Joint with Zelditch

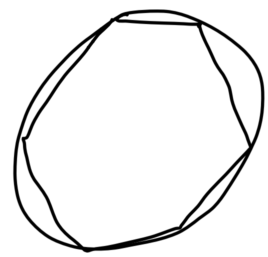
$\Omega$: $C^\infty$ s. convex domain in $\mathbb{R}^2$

$\Delta$ - Spectral problem: Determine $\Omega$ from $\mathrm{Spec}(\Delta_\Omega)$

with Dirichlet or Neumann

Length spectral problem: Determine $\Omega$ from

$L_{sp}(\Omega) = \{\text{length of periodic billiard trajectory}\}$



Anderson-Melrose (1976)

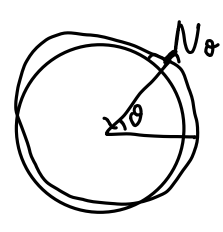
$$\mathrm{S.S.}\ \underbrace{\mathrm{Tr}\cos t\sqrt{-\Delta}}_{w(t)} \subset \overline{\mathcal{L}_{sp}(\Omega)} \underline{\cup \{c\}}$$

$$\cup -\mathcal{L}_{sp}(\Omega)$$

Def: $\Omega$ is called $\underline{\varepsilon\text{-NC in } C^n}$ if

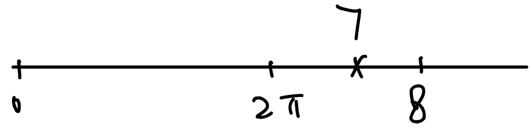
$\partial\Omega = \partial D + f(\theta) N_\theta$ , $D = \{x^2+y^2 < 1\}$

with $\|f\|_{C^n} \leq A_n \varepsilon$ if

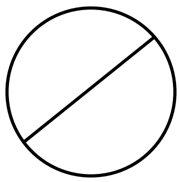


Thm: $\Omega$ is $\varepsilon$-NC in $C^7$. Then

$$\text{S.S. } w(t) \cap [0,7) = \overline{\mathcal{L}_{sp}(\Omega)} \cap [0,7).$$




$\gamma^2$



$\mathcal{L}_{p.q} = \{$lengths of $(p.q)$ types periodic orbits$\}$

$p = \#$ reflection orbits

$\gamma : (1.2)$

$\gamma^2 : (2.4)$

$q =$ winding number

If $p \geq 2$, then $\ell(\gamma) > 7$

$$\text{S.S. } w(t) \cap [0,7) = \overline{\bigcup_{q \geq 2} \mathcal{L}_{1,q}}$$

Proof uses parametrix of Marvizi–Melrose 1982

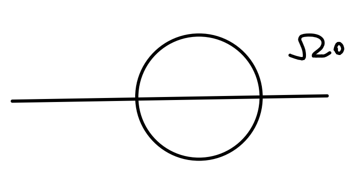
$q \geq q_0$

Thm (De Simoi - Kaloshin - Wei 16)

$\{\Omega_t\}_{0 \leq t \leq 1}$ of $\underline{\varepsilon\text{-NC in } C^8 \text{ with a } \mathbb{Z}_2 \text{ symmetry}}$

$S^8_\varepsilon$

If $\mathcal{L}_{sp}(\Omega_t) = \mathcal{L}_{sp}(\Omega_0)$, then $\Omega_t = \Omega_0$



Cor: If $\Omega_t \in S_\varepsilon^\delta$, and $\mathrm{Spec}(\Delta_{\Omega_t}) = \mathrm{Spec}(\Delta_{\Omega_0})$

then $\Omega_t = \Omega_0$

Thm: Nearly circular ellipses are spectrally uniquel among all smooth domains.

$$\partial E_\varepsilon = \{x^2 + \frac{y^2}{1-\varepsilon^2} = 1\} \qquad 0 \le \varepsilon \le \varepsilon_0 \text{ small}$$

Kac: $D = E_0$ is spectrally unique.

① If $\mathrm{Spec}(\Delta_\Omega) = \mathrm{Spe}(\Delta E_\varepsilon)$, then $\Omega$ is $\varepsilon$-NC in $C^n$ $\forall n \in \mathbb{N}$

Use formulas of Melrose for heat trace invariants.

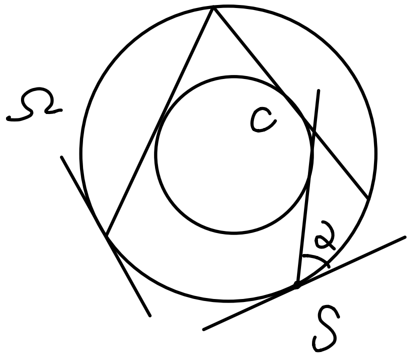
② $\Omega$ is $\varepsilon$-NC in $C^n$. $\mathrm{Spec}(\Delta_\Omega) = \mathrm{Spec}(\Delta E_\varepsilon)$

Then $\Omega$ is Rationally Integrable (R.I.)

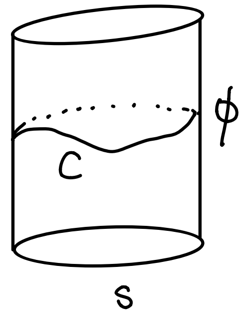
Def: $\Omega$ is called R.I. if $\forall q \ge 3$,

$T_{1,q} = \{$ periodic orbits containing $\mathcal{L}_{1,q}\}$

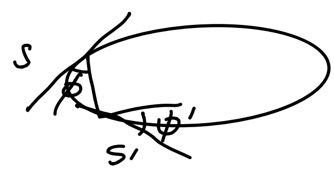
forms a Constic (invariant curve) in $\Omega$.



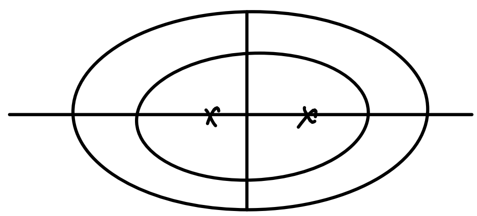

$$\hookrightarrow \Pi = \overset{s}{\mathbb{R}/\ell(\partial\Omega)\mathbb{Z}} \times \overset{\phi}{[0,\pi]}$$




C is invariant under $\begin{cases} \beta : \Pi \to \Pi \\ \beta(s,\phi) = (s',\phi') \end{cases}$




Example: Ellipses



Thm: (Avila-De Simoi-Kaloshin)

If $\Omega$ is $\varepsilon$-NC in $C^{39}$ and R.I., then $\Omega$ is an ellipse.

③ Free: $\Omega$ is an ellipse

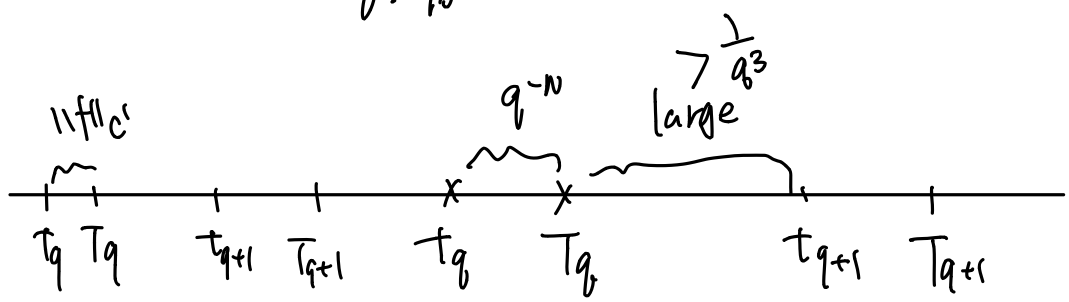
④ $\Omega$ is ellipse, $\mathrm{Spec}(\Omega) = \mathrm{Spec}(E_\varepsilon)$, then $\Omega = E_\varepsilon$.

Proof of ②: $\bigcup_{q \geq 2} \mathcal{L}_{1,q}(\Omega) = \bigcup_{q \geq 2} \mathcal{L}_{1,q}(E_\varepsilon)$

Marvizi-Melrose: $T_q = \sup \mathcal{L}_{1,q}$, $t_q = \sup \mathcal{L}_{1,q}$

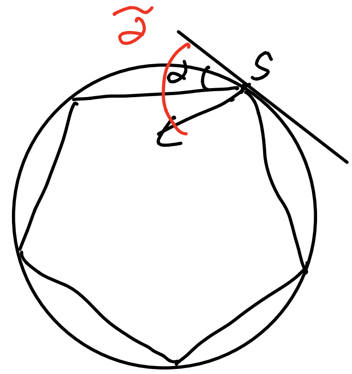
- $T_q - t_q = O_N(q^{-N})$
- $T_q \sim \ell(\partial\Omega) + \frac{C_1(\Omega)}{q^2} + \frac{C_2(\Omega)}{q^4} + \cdots$ $\quad C_1(\Omega) < 0$

$q \geq q_0$



Thm: $\Omega$ is $\varepsilon$-NC in $C^6$. Then $\forall s \in \partial\Omega$,

$\forall q \geq 2$, $\exists!$ loop of type $(1,q)$ at $s$.




$$\beta^q(s,\phi) = (s,\tilde{\phi})$$

$L_q(s)$ = length of this loop  Loop function

Fact: $\tilde{\phi} - \phi = O(q^{-N})$

$$T_q = \max L_q(s) \qquad t_q = \min L_q(s)$$

Lemma: $L_q(s) = 2q \sin\frac{\pi}{q} + O(\|f\|_{C^1})$

RHS: $\bigcup_{q \geq 2} \mathcal{L}_{1,q}(E_\nu) = \{ t_2 < T_2 < T_3 < \cdots \}$

☆ Gaps in this set are s. decreasing.

Fact: $T_q(E_\nu) = t_q(E_\nu)$

Pancelet 1822

$T_q(\Omega) = T_q(\Omega)$ or $L_q(s)$ is constant, $q \geq 3$.

$\mathcal{C} = \{(s, \phi_q(s)) \; ; \; s \in \partial\Omega\}$

Claim: $\mathcal{C}$ is invariant curve consists of $(1, q)$ type periodic orbits.