Using Fourier inversion formula for defining an operator:

$$f(x) = c_0 \int_{\hat{G}} \mathrm{tr}\left(\pi_x^\lambda \widehat{f}(\lambda)\right) |\lambda|^d d\lambda.$$

$$\mathrm{Op}(\sigma)f(x) = c_0 \int_{\hat{G}} \mathrm{tr}\left(\pi_x^\lambda \sigma(x,\lambda)\widehat{f}(\lambda)\right) |\lambda|^d d\lambda, \;\; \sigma(x,\lambda) \in \mathcal{L}(\mathcal{H}_\lambda).$$

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- Symbols : $\sigma(x,\lambda) \in S^m(G)$ iff σ is $\mathcal{C}_c^\infty$ in x and "homogeneous of degree m in λ" + some "differentiability" condition in π^λ (difference operators). Note that $\sigma(x,(0,\omega))$ is a scalar and not an operator.

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- Semi-classical pseudodifferential operators: Let $\sigma \in S^{-\infty}(G)$, $\varepsilon \ll 1$

$$\mathrm{Op}_\varepsilon(\sigma) f(x) = c_0 \int_{\hat{G}} \mathrm{tr}\left(\pi_x^\lambda \sigma(x, \varepsilon \cdot \lambda) \widehat{f}(\lambda)\right) |\lambda|^d d\lambda.$$

[Bahouri, FK & Gallagher, 12], [Fischer & Ruzhansky, 16], [FK & Fischer, 18].

The kernel of a semi-classical pseudodifferential operator.

For $f \in \mathcal{S}(G)$,

$$\begin{aligned}
\mathrm{Op}_\varepsilon(\sigma) f(x) &= c_0 \int_{\widehat{G}} \mathrm{Tr}\left(\pi_x^\lambda \sigma(x, \varepsilon \cdot \lambda) \mathcal{F} f(\lambda)\right) |\lambda|^d \, d\lambda \\
&= c_0 \, \varepsilon^{-Q} \int_{G \times \widehat{G}} \mathrm{Tr}\left(\pi^\lambda_{\delta_{\varepsilon^{-1}}(y^{-1}x)} \sigma(x, \lambda)\right) f(y) |\lambda|^d \, d\lambda \, dy \\
&= \int_G \kappa_x^\varepsilon(y^{-1}x) f(y) dy.
\end{aligned}$$

The convolution kernel of $\mathrm{Op}_\varepsilon(\sigma)$ is obtained by scaling from $\kappa_x(\cdot)$,

$$\kappa_x^\varepsilon(z) = \varepsilon^{-Q} \kappa_x\left(\delta_{\varepsilon^{-1}} z\right)$$

with

$$\kappa_x(z) = c_0 \int_{\hat{G}} \mathrm{Tr}\left(\pi_z^\lambda \sigma(x, \lambda)\right) |\lambda|^d d\lambda \ \text{ i.e. } \ \mathcal{F}(\kappa_x)(\lambda) = \sigma(x, \lambda).$$

Action on $L^2(G)$

The set $\mathcal{A}_0$ is the set of operator-valued symbols $\sigma(x,\lambda) \in \mathcal{L}(\mathcal{H}_\lambda)$ such that

$$\sigma(x,\lambda) = \mathcal{F}\kappa_x(\lambda) = \int_G \kappa_x(z)(\pi_z^\lambda)^* dz,$$

where $x \mapsto \kappa_x(y)$ is in $\mathcal{C}_c^\infty(G, \mathcal{S}(G))$.

Proposition

The set $\mathcal{A}_0$ is an algebra and $\exists C > 0$, $\forall \sigma \in \mathcal{A}_0$, $\forall \varepsilon > 0$,

$$\|\mathrm{Op}_\varepsilon(\sigma)\|_{\mathcal{L}(L^2(G))} \leq C \int_G \sup_{x\in G} |\kappa_x(z)| dz.$$

Proof:

$$|\mathrm{Op}_\varepsilon(\sigma) f(x)| = \left| \int_G f(y) \kappa_x^\varepsilon(y^{-1}x) dy \right|$$
$$\leq \int_G |f(y)| \sup_{x_1 \in G} |\kappa_{x_1}^\varepsilon(y^{-1}x)| \, dy = |f| * \sup_{x_1\in G} |\kappa_{x_1}^\varepsilon(\cdot)|(x)$$

Then Young conv. inequ. and $\| \sup_{x\in G} |\kappa_x^\varepsilon(\cdot)| \|_{L^1(G)} = \| \sup_{x \in G} |\kappa_x(\cdot)| \|_{L^1(G)}$.

Egorov Theorem [FK, Fischer 19]

Let $\psi \in L^2(G)$ and $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$, $\sigma \in \mathcal{A}$ with $\sigma = 0$ close to $\lambda = 0$ and

$$Q_\sigma^\varepsilon(t) = \mathrm{e}^{-i\frac{t}{2\varepsilon^\tau}\varepsilon^2\Delta_G}\,\mathrm{op}_\varepsilon(\sigma)\,\mathrm{e}^{i\frac{t}{2\varepsilon^\tau}\varepsilon^2\Delta_G}.$$

- If $[\sigma, H(\lambda)] \neq 0$, then $\displaystyle\int_{\mathbb{R}} \theta(t)\,(Q_\sigma^\varepsilon(t)\psi, \psi)_{L^2(G)}\,dt = O(\varepsilon^{\min(\tau,1)}\|\psi\|^2)$.
- If $[\sigma, H(\lambda)] = 0$, $\sigma = \Pi_n \sigma \Pi_n$
  1. if $\tau \in (0,2)$, $\displaystyle\int_{\mathbb{R}} \theta'(t)\,(Q_\sigma^\varepsilon(t)\psi, \psi)\,dt = O(\varepsilon^{\min(1,\tau-2)}\|\psi\|^2)$,
  2. if $\tau = 2$, for all $s \in \mathbb{R}$ (transport)
  $$\int_{\mathbb{R}} \theta(t)\,(Q_\sigma^\varepsilon(t)\psi, \psi)\,dt = \int_{\mathbb{R}} \theta(t+s)\left(Q^\varepsilon_{\Phi_n^{-s}(\sigma)}(t)\psi, \psi\right)dt + O(\varepsilon\|\psi\|^2),$$
  3. if $\tau > 2$, for all $s \in \mathbb{R}$ (invariance)
  $$\int_{\mathbb{R}} \theta(t)\,(Q_\sigma^\varepsilon(t)\psi, \psi)\,dt = \int_{\mathbb{R}} \theta(t)\left(Q^\varepsilon_{\Phi_n^{-s}(\sigma)}\psi, \psi\right)dt + O(\varepsilon^{\min(1,2-\tau)}\|\psi\|^2).$$

Above, $\Phi_n^s(\sigma) = \sigma\left(\mathrm{Exp}(\frac{2n+d}{2|\lambda|}\mathcal{Z}^{(\lambda)}s)x, \lambda\right)$.

Proof of the Egorov Theorem

One writes

$$\varepsilon^\tau \frac{d}{dt}\left(\mathrm{op}_\varepsilon(\sigma)\psi^\varepsilon(t), \psi^\varepsilon(t)\right) = \left(\mathrm{op}_\varepsilon([\sigma, H(\lambda)])\psi^\varepsilon(t), \psi^\varepsilon(t)\right)$$
$$+\varepsilon\left(\mathrm{op}_\varepsilon(V.\pi^\lambda(V)\sigma)\psi^\varepsilon(t), \psi^\varepsilon(t)\right) - \varepsilon^2\left(\mathrm{op}_\varepsilon(\Delta_G\sigma)(t)\psi^\varepsilon(t), \psi^\varepsilon(t)\right).$$

2 facts:

1. There exists $\sigma_1 \in \mathcal{A}$ such that
$$V.\pi^\lambda(V)\sigma = [\sigma_1, H(\lambda)].$$
2. For this σ_1, one has
$$\Pi_n\left(V.\pi^\lambda(V)\sigma_1 - \frac{1}{2}\Delta_G\sigma\right)\Pi_n = \frac{2n+d}{2|\lambda|}\Pi_n\mathcal{Z}^{(\lambda)}\sigma\,\Pi_n.$$

Proof of the Egorov Theorem

One writes for $\sigma = \Pi_n \sigma \Pi_n$

$$\varepsilon^\tau \int \theta(t) \frac{d}{dt} \left(\mathrm{op}_\varepsilon(\sigma) \psi^\varepsilon(t), \psi^\varepsilon(t) \right) dt =$$

$$\varepsilon^2 \int \theta(t) \left(\mathrm{op}_\varepsilon \left(\frac{2n+d}{2|\lambda|} \Pi_n \mathcal{Z}^{(\lambda)} \sigma \, \Pi_n \right) \psi^\varepsilon(t), \psi^\varepsilon(t) \right) dt.$$

2 facts:

1. There exists $\sigma_1 \in \mathcal{A}$ such that

$$V.\pi^\lambda(V)\sigma = [\sigma_1, H(\lambda)].$$

2. For this σ_1, one has

$$\Pi_n \left(V.\pi^\lambda(V)\sigma_1 - \frac{1}{2} \Delta_G \sigma \right) \Pi_n = \frac{2n+d}{2|\lambda|} \Pi_n \mathcal{Z}^{(\lambda)} \sigma \, \Pi_n.$$

Semi-classical measures and Schrödinger equation

Existence and structure of semi-classical measures

Let $\mathcal{M}_1^+(G \times \widehat{G})$ be the set of pairs (Γ, γ) consisting of a positive Radon measure γ on $G \times \widehat{G}$ and a family of positive trace 1 operators $\Gamma(x, \lambda)$ on $\mathcal{H}_\lambda$.

Proposition

Let $(\psi^\varepsilon(t))$ be a bounded family in $L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))$. There exist $\varepsilon_k \underset{k \to +\infty}{\longrightarrow} 0$ and $t \mapsto \Gamma_t d\gamma_t \in L^\infty(\mathbb{R}, \mathcal{M}_1^+(G \times \widehat{G}))$ such that for all $\theta \in L^1(\mathbb{R})$ and $\sigma \in \mathcal{A}_0$,

$$\int \theta(t) \left(\mathrm{Op}_\varepsilon(\sigma)\psi^\varepsilon(t), \psi^\varepsilon(t)\right) dt \underset{\varepsilon_k \to 0}{\longrightarrow} \int_{\mathbb{R}\times G \times \widehat{G}} \theta(t) \mathrm{Tr}\left(\sigma(x,\lambda)\Gamma_t(x,\lambda)\right) d\gamma_t(x,\lambda) dt.$$

Remark:

1. $\Gamma_t d\gamma_t$ has two parts:
   - a non commutative part described by an operator valued measure on $G \times \mathfrak{z}^* \setminus \{0\}$ $(\mathcal{H}_\lambda = L^2(\mathfrak{p}_\lambda))$.
   - an Euclidean part described by a (scalar) positive Radon measure on $\mathfrak{v}^*$ $(\mathcal{H}_\lambda = \mathbb{C})$.

Existence and structure of semi-classical measures

Proposition

Let $(\psi^\varepsilon(t))$ be a bounded family in $L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))$. There exist $\varepsilon_k \underset{k\to+\infty}{\longrightarrow} 0$ and $t \mapsto \Gamma_t d\gamma_t \in L^\infty(\mathbb{R}, \mathcal{M}_1^+(G \times \widehat{G}))$ such that for all $\theta \in L^1(\mathbb{R})$ and $\sigma \in \mathcal{A}$,

$$\int \theta(t) \left(\mathrm{Op}_\varepsilon(\sigma)\psi^\varepsilon(t), \psi^\varepsilon(t)\right) dt \underset{\varepsilon_k\to 0}{\longrightarrow} \int_{\mathbb{R}\times G\times\widehat{G}} \theta(t)\mathrm{Tr}\left(\sigma(x,\lambda)\Gamma_t(x,\lambda)\right) d\gamma_t(x,\lambda)dt.$$

Remark:

2. **Link with the weak limit of the energy density**:
   If $\|(-\varepsilon^2\Delta_G)^{\frac{s}{2}}\psi_0^\varepsilon\|_{L^2(G)} \leq C$, then $\forall \phi \in \mathcal{C}_c^\infty(G)$, $\theta \in \mathcal{C}(\mathbb{R})$,

$$\limsup_{k\to+\infty} \int \theta(t) \int_G \phi(x)|\psi^{\varepsilon_k}(t,x)|^2 dxdt = \int_{\mathbb{R}\times G\times\widehat{G}} \theta(t)\phi(x)\mathrm{Tr}\left(\Gamma(x,\lambda)\right) d\gamma(x,\lambda)$$

A few words about the proof...

- **Difficulty** : No Garding's inequality...
  $\Longrightarrow$ Use a "C^*-algebra" approach.
- **The key**: Consider the C^*-algebra $\mathcal{A}$ obtained by completion of $\mathcal{A}_0$ with

$$\sigma \mapsto \sup_{(x,\pi)\in G\times\hat{G}} \|\sigma(x,\pi)\|_{\mathcal{L}(\mathcal{H}_\pi)}.$$

  The set of the states of $\mathcal{A}$ coincide with $\mathcal{M}_1^+(G\times\hat{G})$.
- **The arguments**: The quantities

$$\ell_\varepsilon(\sigma) = (\mathrm{Op}_\varepsilon(\sigma)\psi^\varepsilon, \psi^\varepsilon)$$

  satisfy:
  1. For any $\sigma\in\mathcal{A}_0$, $\ell_\varepsilon(\sigma)$ is bounded and there exists $(\varepsilon_k(\sigma))_{k\in\mathbb{N}}$ such that $\ell_{\varepsilon_k(\sigma)}(\theta,\sigma)$ has a limit $\ell(\sigma)$.
  2. Using the separability of $\mathcal{A}_0$ and a diagonal extraction, one finds $(\varepsilon_k)_{k\in\mathbb{N}}$ such that for all $\sigma\in\mathcal{A}_0$, $(\ell_{\varepsilon_k}(\sigma))_{k\in\mathbb{N}}$ has a limit $\ell(\sigma)$.
  3. The map $\sigma\mapsto\ell(\sigma)$ extends to $\mathcal{A}$ and is a state of $\mathcal{A}$ ($\ell(\sigma^*\sigma)\geq 0$).

Semi-classical measures and Schrödinger equation

Consider solutions of the Schrödinger equation

$$i\varepsilon^\tau \partial_t \psi^\varepsilon = -\varepsilon^2 \Delta_G \psi^\varepsilon, \;\; \psi^\varepsilon(0) = \psi_0^\varepsilon.$$

Let $\Gamma_t d\gamma_t$ be the semi-classical measure of a family $\psi^\varepsilon(t)$.

- The non-commutative part

Theorem (FK & Fischer 2019)

(i) *For $d\gamma_t$-a. a. $(x,\lambda) \in G \times \widehat{G}$ and a.a. $t \in \mathbb{R}$,*

$$\Gamma_t(x,\lambda) = \sum_{n\in\mathbb{N}} \Gamma_{n,t}(x,\lambda) \;\; \textit{with} \;\; \Gamma_{n,t}(x,\lambda) := \Pi_n \Gamma_t(x,\lambda) \Pi_n,$$

where Π_n are spectral projections of $H(\lambda)$ for the eigenvalues $|\lambda|(2n+d)$.

(ii)
- *if $\tau \in (0,2)$, $\partial_t \left(\Gamma_{n,t}(x,\lambda) d\gamma_t\right) = 0$,*
- *if $\tau = 2$, $\Gamma_{n,t}(x,\lambda) d\gamma_t(x,\lambda)$ satisfies $\left(\partial_t - \frac{2n+d}{2|\lambda|} \mathcal{Z}^{(\lambda)}\right) \left(\Gamma_{n,t}(x,\lambda) d\gamma_t\right) = 0$*
- *if $\tau > 2$, $\mathcal{Z}^{(\lambda)} \left(\Gamma_{n,t}(x,\lambda) d\gamma_t\right) = 0$ (invariance).*

$\mathcal{Z}^{(\lambda)} \in \mathfrak{z}$ is the vector corresponding to $\lambda \in \mathfrak{z}^*$.

- The non-commutative part

  (...)

- The Euclidean part

Theorem (FK & Fischer 2019)

(iii) *Above* $\lambda = 0$, *set* $d\varsigma_t(x,\omega) = \Gamma_t(x,(0,\omega))d\gamma_t(x,(0,\omega))$

- *if* $\tau \in (0,1)$, $\partial_t \varsigma_t(x,(0,\omega)) = 0$,
- *if* $\tau = 1$, $(\partial_t - \omega \cdot V)\varsigma_t(x,\omega) = 0$ *(transport)*,
- *if* $\tau > 1$, $\omega \cdot V\varsigma_t = 0$ *(invariance)*.

Proof: Same strategy than the proof of Egorov Theorem + limit $\varepsilon \to 0$ + identification of each part of the measure.

Conclusion

- We have extended the microlocal/semiclassical approach to the setting of graded Lie groups.
- In the non-semiclassical framework, one obtains (without pain...) compensated compactness theorems ([Baldi, Franchi, 13], [Baldi, Franchi, Tchou & Tesi 10], [FK & Fischer 18])
- Application to the analysis of eigenfunctions of (complicated) sub-Laplacians ?

Thank you for your attention !