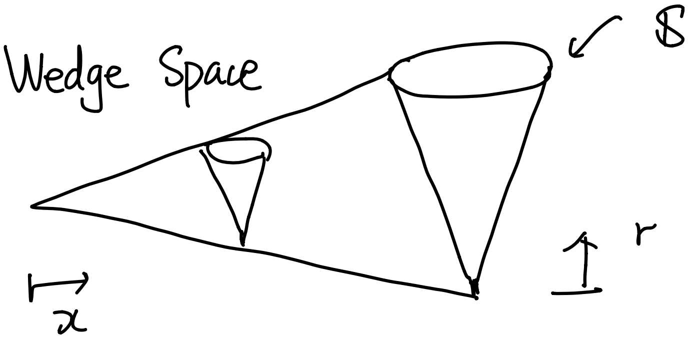
w/ P Albin P. Piazza

(X, g) (Riem) Wedge Space



$[0,1)_x \times [0,1)_r \times S^1$

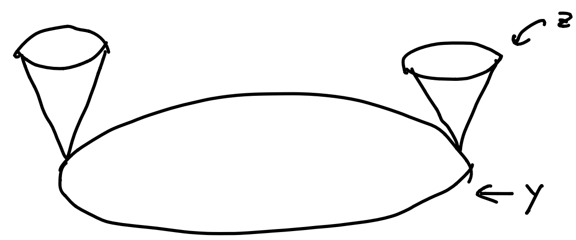
$dx^2 + x^2(dr^2 + \alpha^2 r^2 d\theta^2) \qquad \alpha > 0$

Simple wedge

(X, g)

$\uparrow$

manifold with boundary, $Z - \partial X \xrightarrow{\varphi} Y$



Let x bddary define function for X

g is wedge if $\exists x$ bdf

$$g_w = g = dx^2 + x^2 g_{\partial X/Y} + \varphi_Y^* g_Y + \underbrace{O(x)}_{\text{error term}}$$

g_Y Riem metric on fiber Y

$g_{\partial X/Y}$ restricts to fibers of φ_Y ... Riem

y_i, z_i

$\downarrow \quad \downarrow$

$Y \quad Z$

base fiber

dy, $x\,dz_\alpha$, dy_i $\leftarrow$ bounded length

1 replaced by 2, it's called totally geod wedge metric.

Dirac-type operators:

e.g. $\quad d+d^* = (\varepsilon(e^i) - c(e^i))\nabla_{e_i}$ o.n.f.

$$\varepsilon(e^i) = e^i \wedge, \qquad c = \varepsilon^*$$

$$\varepsilon - c : Cl(T^*X^\circ, g) \longrightarrow \mathrm{End}(\Lambda^* X^\circ)$$

$$\alpha\beta + \beta\alpha = -2\langle\alpha,\beta\rangle.$$

$$(E \to X, {}^E\nabla, cl)$$

$$\cdot\ cl : Cl_w(X, g) \longrightarrow \mathrm{End}(E)$$

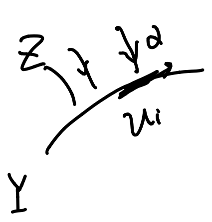
$$\shortparallel$$

$$Cl({}^wT^*X, g)$$

compatibility

$$D = \mathrm{Tr}\ cl \circ \nabla$$

near ∂X



$\partial_x, \ \frac{1}{x}\partial_\alpha, \ u_i$

o.n. frame $\ {}^wTX$

$$D = cl(dx)\nabla_{\partial_x} + cl(xV^\alpha)\nabla_{\frac{1}{x}V_\alpha} + cl(u^i)\nabla_{u_i}$$

$$\eth = x^{\dim Z/2} \, D \, x^{-\dim Z/2}$$

$$= \_\_\_\_ + \frac{1}{x} D_{\partial X/Y} + \_\_\_\_$$

fiber operator

Boundary family

$$x \eth |_{\partial X} = D_{\partial X/Y}$$

Thm: (Albin - G.R. '17)

Assume D has invertible boundary family, let

$$\mathcal{D}_{VAPS} = \overline{x^{\frac{1}{2}} H^1_e \cap \mathcal{D}_{max}}^{\Gamma}$$

then $\eth : \mathcal{D}_{VAPS}(X, E) \to L^2(X, E)$ is Fredholm & self-adjoint

$$\cap$$

$$L^2(X, E) \qquad\qquad E = E_+ \oplus E_-$$

$$\operatorname{ind}(\eth : \mathcal{D}_{VAPS}(X, E_+) \longrightarrow L^2(X, E_-)) = \operatorname{Str} e^{-t\eth^2}$$

$$= \int_X AS_g + \int_Y \hat{A}(Y) \eta(\partial X/Y)$$

$$H^1_e = H^1_e(\nu = dx\,dy\,dz) \ni x^{-\frac{1}{2}+0}$$

$u \in L^2$ and $(x\partial_x, x\partial_y, \partial_z) u \in L^2$ $\qquad (x\partial_x, x\partial_y, \partial_z) u \in L^2$

$$x^{-\frac{1}{2}+0} \in H^e_1$$

Cone : $-cl(dx)\eth = \partial_x - \frac{cl(dx) D_Z}{x}$

Signature Case:

$X = X^{4\ell}$

$i^{\partial}_{*}\, \eth = \pm \eth$

$D_{sig} = d + d^*: C^{\infty}_{comp}(X, \Lambda_+) \to C^{\infty}_{comp}(X, \Lambda_-)$

X closed $\Rightarrow \operatorname{ind}(D_{sig}) = \tau(X) = \int \mathcal{L}$

(X, g) wedge $d+d^*$ $D_{\partial X/Y} = \begin{pmatrix} \eth^{Z_y}_{sig} & N - f/2 \\ N - f/2 & -\eth^{Z_y}_{sig} \end{pmatrix}$ $\quad f = \dim Z$

$\ker(D_{\partial X/Y}) \longleftrightarrow \ker(\eth^{Z_y}_{sig} \cdot \Omega^{f/2}(Z) \to)$

$\| $

$H^{f/2}_{dR}(Z)$

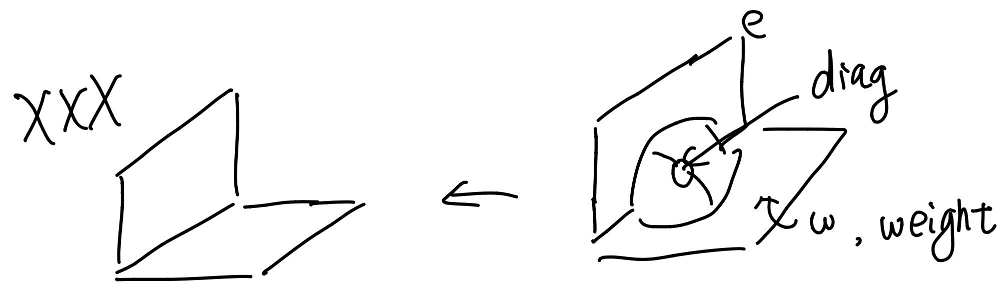
$H^{f/2}_{dR}(Z) \neq \{0\}$ studied by

Albin – Leichtnam – Mazzeo – Piazza

$W \subseteq (\mathcal{H}^{f/2} \to Y)$

$W = *W^{\perp} \rightsquigarrow D_W$

Parametrix Ψ_e^m - edge calculus



$$X_e^2 = [X \times X, \partial Y \times_y \partial X]$$

$$\eth G = Id - R$$

$$\| $$

$$\eth_x (x^{-1} G)$$

$$G \in x \Psi_e^{-1, \omega}$$

$$N_Y(\eth) = cl(dx)\partial_s + \frac{1}{s} D_{Z,y} + i\, cl(\eta)$$

Use Melrose-Piazza

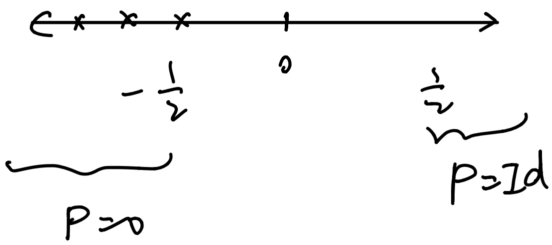
$\beta_y = [cl(dx) D_{\partial X/y}] \in K(Y)$ (commutes with $cl(\eta)$)

MP: If $\beta_Y = 0$, then there exists spectral section for $D_{\partial X/Y}$

Thm: $\beta_Y = 0$

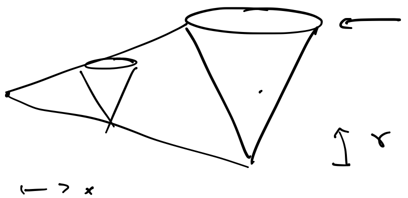
$$\left\{ \begin{array}{l} \text{graded } Cl(T^*Y) \\ \text{spec sections} \\ P \text{ of } D_{\partial X/y} \\ \text{of width } 1/2 \end{array} \right\} \to \left\{ \begin{array}{l} D_P \subseteq L^2 \text{ S.A.} \\ \text{Fredholm domains} \\ \text{of } \eth \end{array} \right\}$$

$P^2 = P$



Analysis on Singular Spaces
(w/ P Albin, P. Piazza)

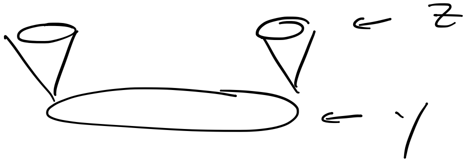
(X, g) (Riem) wedge space



$[0,1)_x \times [0,1)_r \times S^1$

$dx^2 + x^2(dr^2 + \alpha^2 r^2 d\theta^2)$ $\alpha > 0$

simple wedge



X - mwb

$Z - \partial X \xrightarrow{\varphi_Y} Y$

$\exists$ x - bdf

$$g_w = g = dx^2 + x^2 g_{\partial X/Y} + \varphi_Y^* g_Y + O(x^1)$$

g_Y Riem on fib,

$g_{\partial X/Y}$ Riem on fibers of φ_Y

bounded forms: y_i, z_α coords

$y_i \to$ base, $z_\alpha \to$ fib

dx, $x\,dz_\alpha$, dy_i

$1 \mapsto 2$ "totally geod. wedge mf"

Dirac-type operators:

e.g. $d + d^* = \sum_i (\varepsilon(e^i) - \iota(e^i)) \nabla_{e_i}$

ε = ext. mult

ι = ε^* w.r.t. g

$$\varepsilon - \iota : \underbrace{Cl(T^*X, g)}_{\alpha\beta + \beta\alpha = -2\langle \alpha, \beta \rangle g} \longrightarrow End(\Lambda^* X^\circ)$$

$$\left(E \to X,\ {}^{E}\nabla,\ cl\right)$$

- $cl : \mathbb{C}l_w(X, g) \longrightarrow End(E)$

$$\parallel$$

$$\mathbb{C} \otimes Cl({}^{w}\underline{T}^{*}X, g)$$

$$\swarrow$$

$$\langle dx, dy_i, x dz_i \rangle_{C^\infty(X)} \overset{\text{loc over } Y}{=} \Gamma({}^{w}T^{*}X)$$

- $cl(\theta)\nabla_x - \nabla_x cl(\theta) = cl(\nabla_x \theta)$

- $D = \mathrm{Tr}\; cl \circ {}^{E}\nabla$

$\alpha \in C^\infty(X; {}^{w}T^{*}X)$ iff $\alpha|_{\mathcal{B}_Y}(V) = 0$ provided $(\varphi_Y)_* V = 0$.

near ∂X,

$$D = cl(dx)\nabla_{\partial_x} + cl(xV^\alpha)\nabla_{\frac{1}{x}V_\alpha} + cl(u^i)\nabla_{u_i}$$

$$\eth = x^{\dim Z/2}\, D\, x^{-\dim Z/2}$$

$$= \underline{\qquad\qquad} + \frac{1}{x} D_{\partial X/Y} + \underline{\qquad\qquad}$$

$$x\eth\big|_{x=0} = D_{\partial X/Y}$$ "boundary family"

Thm (Albin–G.R. '17) Assume that $\eth$ has invertible boundary family, let

$$\mathcal{D}_{VAPS} = \overline{x^{\frac{1}{2}} H^1_e(X;E) \cap \mathcal{D}_{max}}^{\Gamma}$$

Then $\eth : \mathcal{D}_{VAPS} \to L^2$

is Fredholm & self-adjoint

$$E = E_+ \oplus E_-$$

$$\operatorname{ind}(\eth : \mathcal{D}_{VAPS}(X;E_+) \to L^2(X;E_-))$$

$$= \operatorname{Str} e^{-t\eth^2}$$

$$= \int \hat{A}(X,g)\, ch'(E) + \int \hat{A}(Y)\, \eta(\partial X/Y)$$

$$H^1_e = H^1_e(\nu = dx\,dy\,dz)$$

$\cup$

$u \in L^2 \qquad (x\partial_x, x\partial_y, \partial_z)\, u \in L^2$

$$x^{-\frac{1}{2}+0} \in H^1_e$$

<u>Cone:</u> $\qquad -cl(dx)\eth = \partial_x - \dfrac{(cl(dx)D_z)}{x}$ $\quad \overset{\text{s.a.}}{=: A}$

$$\eth(x^\lambda \varphi) = 0 \qquad A\varphi = \lambda\varphi.$$

$$\eta(\partial X/Y) = \tilde{\eta}(\partial X/Y) + \int_{\partial X/Y} TA(\partial X/Y)\, ch'(E)$$

e.g. $X = X^{4\ell}$

$$D_{sig} = d + d^* : C^\infty_{comp}(X^{4\ell}, \Lambda_+) \to C^\infty_{comp}(X^{4\ell}, \Lambda_-)$$

$\alpha \in \Lambda_\pm$, $\quad i^p * \alpha = \pm \alpha$

X closed $\Rightarrow$

$$\operatorname{ind}(D_{sig}) = \tau(X) = \int L.$$

(X, g) wedge $d + d^*$

$$D_{\partial X / Y} = \begin{pmatrix} \eth^Z_{dR} & \mathbb{1}N - f/2 \\ \mathbb{1}N - f/2 & -\eth^Z_{dR} \end{pmatrix}$$

$f = \dim Z$

$$\ker D_{\partial X / Y} = \ker \left(\eth^Z_{dR} \Big|_{\Omega^{f/2}(Z)} \right)$$

$$= H^{f/2}_{dR}(Z)$$

$D^{-1}_{\partial X / Y}$ exists $\Longleftrightarrow$ $H^{f/2}_{dR}(Z) = \{0\}$

$H^{\ell/2}_{dR}(Z) \neq \{0\}$ studied by

Albin-Leichtnam-Mazzeo-Piazza

$W \subseteq (\mathcal{H}^{\ell/2}_{dR} \longrightarrow Y)$

$\hookrightarrow$ vert. harm. forms

$W = * W^{\perp}$

$\mathcal{D}_W$ Fred s.A. D_{sig}

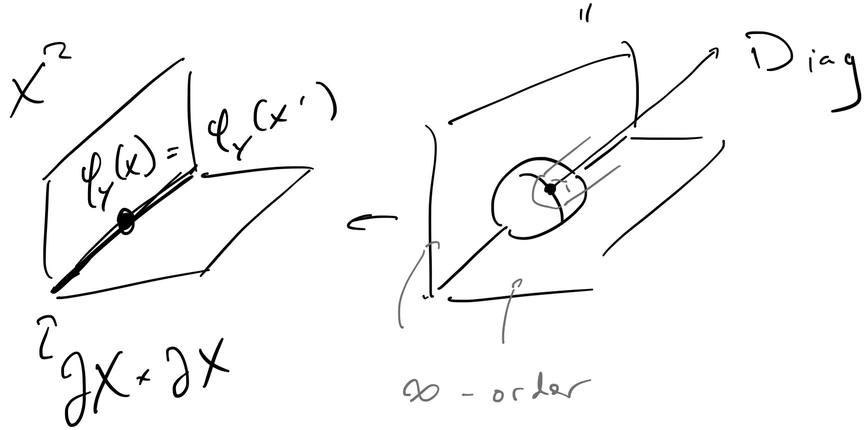
seek conditions on $\{ D_{\partial x/Y} \}$

assuring existence of Fred domain.

Parametrix: $\quad \times\ \eth \in \mathrm{Diff}^1_e(X;E)$

Ψ^m_e - edge ΨDO's

$$X^2_e = [X \times X; \partial X \times_Y \partial X]$$



$$A \in \Psi^m_e = {}^{\infty}_{\int sf} I^m(X^2_e, \mathrm{diag})$$

$$\eth G = I - R$$

$$\eth \times (\Psi^{-1} G)$$

$\exists \; G \in x^{-1,\infty} \Psi_e$, calculus w/ bounds.

$G, R : L^2 \to x^{1/2} H^1_e$

$$\mathcal{N}_y(\gamma) = d(dx)\partial_s + \frac{1}{s} D_{Z,y} + D_{\mathbb{R}^n}$$

on $\mathbb{R}_+ \times Z \times \mathbb{R}^{n = \dim Y}$

$s = x/x'$, $\frac{y-y'}{x'}$ on $\mathbb{R}^n$

$$\widehat{\mathcal{N}}_y(\gamma)(\eta) = d(dx)\partial_s + \frac{1}{s} D_{Z,y} + i\,d(\eta)$$

$$\hat{\mathcal{N}}_y(\gamma)(\eta) = \mathcal{C}\ell(dx)\,\partial_s + \frac{1}{s}\underbrace{D_{Z,y}}_{\mathcal{C}\ell(xV^\alpha)\nabla_{V_\alpha}} + i\,\mathcal{C}\ell(\eta)$$

(This is a family of conic op's, to study we use Melrose-Piazza's approach to families index for b-op's, spec. relate families index of b-fam to existence of spec. sections)

$$\beta_y = \Big[\underbrace{\mathcal{C}\ell(dx)\,D_{\partial X/y}}_{\text{graded, odd } \mathbb{C}\ell(T^*Y)\text{-inv.}}\Big] \in K^{\mathcal{C}\ell(T^*Y)}(Y)$$

(it's the vanishing of this invariant that gives a Fred domain by making a param const. possib., that's the new thm.

<u>Thm</u> (Albin - G.R. - Piazza '19)

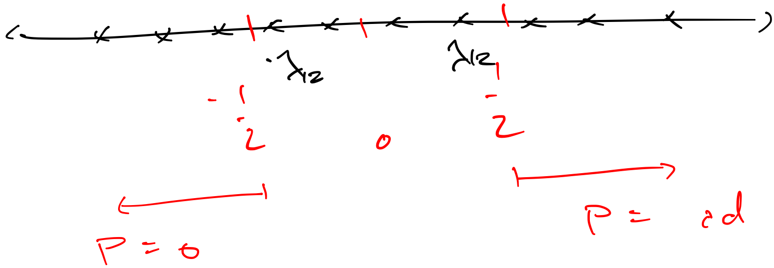
Assume $\beta_Y = 0$.

$$\left\{ \begin{array}{l} \text{graded } \mathbb{C}\ell(T^*Y)\text{-} \\ \text{spectral sections} \\ P \text{ of } D_{\partial X/Y} \\ \text{of width } 1/2 \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \mathcal{D}_P \subseteq L^2 \text{ S.A.} \\ \text{Fredholm domains} \\ \text{of } \eth \end{array} \right\}$$

<u>MP</u>: $\beta_Y = 0 \implies$ exists P.

$$P_y : L^2(Z_y ; E), \quad P_y^2 = P_y$$

$$\mathrm{spec}\,\left(\eth(d_x) D_{Z,y}\big|_{E_y}\right)$$



$$u \in \mathcal{D}_{max}\left(\widehat{N}(\gamma)(\xi)\right)$$

$$\bar{u} = (u_{\lambda_{12}}, \dots, u_{\lambda_1}, u_0, u_{-\lambda_1}, \dots, u_{-\lambda_{12}})$$

$P\bar{u} = 0$ $\quad$ $u \in \ker$

S.A.

$\beta_y = 0 \iff \exists\, \mathrm{C}\ell(T^*Y)$-inv

$Q \in \Psi^{-\infty}(\partial X/Y; E)$

$D_{\partial X/Y} + Q$ <u>invertible</u>

$\rightsquigarrow D + Q$ on X

w/ inv. bdry fam's.