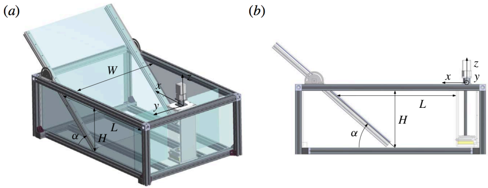
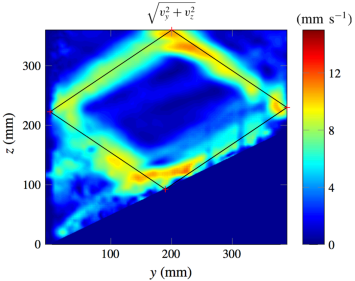
Viscosity limits for 0th order operators

Microlocal Workshop at MSRI

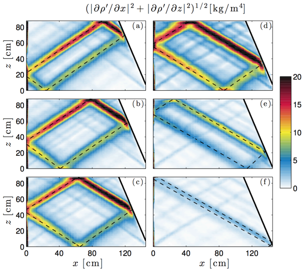
Maciej Zworski, UC Berkeley

Motivation (following Colin de Verdière and Saint-Raymond)



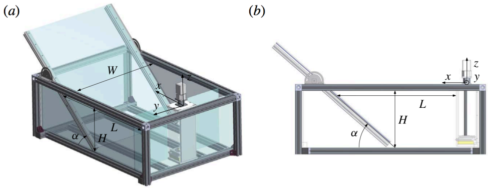


Pillet et al '18



Bouzet '16

Motivation (following Colin de Verdière and Saint-Raymond)

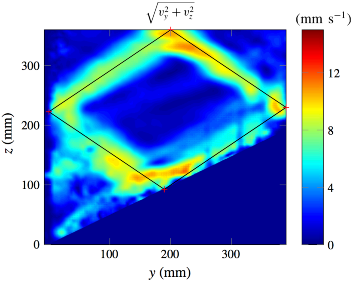


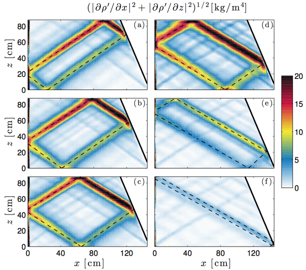
https://www.youtube.com/watch?v=6qgPdszHRzs

Motivation (following Colin de Verdière and Saint-Raymond)

Boussinesq approximation:

$$\begin{cases} \partial_t \eta + \mathbf{u} \cdot \nabla \rho_0 = 0, \quad \operatorname{div} \mathbf{u} = 0, \\ \rho_0 \partial_t \mathbf{u} = -\eta g \mathbf{e}_3 - \nabla P + \mathbf{F} e^{-i\omega_0 t}, \end{cases} \qquad \mathbf{n} \cdot \mathbf{u} = 0.$$





Formal diagonalization gives $\mathbf{u} = u_+ \mathbf{e}_+ + u_- \mathbf{e}_-$

$$i\partial_t u_\pm - P u_\pm = e^{-i\omega_0 t} f_\pm$$

$$P = H_\pm(x, D), \quad H_\pm(x, \xi) = \pm(-g\rho_0'(x)/\rho_0(x))^{\frac{1}{2}} \xi_1 / |\xi|$$

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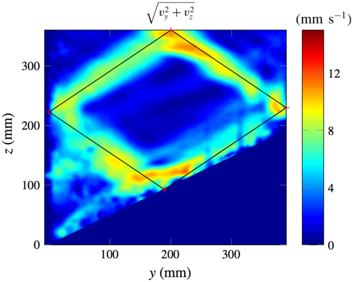
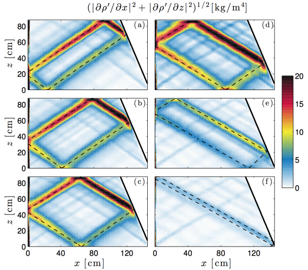




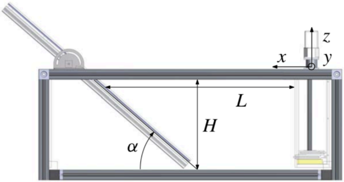
Other related models: rotating fluids Ralston '73

$$\partial_t^2 \Delta_x u = \partial_{x_1}^2 u, \quad u|_{\partial\Omega} = 0$$

$$i\partial_t u - Pu = 0, \quad P = \pm \Delta^{-\frac{1}{2}} \partial_{x_1}$$

Mathematical Model

(very much watered down...)



$\rightarrow$

$$H_{\pm}(x, D) \quad \longrightarrow \quad P \in \Psi^0(\mathbb{T}^2), \quad P^* = P$$

$$p := \sigma(P) \text{ homogeneous of degree } 0 \,, \quad dp|_{p^{-1}(\omega_0)} \neq 0,$$

the flow of $|\xi| H_p|_{p^{-1}(\omega_0)/\sim}$ is Morse–Smale with no fixed points

$$H_p = \partial_\xi p \cdot \partial_x - \partial_x p \cdot \partial_\xi, \quad (x, \xi) \sim (y, \eta) \Leftrightarrow x = y, \ \xi = t\eta, \ t > 0$$

Mathematical Model

$$P \in \Psi^0(\mathbb{T}^2), \quad P^* = P, \quad u|_{t=0} = 0$$

The surface $\Sigma := p^{-1}(\omega_0)/\sim$ lies on the boundary of $\overline{T^*\mathbb{T}^2} \setminus 0$

$|\xi| H_p$ is tangent to Σ.

Morse–Smale flow with no fixed points on Σ:

(i) $|\xi| H_p$ has a finite number of hyperbolic limit cycles;

(ii) every trajectory different from (i) has unique trajectories (i) as its α, ω-limit set.

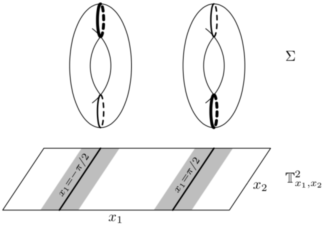
Theorem (Colin de Verdière–Saint-Raymond '18, Dyatlov–Z '18 (no fixed points), Colin de Verdière '18 (fixed points allowed)) *There exists $\delta > 0$ such that*

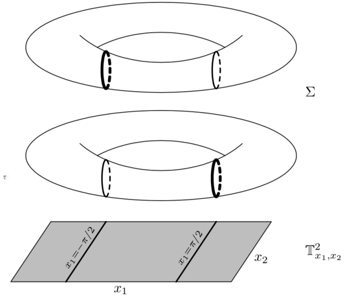
$$[-\delta, \delta] \subset \mathrm{Spec}_{\mathrm{ac}}(P), \quad |\mathrm{Spec}_{\mathrm{pp}}(P) \cap [-\delta, \delta]| < \infty,$$

$$\mathrm{Spec}_{\mathrm{sc}}(P) \cap [-\delta, \delta] = \emptyset.$$

An example

$p = |\xi|^{-1}\xi_2 - 2\cos x_1$ $\qquad$ $p = |\xi|^{-1}\xi_2 - \frac{1}{2}\cos x_1$





Attracting Lagrangians:

$$\Lambda_+ = \{x_1 = \pi/2, \xi_1 < 0, \xi_2 = 0\} \cup \{x_1 = -\pi/2, \xi_1 > 0, \xi_2 = 0\}$$

$$\omega \in [-\delta, \delta] \implies (P - \omega - i0)^{-1} : C^\infty(\mathbb{T}^2) \to I^0(\Lambda_+) \subset H^{-\frac{1}{2}-}(\mathbb{T}^2).$$

An example of an embedded eigenvalue

$$P := p^w(x,\xi), \quad p(x,\xi) := \langle\xi\rangle^{-1}\xi_2 - \alpha\,(1-\chi_k(\xi_1)\psi(\xi_2))\cos x_1,$$
$$\chi_k(k\pm 1) = 1, \quad \psi(\ell) = \delta_{\ell 0}, \quad \chi_k, \psi \in C^\infty_{\rm c}(\mathbb{R}).$$

$$P(e^{ix_1 k}) = 0 \ \text{ i.e. } \ 0 \in \mathrm{Spec}_{\rm pp}(P)$$

Z Tao '19 (undergraduate at UC Berkeley)

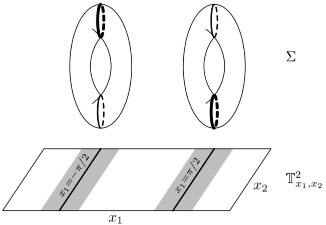
Colin de Verdière '18 suggested a Fermi Golden Rule for embedded eigenvalues: $Pu = 0$, $\Pi : L^2 \to \ker_{L^2} P^\perp$, $V \in \Psi^{-\infty}(\mathbb{T}^2)$

$$\mathrm{Im}\langle Vu, \Pi(P-i0)^{-1}\Pi Vu\rangle \neq 0 \implies \mathrm{Spec}_{\rm pp}(P+\epsilon V)\cap(-\delta,\delta) = \emptyset.$$
$$(\exists\, \epsilon_0,\ \delta\, \forall\, 0 < \epsilon < \epsilon_0 \ \cdots)$$

Applications to forced waves

Theorem (Colin de Verdière–Saint-Raymond '18, Dyatlov–Z '18)
If $\omega_0 \notin \mathrm{Spec}_{\mathrm{pp}}(P)$ *and* $i\partial_t u - Pu = e^{-i\omega_0 t} f \in C^\infty$, $u|_{t=0} = 0$, *then* $u(t) = e^{-i\omega_0 t} u_\infty + b(t) + \epsilon(t)$, $u_\infty \in I^0(\Lambda_+)$, $\|b(t)\|_{L^2} \leq C$, $\|\epsilon(t)\|_{-\frac{1}{2}-} \to 0$.

$$P := \langle D \rangle^{-1} D_{x_2} - 2\cos x_1, \quad f = e^{-3((x+0.9)^2+(y+0.8)^2)+i2x+iy}$$



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$$u(t) = \int_0^t e^{-isP} f \, ds = iP^{-1}(1 - e^{-itP}) f$$

We need to show that

- the limit $(P - \omega - i0)^{-1} f$ exists for ω near 0
- $P^{-1}(1 - e^{-itP})\chi(P) f \xrightarrow{\text{in } H^{-\frac{1}{2}-}} (P - i0)^{-1}\chi(P) f.$
- $(P - i0)^{-1} f \in I^0(\Lambda_+)$

Mathematical tools: Radial propagation estimates (Melrose '94, Vasy '11, Dyatlov–Z '13...), Lagrangian distributions (Hörmander '71...)

Applications to forced waves

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If $\omega_0 \notin \mathrm{Spec}_{\mathrm{pp}}(P)$ *and* $i\partial_t u - Pu = e^{-i\omega_0 t} f \in C^\infty$, $u|_{t=0} = 0$, *then* $u(t) = e^{-i\omega_0 t} u_\infty + b(t) + \epsilon(t)$, $u_\infty \in I^0(\Lambda_+)$, $\|b(t)\|_{L^2} \leq C$, $\|\epsilon(t)\|_{-\frac{1}{2}-} \to 0$.

Theorem (Wang 王健 '19) *There exist invertible maps*

$$G_\pm : C^\infty(\mathbb{S}^1; \mathbb{C}^N) \to S^{\frac{1}{2}}/S^{-\frac{1}{2}}(\Lambda_\pm, \Omega^{\frac{1}{2}}_{\Lambda_\pm} \otimes \mathcal{M}_{\Lambda_\pm})$$

($N =$ *number of components of* $\Lambda_\pm$*) such that for every* $f \in C^\infty(\mathbb{S}^1; \mathbb{C}^N)$ *(fixing* $+$ *or* $-$*)*

$$\exists!\, u = u_- + u_+, \quad u_\pm \in I^0(\Lambda_\pm), \quad Pu = 0, \quad \sigma_{\Lambda_\pm}(u_\pm) = G_\pm(f).$$

The operator $\mathscr{S} := G_-^{-1}(\sigma_{\Lambda_-}(u_-)) \to G_+^{-1}(\sigma_{\Lambda_+}(u_+))$, *extends to a unitary operator on* $L^2(\mathbb{S}^1; \mathbb{C}^N)$.

The operator $\mathscr{S}$ is an analogue of the scattering matrix of Hassell–Melrose–Vasy '04 (for scattering by symbols of order 0) and it has interesting microlocal structure Wang '19.

Viscosity: a non-Hermitian case

$$P \rightsquigarrow P - i\nu\Delta_{\mathbb{T}^2}$$

"The aim of this paper is to present what we believe to be the asymptotic limit of inertial modes in a spherical shell when viscosity tends to zero"

Rieutord–Georgeot–Valdettaro J. Fluid Mech. '01

Theorem (Galkowski–Z, '19?) *Suppose that P satisfies the assumptions above and $(x,\xi) \mapsto P(x,\xi)$ is analytic in a conic neighbourhood of $\mathbb{T}^2 \times \mathbb{R}^2 \subset \mathbb{C}^2/(2\pi\mathbb{Z})^2 \times \mathbb{C}^2$. Then there exists an open neighbourhood of 0, $U \subset \mathbb{C}$ such that*

$$\mathrm{Spec}(P - i\nu\Delta_{\mathbb{T}^2}) \cap U \to \mathscr{R}(P) \cap U, \quad \nu \to 0+$$

where $\mathscr{R}(P) \subset \mathbb{C}_-$ is a discrete *set depending only on P.*

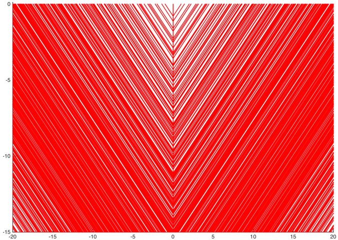
Question: What effect do elements of $\mathscr{R}(P)$ have on long term evolution **?** Not so clear if there is a clean mathematical statement.

Viscosity: previous mathematical results

- Dyatlov–Z '15: X generator of an Anosov flow on M; eigenvalues of $X + \nu\Delta_M$ converge to Ruelle resonances

Earlier results for Anosov maps: Keller–Liverani '99 ... Nakano–Wittsten '15

False for non-Anosov flows:



The limit set of $\mathrm{Spec}(X + i\nu\Delta_{\mathbb{R}^3/\mathbb{Z}^3})$, $\nu \to 0+$ where X generates the geodesic flow of $\mathbb{R}^2/\mathbb{Z}^2$

Based on Galtsev–Shafarevitch '06: $\mathrm{Spec}(-i(hD_\theta)^2 + \sin\theta)$

Viscosity: previous mathematical results

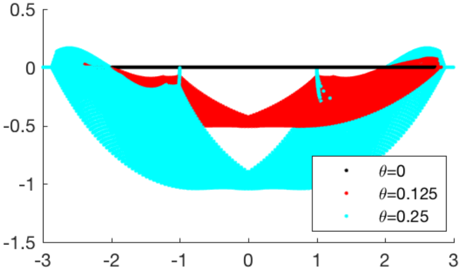
- Dyatlov–Z '15: X generator of an Anosov flow on M; eigenvalues of $X + \nu\Delta_M$ converge to Ruelle resonances
- Z '15: Eigenvalues of $-\Delta + V(x) - i\nu|x|^2$, $V \in L^\infty_{\rm comp}(\mathbb{R}^n)$ converge to resonances of $-\Delta + V$ (justifies the CAP method in computational chemistry)
- Drouot '17: $X + \nu \sum_{i,j} g_{ij}(z)\partial^2_{\zeta_j\zeta_i}\big|_{S_z^*M}$; X generator of geodesic flow on S^*M; convergence to Ruelle resonances (kinetic Brownian motion)
- Frenkel–Losev–Nekrasov '06 (height function on the sphere), Dang–Rivière '18 (general Morse–Smale functions): eigenvalues of $\mathcal{L}_{\nabla f} + \nu\Delta_g$ (Witten Laplacian) converge to Ruelle resonances of the gradient flow.
- Open problem (?): $D_x^2 + x^{-1}\sin x - i\nu|x|^2$

Viscosity: a non-Hermitian case

An Example: $P = \langle D \rangle^{-1} D_{x_2} + 2\cos x_1$

$$\mathscr{R}(P) \cap U \cap \{\operatorname{Im} z > -\theta/C\} = \operatorname{Spec}(P_\theta) \cap U \cap \{\operatorname{Im} z > -\theta/C\}$$

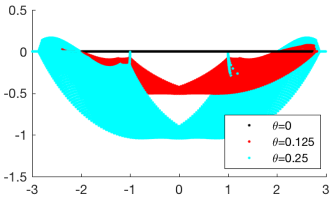
$$P_\theta := P|_{\mathbb{T}^2_\theta}, \quad T^*\mathbb{T}^2_\theta = \{(x_1 + 2i\theta \sin x_1, x_2, (1 - 2i\theta \cos x_1)^{-1}\xi_1, \xi_2)\}.$$

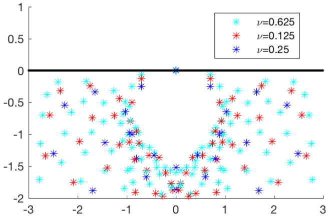


$$\mathscr{R}(P) \cap ((-1+\epsilon, 1-\epsilon) - i[0,\delta)) = \emptyset \quad \textbf{?}$$

Viscosity: a non-Hermitian case

An Example: $P = \langle D \rangle^{-1} D_{x_2} + 2\cos x_1$



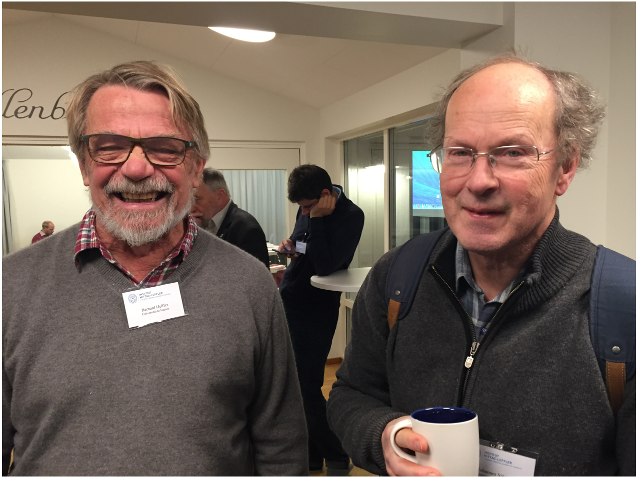


$\mathrm{Spec}(P_\theta)$ $\mathrm{Spec}(P - i\nu\Delta_{\mathbb{T}^2})$

The eigenvalues (or their absence) are visible in complex deformation and in the viscosity limit.

Viscosity: mathematical tools

For the general case one needs to adapt the complex microlocal deformation theory of Helffer–Sjöstrand '86.



Viscosity: mathematical tools

- Boutet de Monvel–Sjöstrand '75: microlocal analysis of projectors on spaces of solutions of systems of operators modeled on $\bar{\partial}$
- Boutet de Monvel–Guillemin '81: theory of generalized Toeplitz operators with the BdM–Sj projector as main example
- Helffer–Sjöstrand '86: calculus for the study of resonances for very general operators in $\mathbb{R}^n$; generalized FBI transforms and corresponding systems of operators; also Martinez '94,'02, Nakamura'96
- Sjöstrand '96: semiclassical compactly supported weights on analytic compact manifolds

For the viscosity limits we follow the roadmap in Sj '96 to extend it to weights which are homogeneous of degree 1 in ξ.

Viscosity: general strategy

There exists G, $G(x, \lambda\xi) = \lambda G(x, \xi)$, $\lambda > 0$ such that

$$H_p G > c_0 \text{ when } p = 0 \text{ and } |\xi| \geq C_0$$

Example: $p(x, \xi) = |\xi|^{-1}\xi_2 + \alpha \cos x_1$, $G(x, \xi) = \alpha\xi_1 \sin x_1$.

(Basis of the argument of Colin de Verdière–Saint-Raymond; in Dyatlov–Z we used general propagation results implicitly involving such G's.)

$P_\theta :=$ "$e^{-\theta G^w(x,D)} P e^{\theta G^w(x,D)}$", $\sigma(P_\theta)$ "$\sim$" $p(x + i\theta G_\xi, \xi - i\theta G_x)$,

and Q "$\in$" $\Psi^{-\infty}$, $P_\theta + iQ$ is invertible.

Then,

$$R_\nu(\omega) := (P_\theta + iQ + i\nu\Delta_\theta - \omega)^{-1} \quad |\omega| < \delta, \quad 0 \leq \nu \leq \nu_0$$

exists.

Continuity of zeros of $\det(I - iR_\nu(\omega)Q)$ as $\nu \to 0$ follows.

Need to justify "$\star$" for weights which are not compactly supported and not linear in ξ.

Complex deformations in phase space

$$Tu(x,\xi) := \int_M K(x,\xi,y)u(y)dy, \quad K(x,\xi,y) \text{ real analytic}$$

$$K(x,\xi,y) = e^{i\varphi(x,\xi,y)}a(x,y,\xi)\chi(x-y) + \mathcal{O}(e^{-\langle\xi\rangle}),$$

$$\varphi = \xi(\exp_x^{-1}(y)) + \langle\xi\rangle d(x,y)^2/2, \quad \xi \in T_x^*M, \ \exp_x^{-1}(y) \in T_x^*M$$

For $\mathbb{T}^2$: $K = c\langle\xi\rangle^{\frac{1}{2}} \sum_{k\in\mathbb{Z}^2} e^{i\langle\xi,x-y-k\rangle - \langle\xi\rangle(x-y-k)^2/2} \langle x-y-k\rangle^{\frac{1}{2}}$.

$$T^*T \equiv I_{L^2(\mathbb{T}^2)}, \quad TT^* \equiv \Pi, \quad \Pi : L^2(T^*\mathbb{T}^2) \xrightarrow{\perp} TL^2(\mathbb{T}^2).$$

(with low frequency modifications – need a small h!)

$$TAT^* \equiv \Pi\widetilde{A}\Pi = \Pi M_{A_0(x,\xi)}\Pi + \Pi\mathcal{O}(\langle\xi\rangle^{-1})\Pi, \quad A \in \Psi^0(\mathbb{T}^2)$$

(In practice, and in preparation for complex microlocal deformations, $T^*T \neq \mathrm{Id}$, and we use an approximate inverse $S \neq T^*$ with differently chosen amplitudes in T and S.)

Complex deformations in phase space

$$Tu(x,\xi) := \int_M K(x,\xi,y)u(y)dy, \quad K(x,\xi,y) \text{ real analytic}$$

$$T^*T \equiv I_{L^2(\mathbb{T}^2)}, \quad TT^* \equiv \Pi, \quad \Pi : L^2(T^*\mathbb{T}^2) \xrightarrow{\perp} TL^2(\mathbb{T}^2).$$

$$TAT^*\Pi \equiv \Pi\widetilde{A}\Pi = \Pi M_{A_0(x,\xi)}\Pi + \Pi\mathcal{O}(\langle\xi\rangle^{-1})\Pi, \quad A \in \Psi^0(\mathbb{T}^2)$$

$$\widetilde{A}(\alpha,\beta) = e^{i\psi_0(\alpha,\beta)}a(\alpha,\beta), \quad \alpha = (x,\xi), \; \beta = (x',\xi')$$
$$\operatorname{Im}\psi_0 \sim \langle\xi\rangle(x-x')^2 + \langle\xi\rangle^{-1}(\xi-\xi')^2$$
$$a_0(\alpha,\alpha) = A_0(x,\xi)$$

$$\zeta_j(x,\xi,D_x,D_\xi)Tu(x,\xi) \equiv 0, \quad [\zeta_k,\zeta_\ell] = 0$$

$$\zeta_j := |\xi|^{-1}(D_{x_j} - \xi_j) - \tfrac{1}{2}|\xi|^{-3}(D_x - \xi)^2\xi_j - iD_{\xi_j} + \mathcal{O}(\langle\xi\rangle^{-1})$$

They are the analogue of the $\bar\partial$ system in BdM–Sj '75

Complex deformations in phase space for tori

We now follow Helffer–Sjöstrand '87, Sj '96 and deform:

$$T_\theta(x,\xi) := Tu(x + i\theta G_\xi(x,\xi), \xi - i\theta G_x(x,\xi))$$

$$S_\theta T_\theta u \equiv u, \quad u \in \mathscr{A} \text{ (analytic functions)} ,$$

$$\mathscr{H}_\theta^s := \bar{\mathscr{A}}, \quad \|u\|_{\mathscr{H}_\theta^s} = \|\langle\xi\rangle^s T_\theta u\|_{L^2(T^*\mathbb{T}^2)}$$

(modification for low frequencies; note that there is no weight since G is homogeneous in ξ – cheating here at low frequencies)

$$\widetilde{\Pi}_\theta := T_\theta S_\theta \text{ extends to } L^2(T^*\mathbb{T}^2)$$

$$\widetilde{\Pi}_\theta : L^2(T^*\mathbb{T}^2) \to \mathscr{H}_\theta^0, \quad \text{not an orthogonal projection}$$

$$\zeta_{j,\theta}\widetilde{\Pi}_\theta = 0, \quad \widetilde{\Pi}_\theta \widetilde{\zeta}^t_{j,\theta} = 0, \quad [\zeta_{j,\theta}, \zeta_{\ell,\theta}] = 0.$$

$$\widetilde{\Pi}_\theta(\alpha,\beta) = e^{i\widetilde{\psi}_\theta(\alpha,\beta)}\widetilde{b}_\theta(\alpha,\beta), \ \operatorname{Im}\widetilde{\psi}_\theta \sim \langle\xi\rangle(x-x')^2 + \langle\xi\rangle^{-1}(\xi-\xi')^2$$

BdM-G '81, Sj 96: $B_{\theta,f} = \widetilde{\Pi}_\theta f \widetilde{\Pi}_\theta^*$, $B_{\theta,f}^2 \equiv B_{\theta,f}$ for a suitable $f \geq c$.

First step: $B_{\theta,f}(\alpha,\beta) = e^{i\psi_\theta(\alpha,\beta)} b_{\theta,f}(\alpha,\beta)$

$$\text{c.v.}_\gamma(\psi_\theta(\alpha,\gamma) + \psi_\theta(\gamma,\beta)) = \psi_\theta(\alpha,\beta)$$

Complex deformations in phase space for tori

$$\widetilde{\Pi}_\theta(\alpha,\beta) = e^{i\widetilde{\psi}_\theta(\alpha,\beta)}\widetilde{b}_\theta, \quad \zeta_{j,\theta}(\alpha, D_\alpha)\widetilde{\Pi}_\theta = 0, \quad \widetilde{\zeta}_{j,\theta}(\beta, D_\beta)\widetilde{\Pi}_\theta = 0$$

$$\psi_\theta(\alpha,\beta) := \mathrm{c.v.}_\gamma(\widetilde{\psi}_\theta(\alpha,\gamma) - \overline{\widetilde{\psi}_\theta(\beta,\gamma)}) \quad (B_{\theta,f} := \widetilde{P}_\theta f \widetilde{P}_\theta^*)$$

Why do we have $\mathrm{c.v.}_\gamma(\psi_\theta(\alpha,\gamma) + \psi_\theta(\gamma,\beta)) = \psi_\theta(\alpha,\beta)$?

$$\mathscr{C} := \{(\alpha, d_\alpha\widetilde{\psi}_\theta; \beta, -d_\beta\widetilde{\psi}_\theta)\} \subset S_1 \times S_2 \subset T^*\mathbb{C}^{2n} \times T^*\mathbb{C}^{2n}$$

$$S_1 = \bigcap_{j=1}^{n} \zeta_{j,\theta}^{-1}(0), \quad S_2 = \bigcap_{j=1}^{n} \widetilde{\zeta}_{j,\theta}^{-1}(0), \quad p_j : S_j \to S_1 \cap S_2 \simeq S_j/S_j^{\sigma_\theta}$$

$$\mathscr{C} = \{(\rho_1,\rho_2) \in S_1 \times S_2 : p_1(\rho_1) = p_2(\rho_2)\}, \quad \mathscr{C} \circ \mathscr{C} = \mathscr{C}$$

Lemma *If $S_1 \cap T^*\Lambda = S_2 \cap T^*\Lambda$, $\Lambda := \{(x + iG_\xi, \xi - iG_x)\}$ then $\mathscr{C} \circ \overline{\mathscr{C}}^t$ is idempotent.*

(Of course everything has to be done in the almost analytic category...)

We then follow the strategy of He–Sj '87 and Sj '96...

Viscosity: general strategy

$$\Psi^m_{\mathrm{hol}}(\mathbb{T}^2) \ni A : \mathscr{H}_\theta \to \mathscr{H}_\theta, \quad \Pi_\theta : L^2(T^*\mathbb{T}^2) \xrightarrow{\perp} T_\theta \mathscr{H}_\theta$$

$$T_\theta A S_\theta \Pi_\theta(\alpha, \beta) = e^{i\psi_\theta(\alpha,\beta)} a_\theta(\alpha, \beta)$$

$$\sigma_\theta(A) := a_\theta|_\Delta = a(x + i\theta G_\xi, \xi - i\theta G_x) + \mathcal{O}(\langle \xi \rangle^{m-1})$$

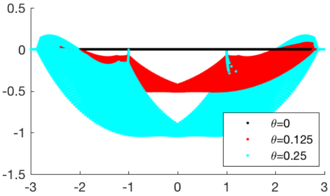
$$\mathscr{H}^s_\theta \hookrightarrow \mathscr{H}^r_\theta \quad \text{is compact if } s > r$$

$$H_p G > c_0 \Rightarrow |\sigma_\theta(P) - \omega| > c_1 \theta \Rightarrow P - \omega : \mathscr{H}^0_\theta \xrightarrow{\text{Fredholm}} \mathscr{H}^0_\theta$$

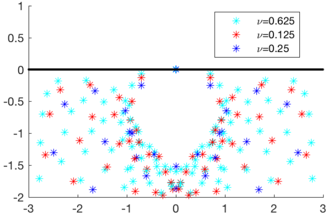
$$\exists\, Q = S_\theta \Pi_\theta q \Pi_\theta T_\theta, \quad q \in C^\infty_{\mathrm{c}}(T^*\mathbb{T}^2)$$

$$R_\nu(\omega) = (P + iQ + i\nu\Delta - \omega)^{-1} : \mathscr{H}^0_\theta \to \mathscr{H}^0_\theta, \ \ 0 \le \nu \le \nu_0, \ \ |\omega| < \delta$$

Study of $\det_{\mathscr{H}^0_\theta}(I - iR_\nu(\omega)Q)$ shows the continuity of spectra.



$\mathrm{Spec}_{\mathscr{H}_\theta}(P)$



$\mathrm{Spec}(P - i\nu\Delta_{\mathbb{T}^2})$

Thank you for your attention!