L^p norms via geodesic beams

Joint work with Y. Canzani

Jeffrey Galkowski

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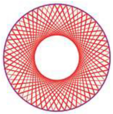

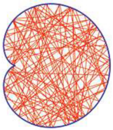

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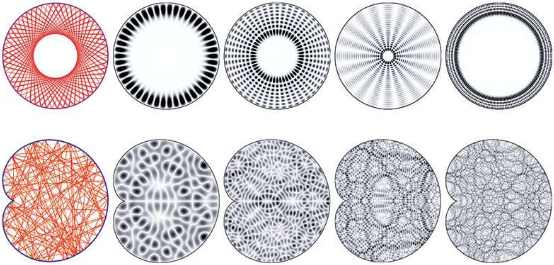

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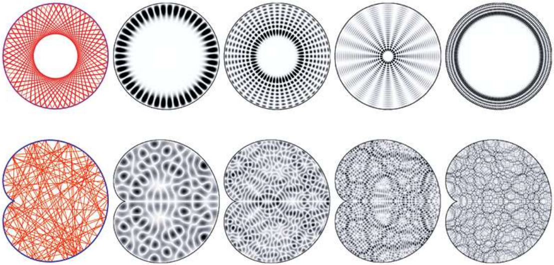

Question: How does ϕ_{λ_j} concentrate as $\lambda_j \to \infty$?

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Question 2: Let $2 < p \leq \infty$. The behavior of

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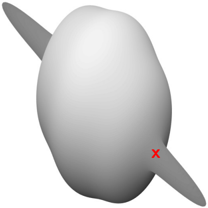
(second part of talk)

Sup-norms



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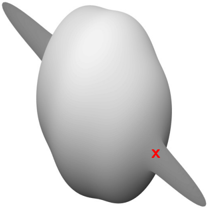

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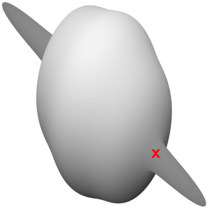

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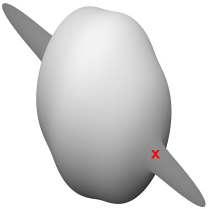

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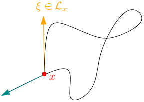

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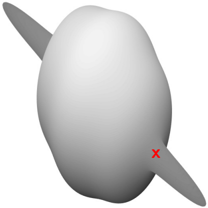

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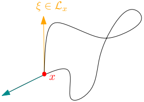

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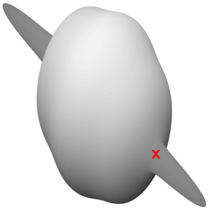

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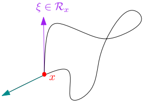

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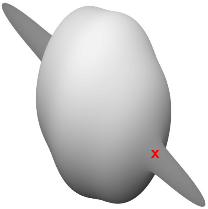

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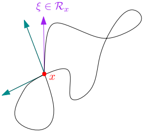

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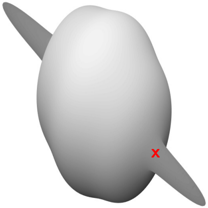

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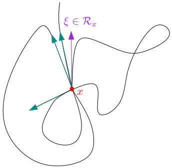

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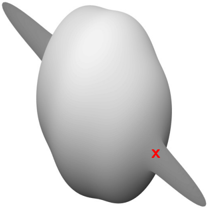

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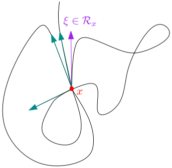

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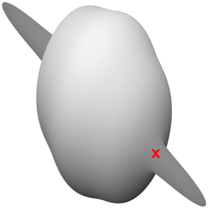

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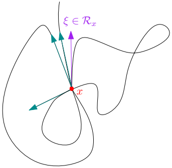

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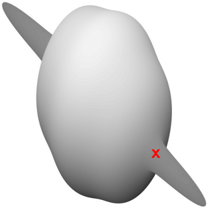

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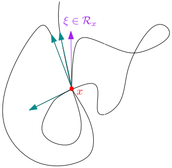

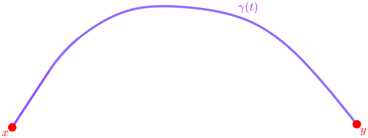

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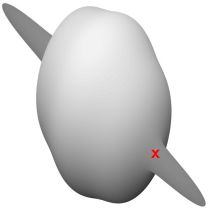

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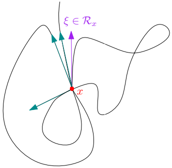

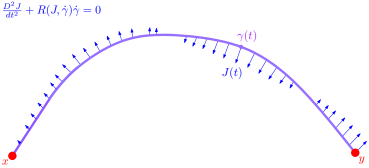

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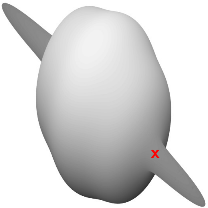

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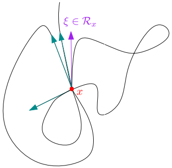

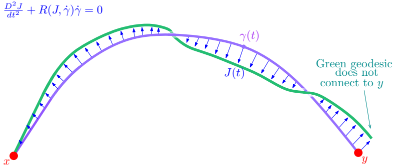

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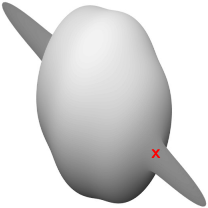

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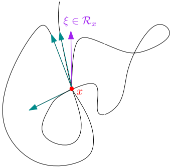
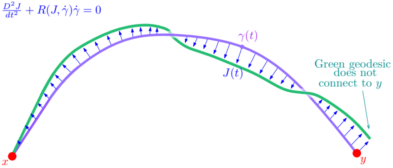




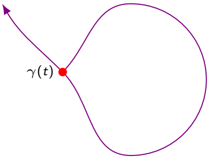
"no conjugate points" means $J \neq 0$ at y

Understanding concentration improves sup-norm bounds

Theorem (G ' 17)

Suppose $x \in M$ is **not** *maximally self-conjugate. Then, for $r_\lambda \to 0$,*

$$\|\phi_\lambda\|_{L^\infty(B(x,r_\lambda))} = o\left(\lambda^{\frac{n-1}{2}}\right).$$



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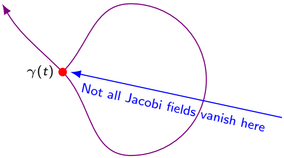

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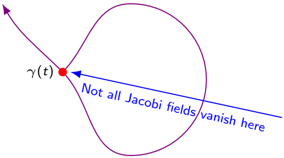

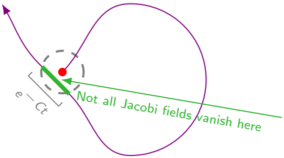

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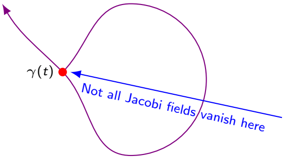
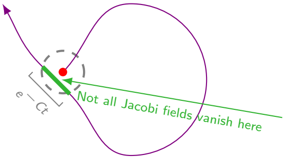
Theorem (G ' 17)

Suppose $x \in M$ *is* **not** *maximally self-conjugate. Then, for* $r_\lambda \to 0$,

$$\|\phi_\lambda\|_{L^\infty(B(x,r_\lambda))} = o\Big(\lambda^{\frac{n-1}{2}}\Big).$$



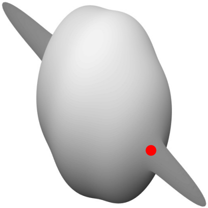


Theorem (Canzani-G '18)

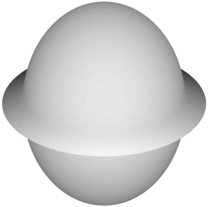
Suppose $x \in M$ *is* **not uniformly** *maximally self-conjugate. Then, for* $r_\lambda = \lambda^{-\delta}$ *with* $0 < \delta < \frac{1}{2}$,

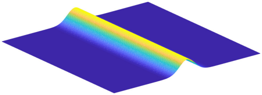
$$\|\phi_\lambda\|_{L^\infty(B(x,r_\lambda))} = O\Big(\frac{\lambda^{\frac{n-1}{2}}}{\sqrt{\log \lambda}}\Big).$$

Eigenfunctions are composed of geodesic beams



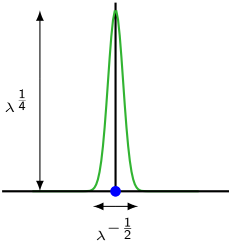
Eigenfunctions are composed of geodesic beams

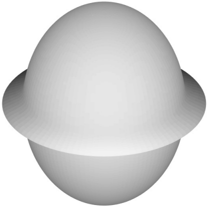


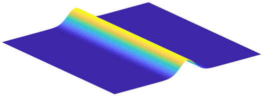


Eigenfunctions are composed of geodesic beams

Profile across a geodesic beam

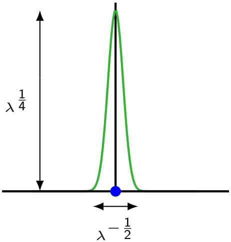


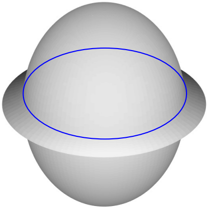


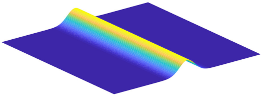


Eigenfunctions are composed of geodesic beams

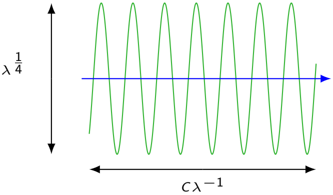
Profile across a geodesic beam







Profile along a geodesic beam



Eigenfunctions are composed of geodesic beams

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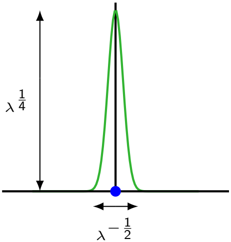
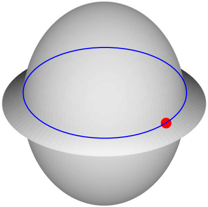
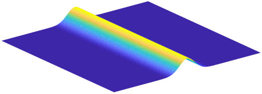
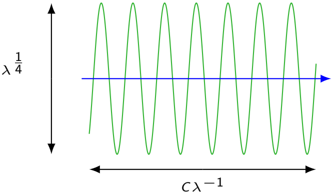






Profile along a geodesic beam

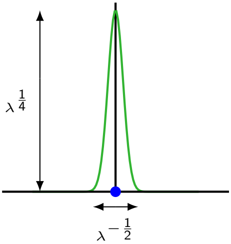


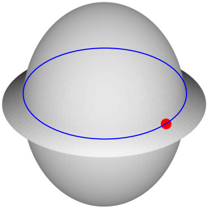
$\|\phi_\lambda\|_{L^2} = 1$

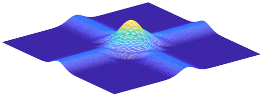
$\|\phi_\lambda\|_{L^\infty} \sim \lambda^{\frac{1}{4}}$

Eigenfunctions are composed of geodesic beams

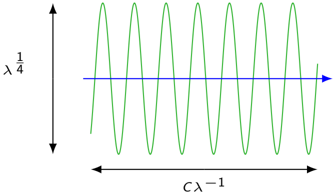
Profile across a geodesic beam

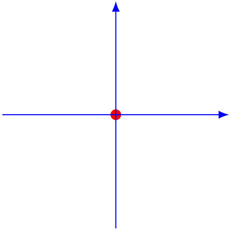






Profile along a geodesic beam



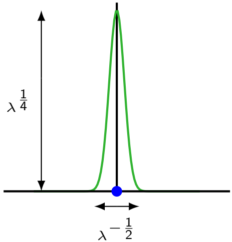


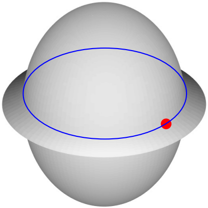
$\|\phi_\lambda\|_{L^2} = \sqrt{2}$

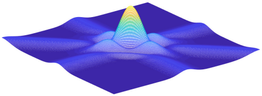
$\|\phi_\lambda\|_{L^\infty} \sim 2\lambda^{\frac{1}{4}}$

Eigenfunctions are composed of geodesic beams

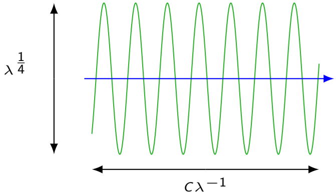
Profile across a geodesic beam

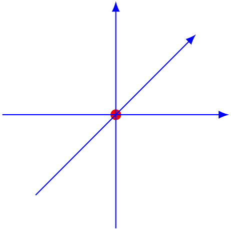






Profile along a geodesic beam



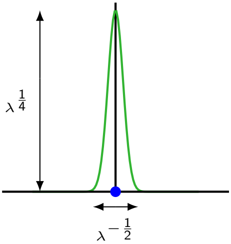


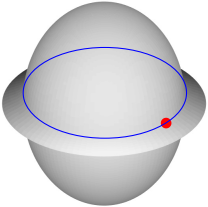
$\|\phi_\lambda\|_{L^2} = \sqrt{3}$

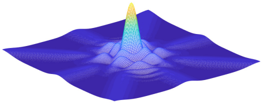
$\|\phi_\lambda\|_{L^\infty} \sim 3\lambda^{\frac{1}{4}}$

Eigenfunctions are composed of geodesic beams

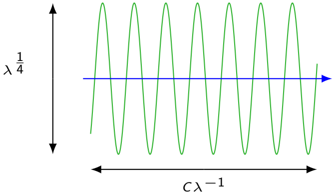
Profile across a geodesic beam

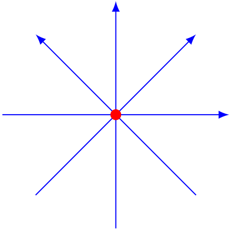






Profile along a geodesic beam



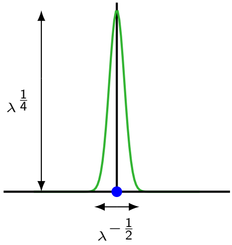


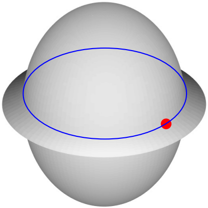
$$\|\phi_\lambda\|_{L^2} = \sqrt{4}$$

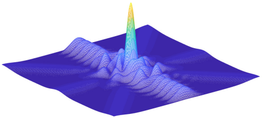
$$\|\phi_\lambda\|_{L^\infty} \sim 4\lambda^{\frac{1}{4}}$$

Eigenfunctions are composed of geodesic beams

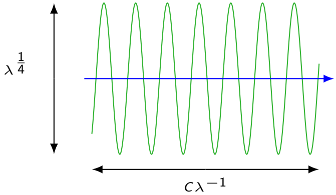
Profile across a geodesic beam

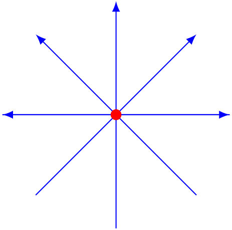






Profile along a geodesic beam



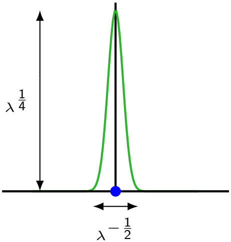


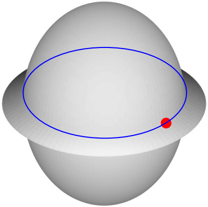
$\|\phi_\lambda\|_{L^2} = \sqrt{5}$

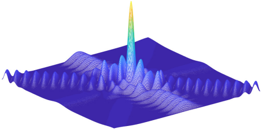
$\|\phi_\lambda\|_{L^\infty} \sim 5\lambda^{\frac{1}{4}}$

Eigenfunctions are composed of geodesic beams

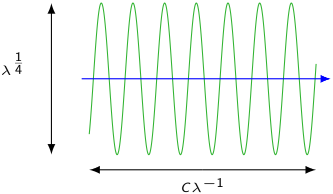
Profile across a geodesic beam

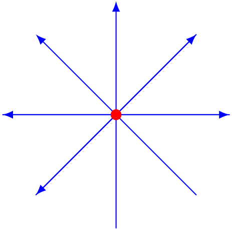






Profile along a geodesic beam



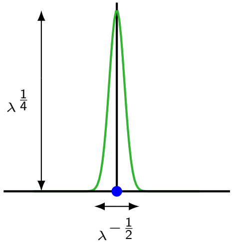


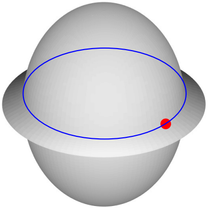
$$\|\phi_\lambda\|_{L^2} = \sqrt{6}$$

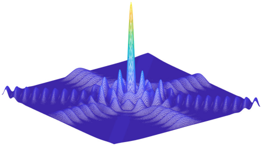
$$\|\phi_\lambda\|_{L^\infty} \sim 6\lambda^{\frac{1}{4}}$$

Eigenfunctions are composed of geodesic beams

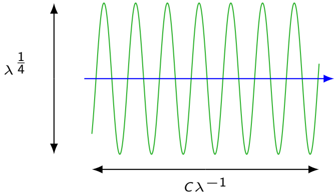
Profile across a geodesic beam

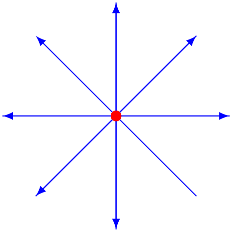






Profile along a geodesic beam



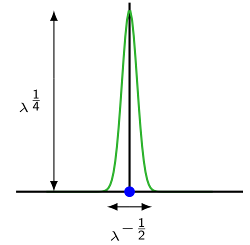


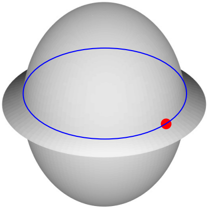
$$\|\phi_\lambda\|_{L^2} = \sqrt{7}$$

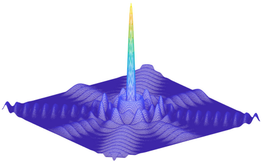
$$\|\phi_\lambda\|_{L^\infty} \sim 7\lambda^{\frac{1}{4}}$$

Eigenfunctions are composed of geodesic beams

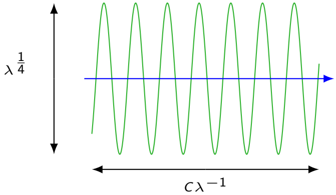
Profile across a geodesic beam

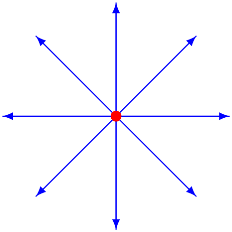






Profile along a geodesic beam



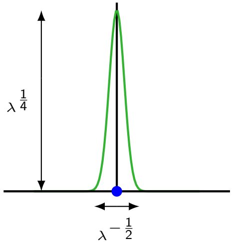


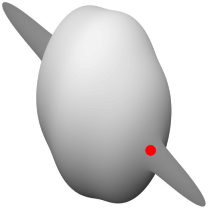
$$\|\phi_\lambda\|_{L^2} = \sqrt{8}$$

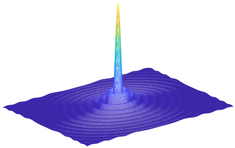
$$\|\phi_\lambda\|_{L^\infty} \sim 8\lambda^{\frac{1}{4}}$$

Eigenfunctions are composed of geodesic beams

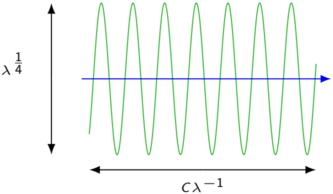
Profile across a geodesic beam

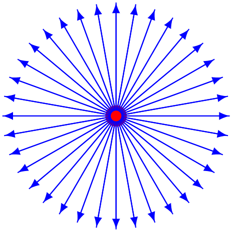






Profile along a geodesic beam

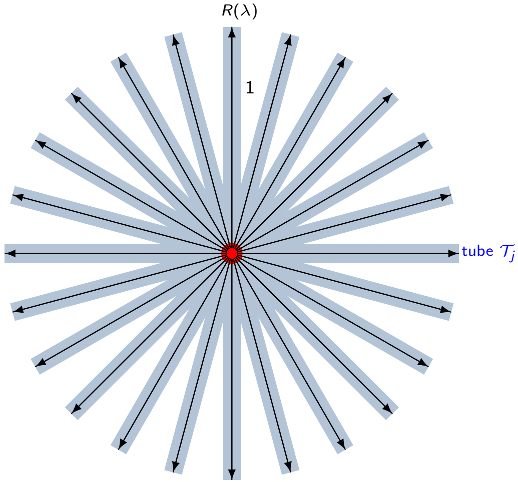




$$\|\phi_\lambda\|_{L^2} = \sqrt{\lambda^{\frac{1}{2}}}$$

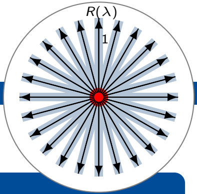
$$\|\phi_\lambda\|_{L^\infty} \sim \lambda^{\frac{1}{2}}\lambda^{\frac{1}{4}}$$

Fine microlocalization - Tubes on S_x^*M



Note: $\mathrm{vol}(\mathcal{T}_j) = R(\lambda)^{n-1}$

Understanding concentration gives quantitatively improved sup-norm bounds

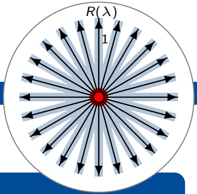


Theorem (Canzani–G '18 (main estimate, no assumptions!))

Let $x \in M$. There exists $C_n > 0$ so that

$$\|\phi_\lambda\|_{L^\infty(B(x,R(\lambda)))} \leq C_n \lambda^{\frac{n-1}{2}} R(\lambda)^{\frac{n-1}{2}} \sum_{j \in \mathcal{I}} \|\chi_{\mathcal{T}_j} \phi_\lambda\|_{L^2(M)}$$

Understanding concentration gives quantitatively improved sup-norm bounds



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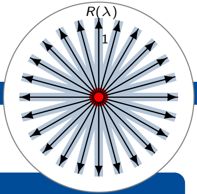
Theorem (Canzani–G '18)

Suppose $\mathcal{I} = \mathcal{B} \cup \mathcal{G}$ and that $\bigcup_{j \in \mathcal{G}} \mathcal{T}_j$ is non-self looping for $\log \lambda$ times.

Understanding concentration gives quantitatively improved sup-norm bounds



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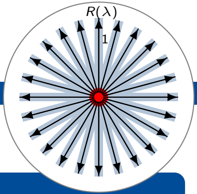
Theorem (Canzani–G '18)

Suppose $\mathcal{I} = \mathcal{B} \cup \mathcal{G}$ and that $\bigcup_{j \in \mathcal{G}} \mathcal{T}_j$ is non-self looping for $\log \lambda$ times.

Then,

$$\|\phi_\lambda\|_{L^\infty(B(x,R(\lambda)))} \leq C_n \lambda^{\frac{n-1}{2}} R(\lambda)^{\frac{n-1}{2}} \left(|\mathcal{B}|^{\frac{1}{2}} + \frac{|\mathcal{G}|^{\frac{1}{2}}}{(\log \lambda)^{\frac{1}{2}}} \right) \|\phi_\lambda\|_{L^2(M)}$$

Understanding concentration gives quantitatively improved sup-norm bounds



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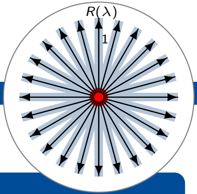
Understanding concentration gives quantitatively improved sup-norm bounds



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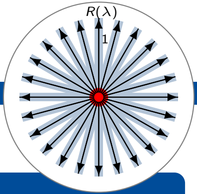

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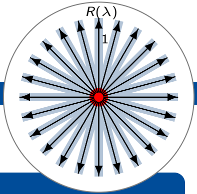

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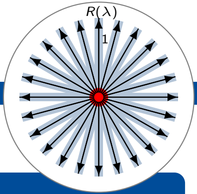

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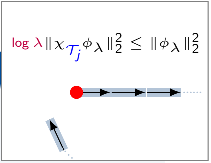

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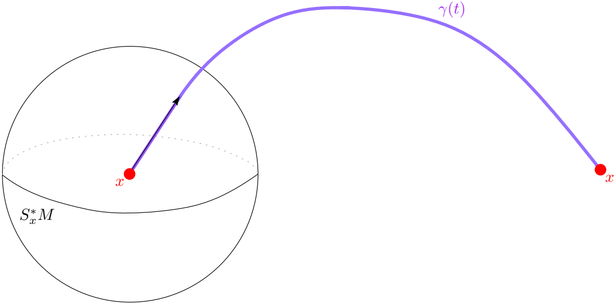

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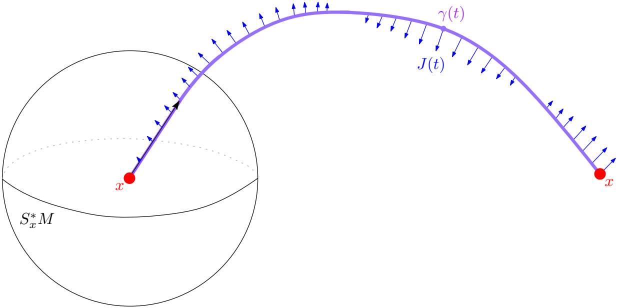

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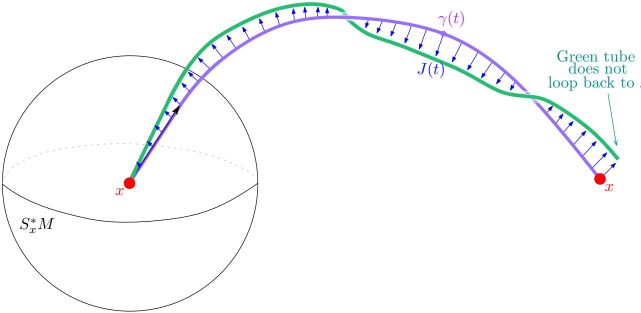

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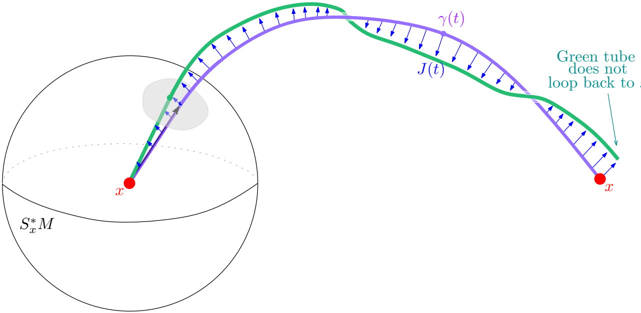

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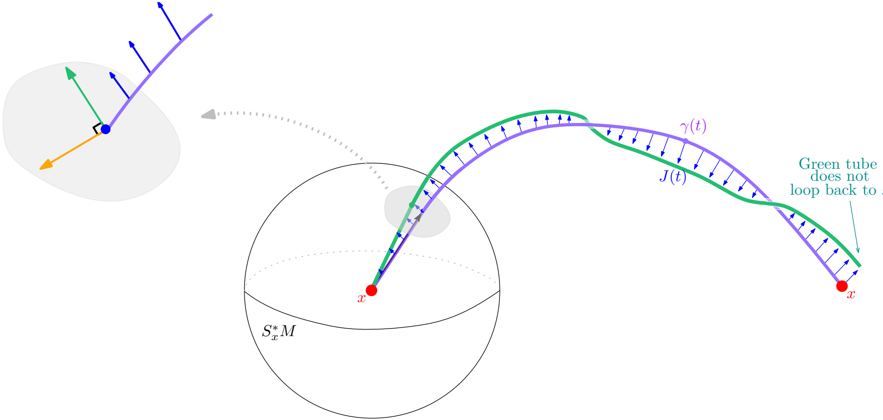

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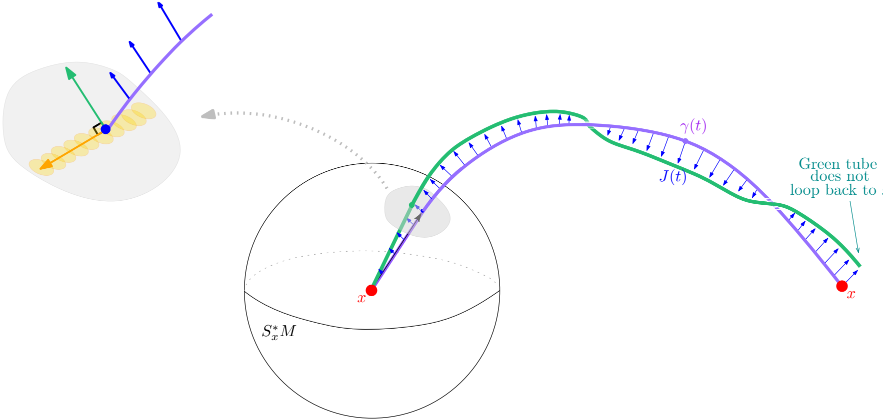

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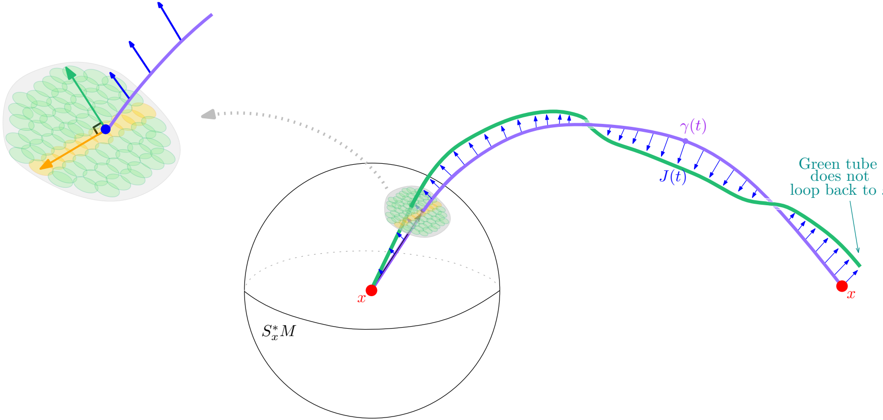

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Let $x \in M$ and suppose x is **not uniformly** *self-conjugate with maximal multiplicity,*

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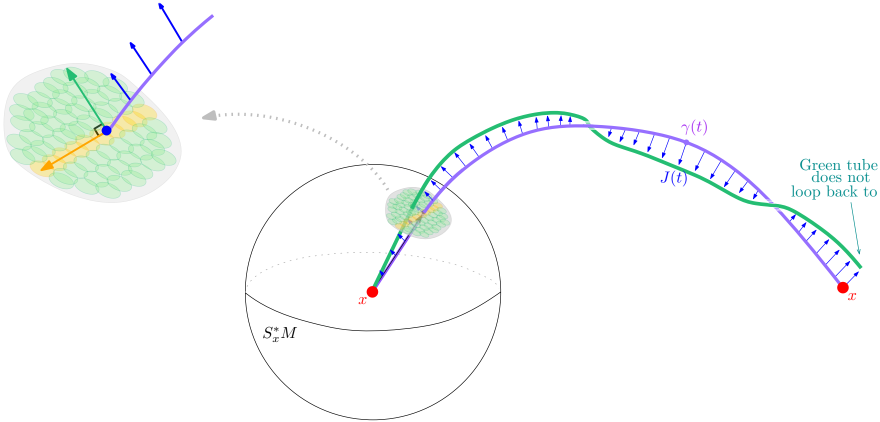

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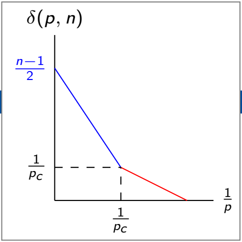

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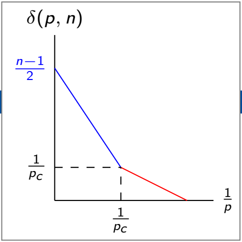

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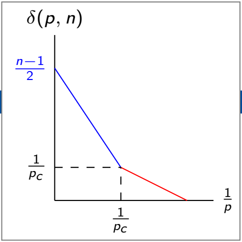

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Understanding concentration gives logarithmic improvements for high L^p norms

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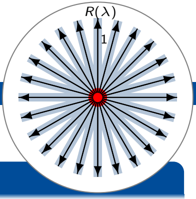

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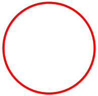

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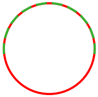

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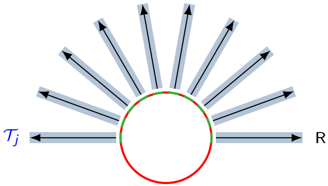

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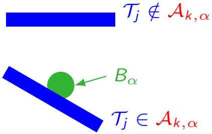

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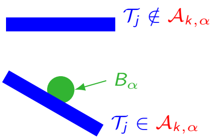

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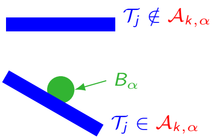

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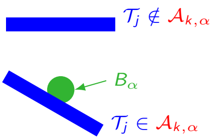

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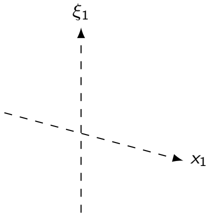

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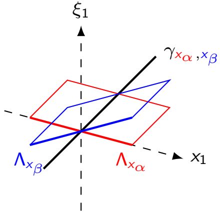

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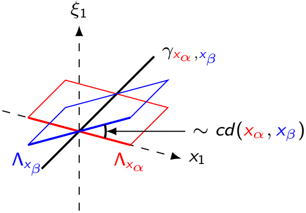

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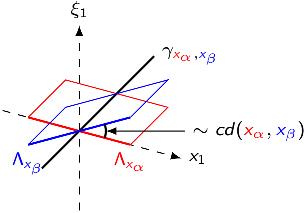

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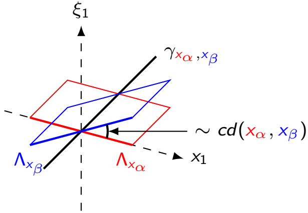

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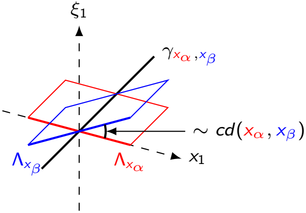

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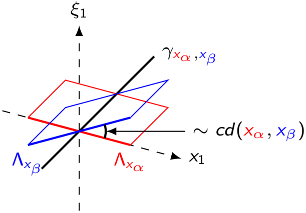

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$$\approx 2^{2m} R^{n-1} 2^{-2k} |\mathcal{I}_{k,m}| \|u\|^2_{L^2} \leq \sum_{\alpha \in \mathcal{I}_{k,m}} \|\chi_{\Lambda_{x_\alpha}} u\|^2_{L^2} \leq \|u\|^2_{L^2}.$$

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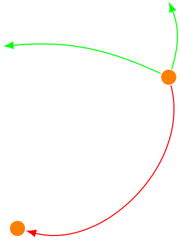

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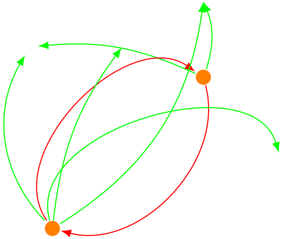

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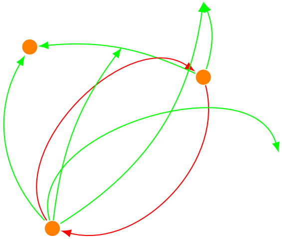

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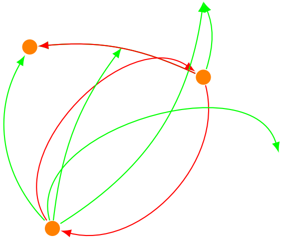

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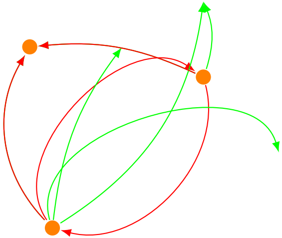

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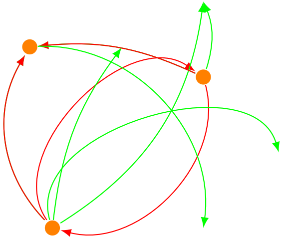

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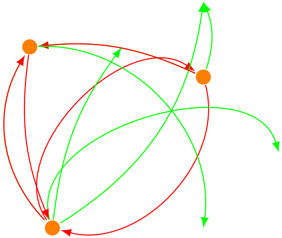

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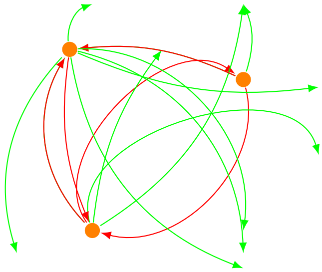

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- Since the tubes in $\mathcal{G}$ are T non-looping implies
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- Let $|\mathcal{B}_{xy}|$ be the maximal number of looping tubes between a pair (x, y).
-
$$\|\chi_{\mathcal{B}} w_k\|_{L^\infty(U_{k,m})} \leq h^{\frac{1-n}{2}} R^{\frac{n-1}{2}} |\mathcal{B}_{xy}||\mathcal{I}_{k,m}|2^{-k}\|u\|_{L^2}$$
$$\|\chi_{\mathcal{B}} w_k\|_{L^2} \leq h^{\frac{1-n}{2}} R^{\frac{n-1}{2}} |\mathcal{B}_{xy}||\mathcal{I}_{k,m}|^2 2^{-k}\|u\|_{L^2}$$
- Since the tubes in $\mathcal{G}$ are T non-looping implies
$$|\mathcal{G}| \leq 2^{2k} T^{-1}$$
-
$$\|\chi_{\mathcal{G}} w_k\|_{L^2(U_{k,m})} \leq \frac{\|u\|_{L^2}}{T^{\frac{1}{2}}}.$$

Concentration can be measured in many ways

What does concentration of ϕ_{λ_j} in position and momentum say about:

Question 1: Let $\mathbf{x} \in M$. What is the behavior of

$$\lim_{\lambda_j \to \infty} \left|\phi_{\lambda_j}(\mathbf{x})\right| \quad ?$$

(first part of the talk)

Question 2: Let $2 < p \leq \infty$. The behavior of

$$\lim_{\lambda_j \to \infty} \left\|\phi_{\lambda_j}\right\|_{L^p(M)} \quad ?$$

(second part of talk)

Future directions

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Thank you!