Recent progress on the Fried conjecture.

Workshop *Recent developments in microlocal analysis*, MSRI

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Motivation.

Algebra	Topology	Dynamics
$\dim(V)$	Euler $\chi(V,d)$	zeros vector fields $\sum_{c\in Crit(V)}(-1)^{\text{ind}_V(c)}$
trace(T)	Lefschetz $\mathcal{L}(T)$ $\sum_{i=0}^{\dim(M)}(-1)^i Tr(T\|_{H^i(M)})$	fixed points of maps $\sum_{x=T(x)}\text{ind}_T(x)$
determinant	Torsion τ	periodic orbits flows $\prod_{\gamma\in\text{ prime}}\det\left(Id-\rho(\gamma)\Delta(\gamma)\right)^{(-1)^{\text{ind}(\gamma)}}$

Geometric context.

1. (M, θ), $\dim(M) = 2d + 1$, θ contact 1-form : $\theta \wedge d\theta^{\wedge d}$ **volume form**. Ex :$S^*\mathcal{M}$.

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2. X Reeb field, $\theta(X)=1$. Assume X **Anosov** i.e. $TM=E_s\oplus E_u\oplus\langle X\rangle$, (E_s,E_u) called stable, unstable bundles $\exists C,\lambda>0$ s.t. $\forall t\geqslant 0$:

$$\|de^{tX}(v)\|\leqslant Ce^{-\lambda t}\|v\|,\forall v\in E_s,\ \ \|de^{-tX}(v)\|\leqslant Ce^{-\lambda t}\|v\|,\forall v\in E_u.$$

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3. Representation $\rho=e^{\langle\alpha,.\rangle}:\pi_1(M)\mapsto\mathbb{C}^*$, $[\alpha]\in H^1(M,\mathbb{R})$.
   α a closed 1-form, then $\rho(\gamma)=\exp\left(\int_\gamma\alpha\right)$ is a **character** on $\pi_1(M)$:
   $\rho(\gamma_1+\gamma_2)=\exp\left(\int_{\gamma_1\circ\gamma_2}\alpha\right)=\exp(\int_{\gamma_1}\alpha)\exp(\int_{\gamma_2}\alpha)=\rho(\gamma_1)\rho(\gamma_2)$ hence $\rho:\pi_1(M)\mapsto\mathbb{C}^*$.

Main object : twisted Ruelle zeta

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Using (X, ρ), we can form the twisted Ruelle zeta function (dynamical L functions)

$$\boxed{\zeta_{X,\rho}(s) = \prod_{\gamma \in \mathcal{P}} \left(1 - \rho(\gamma) e^{-s\ell(\gamma)}\right)}$$

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Example

On $\mathbb{S}^1$ *of length* ℓ*, flow* ∂_θ*,* u *generator of* $\pi_1(M)$*,* **monodromy** $\rho(u) \in \mathbb{C}^*$,
$\boxed{\zeta_{X,\rho}(s) = (1-\rho(u)e^{-s\ell})}$.

Some questions on $\zeta_{X,\rho}$.

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Both problems deeply related.

Topological content of ζ.

Simple to state :

Theorem (Dyatlov–Zworski, Hadfield with boundary)

For a surface $\mathcal{M}$ of variable negative curvature, X generates the geodesic flow on $S^*\mathcal{M}$ then :

$$\zeta_{X,Id}(s) = \prod_{\gamma}(1 - e^{-s\ell(\gamma)}) = s^{2g-2}\left(c + \mathcal{O}(s)\right) \quad (1)$$

g **genus** *of $\mathcal{M}$. In particular, the* **length spectrum determines the genus**.

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Dynamical meaning of zeroes and poles of $\zeta_{X,\rho}$?

Zeroes and poles of $\zeta_{X,\rho}$ as Pollicott–Ruelle resonances.

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Other method : poles of $z \mapsto \int_0^\infty e^{-ts} \left\langle \Psi_2, e^{-tA}\Psi_1 \right\rangle dt = \sum_{\lambda\in\sigma(A)} \frac{1}{\lambda+s} \left\langle \Psi_2, \Pi_\lambda(\Psi_1) \right\rangle$ for all Ψ_1, Ψ_2 test vectors and Π_λ projector on eigenspaces.

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Zeroes and poles of $\zeta_{X,\rho}$ have **deep dynamical meaning** as Pollicott–Ruelle resonances.

Definition (Dynamical correlators)

Let Ψ_1, Ψ_2 *two test forms,* $u(t) = e^{-tX*}\Psi_1$ *solves* **transport equation by Anosov flow** $\boxed{\partial_t u(t) + \mathcal{L}_X u(t) = 0}$ *with Cauchy data* $u(0) = \Psi_1$.

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$$C(\Psi_1, \Psi_2, t) = \int_M \Psi_2 \wedge \left(e^{-tX*}\Psi_1\right) \qquad (2)$$

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Definition (Pollicott–Ruelle resonances)

Poles of the Laplace transformed correlators

$$\mathcal{L}C(\Psi_1, \Psi_2, .)(s) = \int_0^\infty e^{-st} C(\Psi_1, \Psi_2, t)dt.$$

Capture long time behaviour of the dynamics.

Transport equation with potential.

Cheated in previous slides, implement representation ρ!

A representation $\boxed{\rho = e^{\langle \alpha, . \rangle} : \gamma \in \pi_1(M) \mapsto e^{\int_\gamma \alpha} \in \mathbb{C}^* \Leftrightarrow M \times \mathbb{C} \mapsto M}$ with flat connection $\nabla = d + \alpha$, α **closed** 1-form.
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Zeroes, poles of $\zeta_{X,\rho}(s)$ related to **asymptotics** of $u(t,.)$ sol. of **transport equation** $\boxed{\partial_t u + \left(\mathcal{L}_X + \underbrace{\alpha(X)}_{\text{potential}}\right) u = 0.}$ Solution $\boxed{\underbrace{e^{-\int_0^t \alpha(X)(e^{(s-t)X}(x))ds}}_{\text{weight}} u_0(\underbrace{e^{-tX}(x)}_{\text{transport}}).}$

Topology in the kernel.

Theorem (D–Rivière)

For X Anosov or Morse–Smale, $C^k(0)$ currents of degree k s.t. $(\mathcal{L}_X + \alpha(X))^p u = 0$ for some $p \in \mathbb{N}$ and $WF(u) \subset \mathcal{D}'_{E_u^*}$.
$(C(0), d + \alpha)$ **chain complex** *is quasi–isomorphic to the De Rham complex.*
In particular, $\underbrace{\dim\left(C^k(0)\right)}_{\text{dim of kernel on } k\text{-forms}} \geqslant \underbrace{b_k}_{\text{Betti}}$.

Abstract torsion of chain complexes.

Example

$T : E \mapsto F$ isomorphism, corresponding complex $0 \mapsto E \mapsto F \mapsto 0$. How from T do we get **numbers** *?*

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In general, for an **acyclic** cochain complex $(C^\bullet, d)$

$$0 \mapsto C^0 \overset{d}{\mapsto} C^1 \mapsto \dots \overset{d}{\mapsto} C^N \mapsto 0$$

$d \circ d = 0$, $Im(d) = \ker(d)$, choosing a volume element $[b]$ in $C^\bullet$ associates a number

$$\boxed{\tau(C^\bullet, d) = \prod_{i=0}^{N-1} \left|\det{}_{C^i_{coex} \mapsto C^{i+1}_{ex}}(d)^{(-1)^i}\right|.} \quad (3)$$

Torsion of manifolds.

Recipe, on M with closed α, choose Morse function f. Morse complex generated by $\textbf{Crit}(f)$, twisted by $\rho = e^{\langle \alpha, . \rangle}$. Differential

$$\partial a = \sum_{\gamma : a \mapsto b} \pm \underbrace{e^{\int_\gamma \alpha}}_{\text{twisting}} b \tag{4}$$

sum runs over instantons connecting (a, b) s.t. $ind(b) = ind(a) + 1$.

Theorem

M C^∞ manifold, ρ **unitary** *reps s.t. twisted Morse complex $(C_f^\bullet, d_\rho)$* **acyclic***. Then $\tau_R(\rho) := \tau(C_f^\bullet, d_\rho)$ does not depend on f.* **Topological invariant** *of (M, ρ).*

Back to our friend $\mathbb{S}^1$.

Example

Acyclicity. *On* $\mathbb{S}^1$, $\alpha \in i\mathbb{R}$. *Differential* $d + \alpha d\theta$ *corresponding to* **unitary reps** $\rho : \gamma \mapsto e^{\int_\gamma \alpha d\theta} \in \mathbb{S}^1$.

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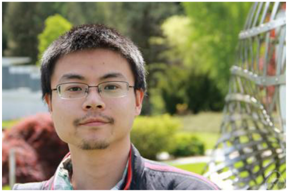
Then $\partial_\theta u + \alpha u = 0$ *with* $u(0) = u(1)$ *solution* $u(\theta) = u(0)e^{\alpha\theta}$. *But periodicity and* $e^\alpha \neq 1 \implies u = 0$.
Finally $\ker(\partial_\theta + \alpha) = \{0\} \implies$ **acyclicity** *of* $d + \alpha d\theta$.
Torsion. $\mathbb{S}^1$ *North South dynamics. Basis* (a, b). *Differential*

$$\partial a = e^{\frac{\alpha}{2}} b - e^{-\frac{\alpha}{2}} b = (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}})b \implies |\det(\partial)| = |1 - e^\alpha|.$$

$$\boxed{\tau_R(\rho) = |1 - e^\alpha| = |\zeta_{X,\rho}(0)|.} \tag{5}$$

Fried





The Fried conjecture.

Relate $|\zeta_{X,\rho}(0)|$ to $\tau_R(\rho)$.

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Theorem

1. *when* $M = S^*\mathcal{M}$ *for* **hyperbolic** $\mathcal{M}$, ρ **unitary**, *then Fried(1986) showed*
$$\tau_R(\rho) = |\zeta_{X,\rho}(0)|^{(-1)^{d-1}}. \tag{6}$$
2. *Extended to locally symmetric spaces by Moscovici–Stanton, Shen(2018)*
3. *Sanchez–Morgado(1996) for* X **analytic Anosov** *in* $3d$.
4. *D–Guillarmou–Rivière–Shen, if for some flat connection* ∇ *and Anosov* X_0*, we have* $\ker(X_0) = \{0\}$ *then*
$$\zeta_{X,\rho} = \zeta_{X_0,\rho} \tag{7}$$
for all X *near* X_0*. In particular, the Fried conjecture holds true for* X *Anosov in* $3d$ *if* $b_1(M) > 0$ *and in* $5d$ *near geodesic flows of hyperbolic manifolds.*

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Proposition (Lepage 1946)

isomorphisms $\mathrm{L}^k : \varphi \in \Omega^k(M) \cap \ker(\iota_X) \mapsto \varphi \wedge d\theta^k \in \Omega^{2d-k}(M) \cap \ker(\iota_X), \forall k \leqslant d$

Definition

Every k-form $\varphi = f \wedge \theta + g, (f,g) \in \ker(\iota_X)$ *and chirality* Γ *unique involution satisfying :*

$$\Gamma\varphi = \mathrm{L}^{d-k} g \wedge \theta + \mathrm{L}^{d-k+1} f, k \leqslant d. \tag{8}$$

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Key observation : if X is **contact Anosov**, **canonical involution** Γ on $C(0)$.

Yann Chaubet.

The main Theorem.

Proposition (Braverman–Kappeler)

Γ-*invariant basis* $[b]$ *of* $\ker_{gen}(X)$ *then* $\tau(\ker_{gen}(X), d+\alpha)$ **does not depend** *on* $[b]$, *only on* $\Gamma \implies$ *Intrinsic finite dim torsion* $\tau_\Gamma(X)$.

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Theorem (Chaubet–D)

$(\mathcal{M}, g)$ *hyperbolic of odd dimension* d*.* X_0 *generates geodesic flow on* $S^*\mathcal{M}$*. For any contact Anosov flow* X **path connected** *to* X_0 **among contact Anosov flows***,* ρ **acyclic unitary reps** *:*

$$\boxed{|\zeta_{X,\rho}(s)| = |s^m| \underbrace{\tau_R(\rho)}_{R-torsion} \left(|\frac{\tau_\Gamma(X_0)}{\tau_\Gamma(X)}| + O(s) \right)} \tag{9}$$

m depends on (X, ρ)*.*

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Fix ambiguities in τ_R, consider torsion as holomorphic functions of nonunitary reps.

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Acyclic *part* $Rep_0 = \mathbb{C}^* \setminus \{1\}$.

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On $\mathbb{S}^1$, representation variety $Rep = Hom(\pi_1(\mathbb{S}^1), \mathbb{C}^*) \simeq \mathbb{C}^*$.
Acyclic *part $Rep_0 = \mathbb{C}^* \setminus \{1\}$.*
Choose Euler structure $\mathfrak{e} \in \mathbb{Z}$, $u \in \mathbb{C}^* \setminus \{1\} \mapsto \tau_{\mathfrak{e}}(u) = u^{\mathfrak{e}}(1-u)$ is **holomorphic***.*
Observe that for $u \in \mathbb{S}^1 \setminus \{1\}$, acyclic unitary reps, $|\tau_{\mathfrak{e}}(u)| = |1-u| = \tau_R(u)$ hence $\tau_{\mathfrak{e}}$ **extends and refines** τ_R.

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$Rep = Hom(\pi_1(M), \mathbb{C}^*)$, $\rho \in Rep_0 \mapsto \tau_{\mathfrak{e}}(\rho) \in \mathbb{C}$ **holomorphic function** on the acyclic part $Rep_0 \subset Rep$.

Chern–Simons class.

Definition (Hutchings, Burghelea–Haller)

(X_0, X_1) pair of vector fields, $CS(X_0, X) \in H_1(M, \mathbb{Z})$ is a **defect measuring** *the obstruction to deform continuously X_0 into X_1.*

Example

Try to deform ∂_θ continuously to $-\partial_\theta$.

Second main Theorem

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Theorem (Chaubet–D)

X_0 contact Anosov. For every **connected** *open subsets $\mathcal{U} \subset Rep_0$ and $\mathcal{V} \subset \mathcal{A}$, $\exists C$ s.t. for every vector field $X \in \mathcal{V}$ and every $e^{\langle .,\alpha \rangle} \in \mathcal{U}$,*

$$\boxed{\zeta_{X,e^{\langle .,\alpha\rangle}}(s) = s^m C \underbrace{\tau_{\mathfrak{e}_X}\left(e^{\langle .,\alpha\rangle}\right)}_{\text{Turaev torsion}} \underbrace{e^{\langle CS(X_0,X),\alpha\rangle}}_{\text{defect}} (1 + O(s))}$$

where the constant C does not depend on $X, e^{\langle .,\alpha\rangle}$.

Idea of proof.

- introduce **dynamical torsion** :

$$\tau_X(\rho) = \underbrace{\tau_\Gamma(X)}_{\text{correction}} \times \underbrace{\lim_{s\to 0^+} s^{-m}\zeta_{X,\rho}(s)}_{\text{renormalized } \zeta}$$

  where $\tau_\Gamma(X)$ = torsion of kernel $C(0)$ for chirality Γ.

- Prove $\rho \mapsto \tau_X(\rho)$ holomorphic and $X \mapsto \tau_X(\rho)$ is C^1.
- Show that $\partial_X \log \tau_X(\rho) = 0$ "topological invariant" and differentiate on *Rep*

$$\frac{d}{dt} \log \tau_X(\rho e^{t\alpha})|_{t=0} = Tr_s^\flat(\alpha K_\varepsilon)$$

  where $[d, K_\varepsilon] = e^{-\varepsilon \mathcal{L}_X}$.

- Compare $\frac{d}{dt} \log \tau_X(\rho e^{t\alpha})|_{t=0}$ with log derivative of Turaev's torsion $\frac{d}{dt} \log \tau_{\mathfrak{e}}(\rho e^{t\alpha})|_{t=0}$ yields the result.

Thanks for your attention !