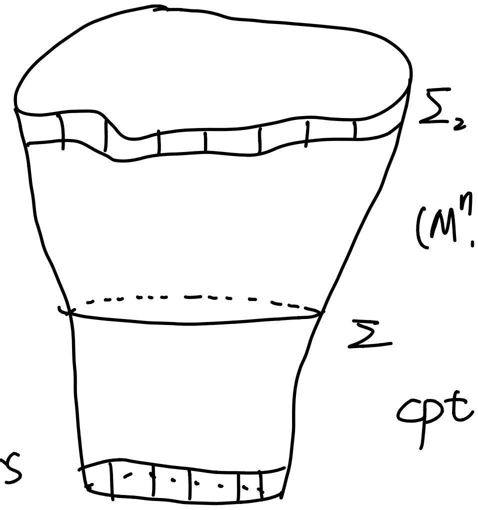
Local Index Theory

(Joint work with C. Bär)



Σ Cauchy

n even

$S_L M$ bundle of lefthanded spinors twisted by E

$\not{D}_L : C^\infty(M, S_L M) \to C^\infty(M, S_R M)$

$\not{D}_R : C^\infty(M, S_R M) \to C^\infty(M, S_L M)$

A_i : Dirac operator on $S\Sigma_i = S_L M|_\Sigma$

APS - Boundary Condiction:

$\phi|_{\Sigma_1}$ in positive spectral subspace for A_1

$\phi|_{\Sigma_2}$ in neg spectral subspace of A_2

Thm (A.S. Bar 2015)

$\not{D}_L$ is Fredholm with APS $\underset{\text{boundary condition}}{\text{b.c}}$

$$\mathrm{ind}(\not{D}_L) = \int \hat{A}(\Omega) \wedge ch(E) - \frac{h(A_1) - h(A_2)}{2} - \frac{\eta(A_1) - \eta(A_2)}{2}$$

$h(A_i) = \dim\ker A_i$, $\eta(A_i) = \eta_{A_i}(0)$; $\eta_{A_i}(s) = \sum \mathrm{sgn}(\mu_i)|\mu_i|^{-s}$

Proved by Matoni, Bunke - Hirschmann (g trivial)

Broverman - Collios selling

Q: time evalution $L^2(\Sigma_1) \to L^2(\Sigma_2)$

$$Q = \begin{pmatrix} Q_{++} & Q_{+-} \\ Q_{-+} & Q_{--} \end{pmatrix}$$

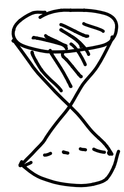
$$\mathrm{ind}(\not{D}) = \mathrm{ind}(Q_{--}) = \dim\ker Q_{--} - \dim\ker Q_{++}$$

Riesz distribution

$\mathbb{R}^n$; $\qquad -\gamma(x) = -x_1^2 + x_2^2 + \cdots + x_n^2$

$$R_s^{\pm} = 2C(s)\,(\gamma(x))_+^{s}\,\chi_{[0,+\infty)}(\pm x_1)$$

$$C(s) = \frac{2^{-n-2s}\,\pi^{\frac{2-n}{2}}}{\left(s+\frac{n-2}{2}\right)!\,s!}$$



wave eqn

$$\Box R^{+}_{-\frac{n}{2}+1} = \delta_0$$

On M: $\mathcal{U}$ nbhd of $\Delta \subset M \times M$

Γ geodesic. $R_s^{\pm \mathcal{U}}$ pull back of $R_s^{\pm}$

(a) $\overset{G_s}{\Gamma R_s^{\pm \mathcal{U}}} = (2s+2)(2s+n) \overset{G_s}{R_{s+1}^{\pm \mathcal{U}}} + \frac{i}{\pi}(8s+n+4) F_{s+1}^{\mathcal{U}}$

(b) $\overset{G_s^{\mathcal{U}}}{(\operatorname{grad}\Gamma) R_{s+1}^{\pm \mathcal{U}}} = 2(2s+n) \operatorname{grad} \overset{G_s^{\mathcal{U}}}{R_{s+1}^{\pm \mathcal{U}}} + \frac{4i}{\pi} \operatorname{grad} F_{s+1}^{\mathcal{U}}$

(c) $\square R_{s+1}^{\pm \mathcal{U}} = \left(\frac{\square \Gamma - 2n}{2n+4s} + 1 \right) R_s^{\pm \mathcal{U}} + \frac{i}{\pi} \left(\frac{\square \Gamma - 2n}{(n+2s)^2} \right) F_s^{\mathcal{U}}$

(d) $R_{-n/2}^{\pm \mathcal{U}} = \delta$

Hördomord parametrix

$$G_{ret}(x, y) = \sum_{k=0}^{\infty} V_k(x, y) R^{\pm \mathcal{U}}_{-\frac{n}{2}+1+k}$$

$$\nabla_{\operatorname{grad}\Gamma} V_k - \left(\tfrac{1}{2} \square \Gamma - n + 2k\right) V_k = 2k \square V_{k-1}$$

$$V_0(x, x) = \mathrm{id}.$$

Feynman parametrix

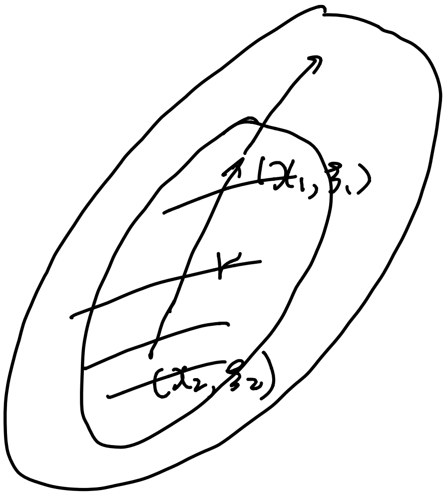
Duistermaut-Hörmander: G Feynmann parametrix

$$\square G = \mathrm{id} \mod C^\infty$$

$$WF'(G) \subset \Delta^* \cup \{ (x_1, \xi_1, x_2, \xi_2) \in T^*(M \times M) \setminus 0 \mid$$

$$\xi \text{ lightlike}, \exists t > 0, \Phi_t(x_1, \xi_1) = (x_2, \xi_2) \}$$

Φ_t geodesic flow



G left parametrix

$\tilde{G}$ right parametrix

$$G[\Box, \chi]\tilde{G} = \underbrace{\chi\tilde{G} - G\chi}_{(x_1,\xi_1,x_2,\xi_2)\notin WF}$$

$$F_s = \lim_{\varepsilon\to 0+} C(s)\,(\gamma(x) + i\varepsilon)^s$$

$$\Downarrow$$

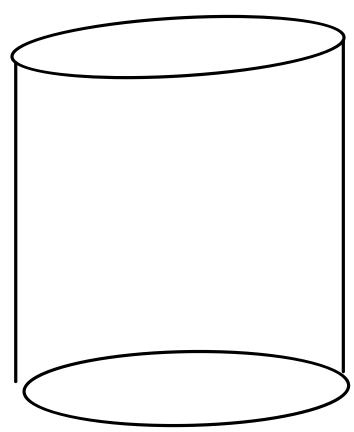
$$F_s^x$$

In odd dimension Feynman parametrix

In even dimension $G_s = \frac{i}{\pi}\frac{d}{ds}F_s + F_s$

Product case

$$M = \mathbb{R}\times\Sigma$$



Σ

$$\Box = \frac{\partial^2}{\partial t^2} + \Delta$$

$$G_F = \frac{i}{2}\Delta^{-\frac{1}{2}} e^{-i|t-t'|\Delta^{\frac{1}{2}}} \quad \text{if } \ker\Delta = \{0\}$$

$$g = -dt^2 + h$$

$\Box = \not{D}_R \not{D}_L \qquad \Delta = A^2$

$$D_F = (\chi_{[0,\infty)}(t) P_{\geq} - \chi_{(-\infty,+\infty)} P_<) e^{-it\not{D}_\Sigma}$$

Thm: (A.S. Bär)

$G^{reg} = G_F - G^{loc}$ where G^{loc} – local Feynmann parametrix

$x = (t, y)$

Then $G^{reg}(x.x) = \frac{1}{2}(h_y + \eta_y)$

h_y = diagonal of kernel of P_0

$\eta_y = \lim_{t \to 0+} \mathrm{Tr}_y(\mathrm{sgn}(A) e^{-tA^2})$

T operator with smooth kernel

$[T]$ restriction of kernel T to Δ

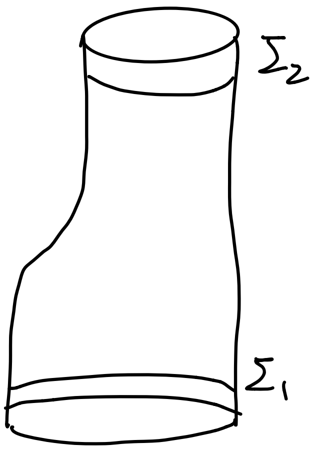
Thm: (A.S. Bär)

Hodamard coeff of $\not{D}_L \not{D}_R \oplus \not{D}_R \not{D}_L$

$[\Box G^{loc}_{Feyn} - id] = 0$

$$\mathrm{Tr}[\not{D} G^{loc}_{Feyn} \not{D}_R - id] = \frac{1}{(4\pi)^{\frac{n}{2}} (\frac{n}{2})!} \mathrm{tr}_s (V_{\frac{n}{2}})$$

$$= \hat{A}(\Omega) \wedge ch(E)(x)$$



$G_{1,\mathrm{Feyn}}$, $G_{2,\mathrm{Feyn}}$ fund sol assoc Σ_1, Σ_2

$$\mathrm{tr}(\xi[\not{D}_L(G_{1,\mathrm{Feyn}} - G_{2,\mathrm{Feyn}})](x)) = J(\xi)$$

$$\delta J = 0;$$

Thm: $\mathrm{ind}(\not{D}_L) = \int_\Sigma *J$

$$\mathrm{index}(\not{D}_L) = \int_{\Sigma_2} *(J^2 - J^1) = \int_{\Sigma_2} *J^2 - \int_{\Sigma_1} *J^1 - \int_M \delta J^2$$

where $J^i = \mathrm{tr}[\xi \not{D}_L G^{\mathrm{reg}}_{i,\mathrm{Feyn}}](x))$

$$= \int \hat{A}(\Omega) \wedge \mathrm{ch}(E) - \frac{h(A_1) - h(A_2) + \eta(A_1) - \eta(A_2)}{2}$$