"Classical" Boundary Problem theory

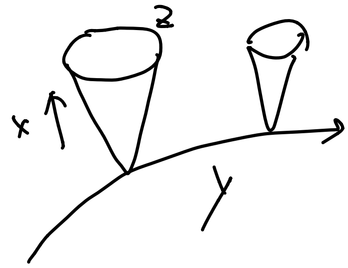
Lions - Magenes

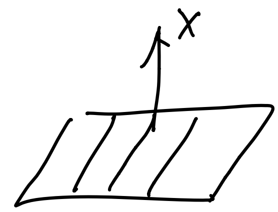
Calderón

Boutet de Monvel

Grubb

Parallel approach via edge theory



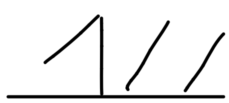


$$\mathcal{V}_e = C^\infty\text{-sp}\{x\partial_x, x\partial_{y_i}, \partial_{z_j}\}$$

$$= \{\text{all smooth v.f. tangent to fibers}\}$$

$$\mathrm{Diff}_e^m \ni L = \sum_{j+|\alpha|+|\beta|} a_{j\alpha\beta}(x\partial_x)^j(x\partial_y)^\alpha\partial_z^\beta$$

$$\rightsquigarrow {}^e\sigma^*$$



$x^2\Delta_g$

$\bar{g}$ nondeg metric

$(x, y) \longmapsto (\lambda x, \lambda y)$ leaves $x\partial_x$, $x\partial_y$ invariant.

- edge- ellipticity

$$\sigma_m(L) = \sum_{j+|\alpha|+|\beta|=m} a_{j\alpha\beta}(x,y,z)\, \xi^j \eta^\alpha \zeta^\beta \quad \text{is invertible}$$

- Full ellipticity

$$N(L) = \sum_{j+|\alpha|+|\beta|} a_{j\alpha\beta}(0, y_0, z)\,(s\partial_s)^j (s\partial_w)^\alpha \partial_z^\beta$$

↑ normal operator

$$\mathbb{R}_+^{b+1} \times Z \qquad s \geq 0, \quad w \in \mathbb{R}^b$$

full ellipticity : $N(L)$ is invertible at each pt of boundary.

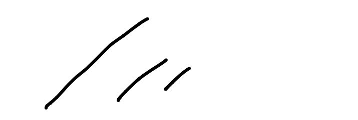
$L(u) = 0, \quad u \in x^\gamma L^\infty$

→ u has an "asymptotic expansion".

Indicial roots of L

$$\gamma \ \text{s.t.}\ \exists \phi \in C^\infty \quad L(x^\gamma \phi(y,z)) = O(x^{\gamma+1})$$

$$\sum a_{j\alpha\beta}(0, y_0, z)\, \gamma^j \partial_z^\beta \phi(z) = 0$$



$$(x^2\partial_x^2 + x^2\Delta_y)(x^\gamma\phi) = \gamma(\gamma-1)x^\gamma + O(x^{\gamma+1})$$

$$(x^2\partial_x^2 + x^2\Delta_y + \Delta_z)(x^\gamma \phi) = (\gamma(\gamma-1) + \Delta_z)\phi + O(x^{\gamma-1})$$

$\{\gamma_j\}$

$$Lu = 0 \Rightarrow u \sim \sum_{\gamma_j \geq \gamma} x^{\gamma_j} a_j(y, z)$$

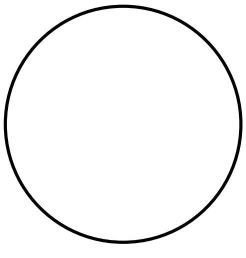
a_j distributional in y

$$\langle u, \chi(y) \rangle \sim \sum x^{\gamma_j} \langle a_j(\cdot, z), \chi \rangle$$

Fact: If $\gamma > \bar{\gamma} \Rightarrow u \sim \sum_{\gamma_j \geq \gamma} x^{\gamma_j} a_j(y, z)$ is a strong expansion.

$\uparrow$

If $N(L)$ is injective on $x^{\bar{\gamma}} L^2$ (globally)



Δ_{Euc} in D

$\gamma_0 = 0, \quad \gamma_1 = 1$

$$N(x^2 \Delta_{Euc}) = s^2 \partial_s^2 + s^2 \Delta_w$$

$u \sim a_0 x^0$
$+ b_0 x^1$

Dirichlet to Neumann
$a_0 \to b_0$

$\mathbb{H}^n$:

$$\Delta_{\mathbb{H}^n} + \lambda(n-1-\lambda)$$

Here $\Delta_{\mathbb{H}^n} = x^2\partial_x^2 + (2-n)x\partial_x + x^2\Delta_y + \lambda(n-1-\lambda)$

$$\gamma_0 = \lambda \quad , \quad \gamma_1 = n-\lambda$$

$$\boxed{\begin{aligned} &(\Delta - \lambda(n-1-\lambda))u = 0 \\ &\Rightarrow \ u \sim a_0(y)x^{\lambda} + \cdots + b_0(y)x^{n-1-\lambda} \end{aligned}}$$

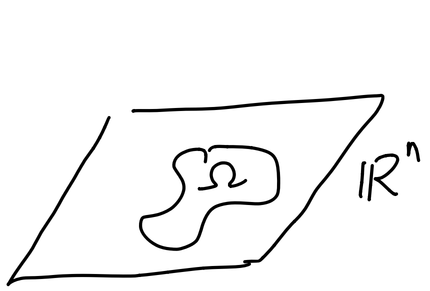
$$S(\lambda)(a_0) = b_0$$

Scattering operators $\in \Psi^{n-1-2\lambda}(\partial M)$

People working on this: Sa Barneto
Perry
Graham-Zworski

Fractional Laplacians: three definitions

① use function calculus



$$u = \sum a_j \phi_j$$

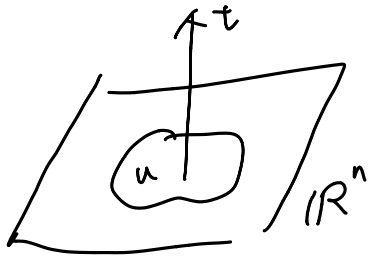
$$\Delta^s u = \sum a_j \lambda_j^s \phi_j$$

② Fourier transform

$$u^0 = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

$$\mathcal{F}^{-1}(|\xi|^{2s} \hat{u}^0)$$

③ $\mathbb{R}^{n+1}_+$



$$L_t = \partial_t^2 + \frac{1-2s}{t} \partial_t + \Delta_x$$

Calculate : $\gamma_0 = 0$, $\gamma_1 = 2s$

Solve $\begin{cases} L_t U = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ U|_{t=0} = u^0 \end{cases}$

$$U \sim u^0 t^0 + \cdots + V^0 t^{2s} + \cdots$$

$$V^0 = \Delta^s u^0$$

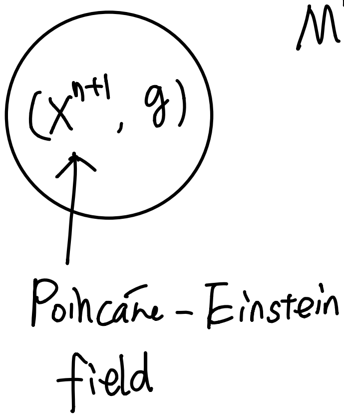
$$L_t U = 0 \xrightarrow{X\text{-F.T.}} \hat{L}_t \hat{U} = 0$$

$$\partial_t^2 + \frac{1-2s}{t} \partial_t - |\eta|^2 \qquad \hat{U} = a(\eta)\, t^s K_s(t|\eta|)$$

Caffarelli – Silvestre $\qquad \sim a(\eta)\,(t^0 + \cdots$

$$+ |\eta|^{2s} t^{2s} + \cdots)$$

Graham–Zworski



M^n

Problems:

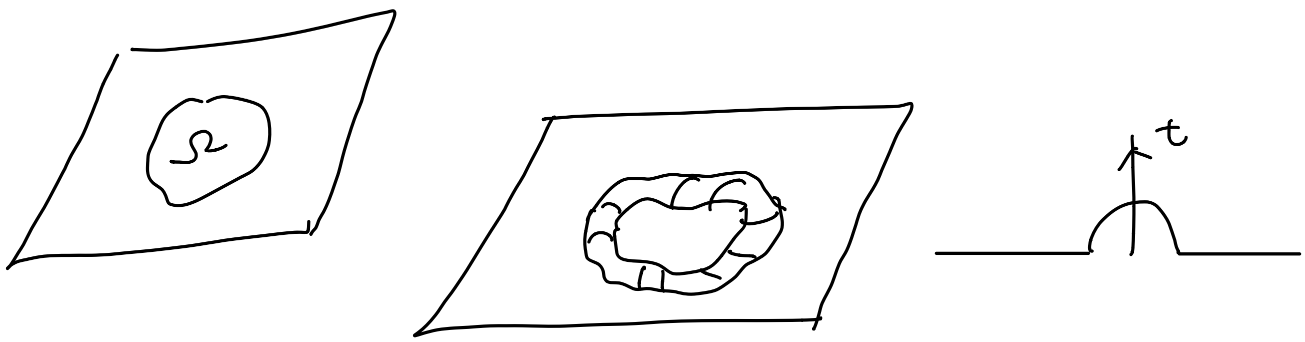
$$\Delta^s u = f$$

1) Solvability

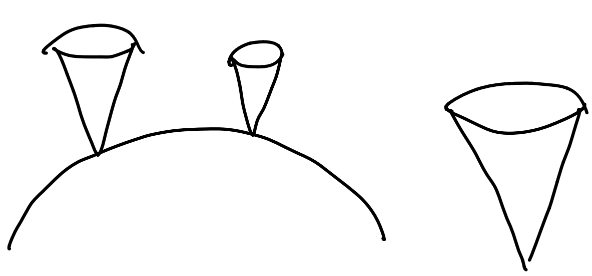
2) Regularity (at $\partial\Omega$)

Joint with Gingerlein, Louca, M

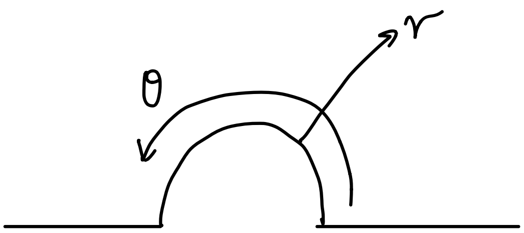
$$[\mathbb{R}^{n+1}, \partial\Omega \times \{0\}]$$



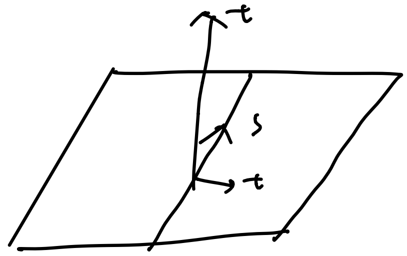
A boundle of cones.



$$L_t = \partial_r^2 + \frac{2-2s}{r}\partial_r + \frac{1}{r^2}P_s + \Delta_w , \quad w \in \partial\Omega$$



$$P_s = \partial_t^2 + (1-2s)\cos\theta\,\partial_t$$



$w \mapsto \eta$

$\rho = r|\eta|$

$$B(L_s) = \partial_\rho^2 + \frac{2-2s}{\rho}\partial_\rho + \frac{1}{\rho^2}P_s - 1$$

Problems want to study:

a) $B(L_s)U = 0$

$U(\rho, 0) = 1, \quad U(\rho, \pi) = 0$

b) $B(L_s)U = 0$

$\lim_{\theta \to 0} \theta^{1-2s}\partial_\theta U = \rho^{2s}$

$U(\rho, \pi) = 0$

c) $B(L_s)U = F, \quad U(\rho, 0) = U(\rho, \pi) = 0$

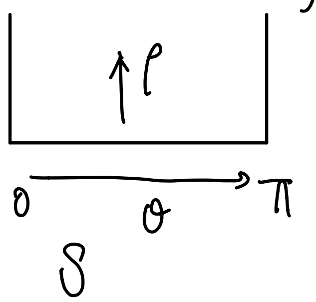
d) $B(L_s)U = F, \quad \lim_{\theta \to 0} \theta^{1-2s}\partial_\theta U = 0, \quad U(\rho, \pi) = 0$

Correspondence for a) → d)

a) define $\Delta^s u$

b) Solve $\Delta^s u = f$ given f

$\mathcal{F}^{-1}(U(r|\eta|))$



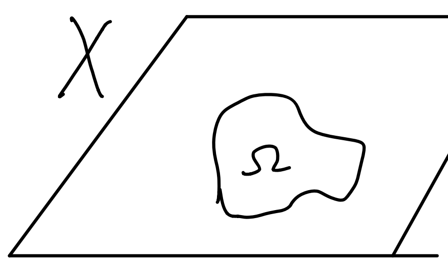
$G(\rho, \tilde{\rho}, \theta, \tilde{\theta})$

↑ Green's function

$G(\lambda\rho, \lambda\tilde{\rho}, \theta, \tilde{\theta}) \sim O(e^{-\lambda(\cdots)})$ $\lambda \to +\infty$

$G(\rho, \tilde{\rho}, \theta, \tilde{\theta})$ distribution on $[S \times S; \cdots]$

$[S \times S, \{\rho = \tilde{\rho} = 0\}, \{\theta = \tilde{\theta} = 0, \rho = \tilde{\rho}\}, \{\theta = \tilde{\theta} = \pi, \rho = \tilde{\rho}\}$



$X^2_{ie} = [X^2; \{r = \tilde{r} = 0, \omega = \tilde{\omega}\}, \cdots]$

P_s $\lambda_j^{p,s} = (s + j + \frac{1}{2})^2 - (s - \frac{1}{2})^2$

$r_j^{\pm} = 2s + j, -j - 1$

$$\lambda_j^{m,s} = (j+\tfrac{1}{2})^2 - (s-\tfrac{1}{2})^2$$

$$\gamma_{j,m}^{1} = s+j \ , \ s-j-1$$

Thm: (Grubb)

$f \in C^\infty(\bar{\Omega})$

$\Delta^s u = f \quad \Rightarrow \quad u \in x^s C^\infty(\bar{\Omega})$

Typical function space you want to work with

$\Lambda_+ = Op(\langle \xi' \rangle + i\xi_n)$

define $M^{s(\sigma)} = \{u : \Lambda^+ u \in M^\sigma\}$

$\Delta^s : M^{s(\sigma)}(\Omega) \to \bar{H}^{\sigma-2s}(\Omega)$