MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Malgorzata Marciniak **Email/Phone:** mmarciniak@lagcc.cuny.edu 5734620411

Speaker's Name: Martin Cramer Pedersen

Talk Title: Polyhedra and packings from hyperbolic honeycombs

Date: 10 /02 /2018 **Time:** 11 :00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences:

The EPINET project explores 2D hyperbolic tilings as a source of crystalline frameworks in 3D Euclidean space. The goal is to establish the simplest nets in hyperbolic space, from which Euclidean counterparts can be generated. The guiding principal is one of hyperbolic surface tiling, where the 3D crystallinity of an underlying surface induces 3–periodic networks. The extraordinary wealth of hyperbolic tilings allows us to enumerate networks and their spatial realizations with greater breadth than conventional approaches.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

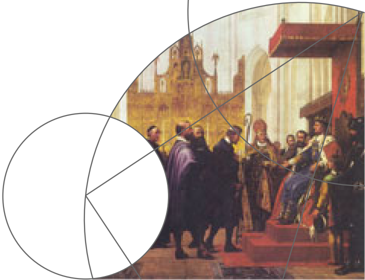
- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
    - **Computer Presentations**: Obtain a copy of their presentation
    - **Overhead**: Obtain a copy or use the originals and scan them
    - **Blackboard**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
    - **Handouts**: Obtain copies of and scan all handouts
- ☑ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Polyhedra and packings from hyperbolic honeycombs

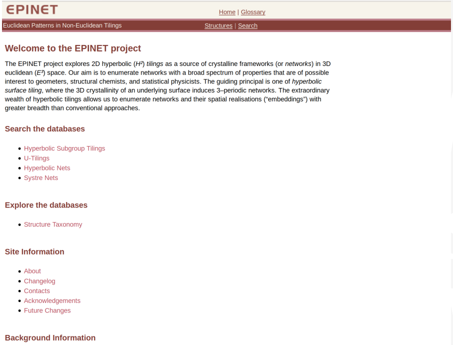
Hot Topics: Shape and Structure of Materials

MSRI, UC Berkeley, 1st - 5th of October, 2018

Martin Cramer Pedersen & Stephen Hyde
Niels Bohr Institute & Australian National University



In this talk



EPINET

Home | Glossary

Euclidean Patterns in Non-Euclidean Tilings

Structures | Search

Welcome to the EPINET project

The EPINET project explores 2D hyperbolic (H^2) *tilings* as a source of crystalline frameworks (or *networks*) in 3D euclidean (E^3) space. Our aim is to enumerate networks with a broad spectrum of properties that are of possible interest to geometers, structural chemists, and statistical physicists. The guiding principal is one of *hyperbolic surface tiling*, where the 3D crystallinity of an underlying surface induces 3–periodic networks. The extraordinary wealth of hyperbolic tilings allows us to enumerate networks and their spatial realisations ("embeddings") with greater breadth than conventional approaches.

Search the databases

- Hyperbolic Subgroup Tilings
- U-Tilings
- Hyperbolic Nets
- Systre Nets

Explore the databases

- Structure Taxonomy

Site Information

- About
- Changelog
- Contacts
- Acknowledgements
- Future Changes

Background Information

EPINET - Euclidean Patterns in Non-Euclidean Tilings

Ramsden *et al.* (2009), EPINET, http://epinet.anu.edu.au/

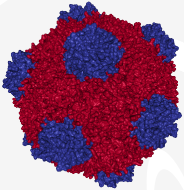
In this talk

The goal of this talk is to establish the simplest nets in hyperbolic space, from which we can generate Euclidean counterparts.

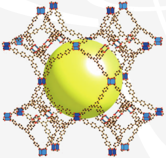
Specifically, we chase highly symmetric, infinite (i.e. 3-periodic) deltahedra in Euclidean space.

Our motivation:

- Templates for reticular chemistry
- Optimal packing geometries
- Enumeration of these structures



Cowpea Mosaic Virus
PDB **2bfu**

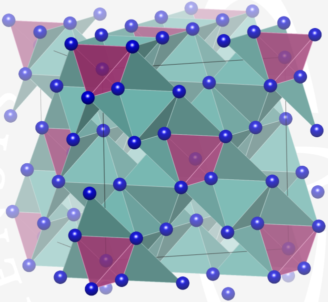


MOF-399
Furukawa *et al.* (2012)

In this talk

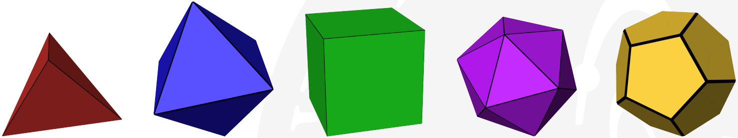
Agenda:

- Establish a bit of notation about nets and their symmetries
- Present a pipeline for generating nets in Euclidean space from nets in hyperbolic space
- Show a bunch of interesting nets generated from this pipeline
- Analyze and compare a subset of the generated nets



Platonic solids

Described by Plato around 360 B.C. (regular nets on $\mathbb{S}^2$):



Tetrahedron	Octahedron	Cube	Icosahedron	Dodecahedron
Fire	Air	Earth	Water	Aether

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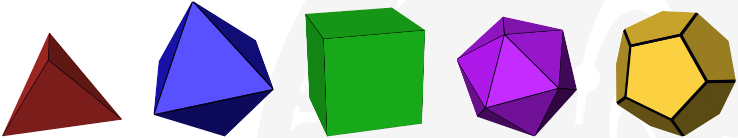

Tetrahedron	Octahedron	Cube	Icosahedron	Dodecahedron
Fire	Air	Earth	Water	Aether
$\{3,3\}$	$\{3,4\}$	$\{4,3\}$	$\{3,5\}$	$\{5,3\}$
Td	*Oh*	*Oh*	*Ih*	*Ih*

Platonic solids

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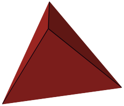
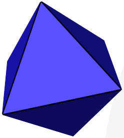


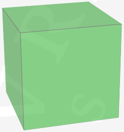
Tetrahedron
Fire

$\{3,3\}$
Td



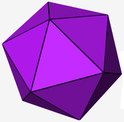
Octahedron
Air

$\{3,4\}$
Oh



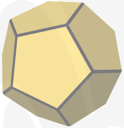
Cube
Earth

$\{4,3\}$
Oh



Icosahedron
Water

$\{3,5\}$
Ih



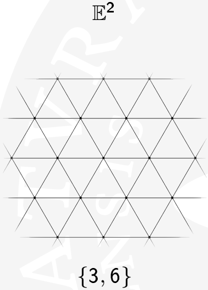
Dodecahedron
Aether

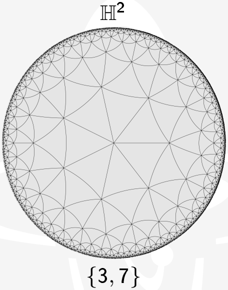
$\{5,3\}$
Ih

In this talk, we are particularly interested in the tetrahedron, the octahedron, and the icosahedron as they are composed of equilateral triangles; they are (regular) “deltahedra”.

Spaces and symmetries

$\mathbb{H}^2$ and $\mathbb{S}^2$ allow for nets that are forbidden in $\mathbb{E}^2$ - i.e. so-called non-crystallographic symmetries.



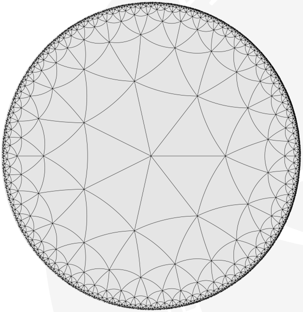


Symmetries are described by:

- 14 point groups in $\mathbb{S}^2$
- 17 wallpaper groups in $\mathbb{E}^2$
- Hyperbolic groups in $\mathbb{H}^2$

Spaces and symmetries

A few words about the Poincaré disk model of $\mathbb{H}^2$:



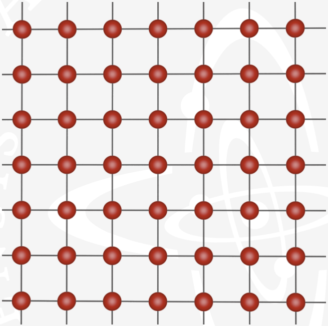
- A conformal model of the hyperbolic plane
  Angles are preserved!
- Geodesics appear as circle arcs (or straight lines through the center)
- These triangles are congruent and equilateral

Spaces and symmetries

The usual crystallographic notation will not work for $\mathbb{H}^2$.

We use the orbifold notation for symmetry groups:

- N means that our pattern has a unique N-fold symmetry point
- A $*$ means a mirror symmetry. $*$N means that N mirror lines meet in a point
- $\times$ means that our pattern has glide symmetry
- $\circ$ means that there is a translational symmetry in our pattern



Thurston (1980), Conway & Huson (2002)

Spaces and symmetries

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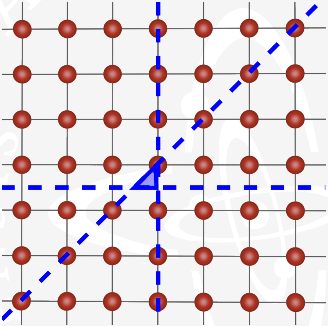


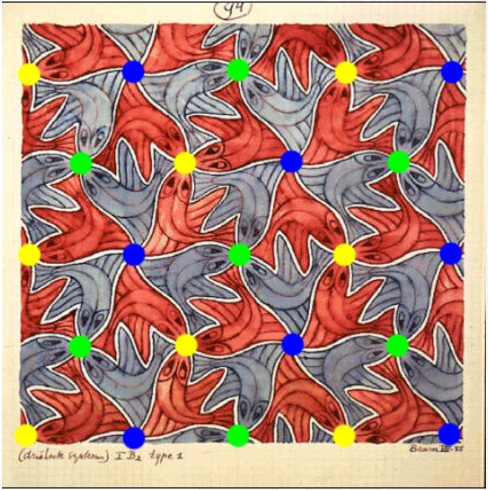
$*244$ or $p4mm$

Thurston (1980), Conway & Huson (2002)

Spaces and symmetries

Escher (1955)

Spaces and symmetries



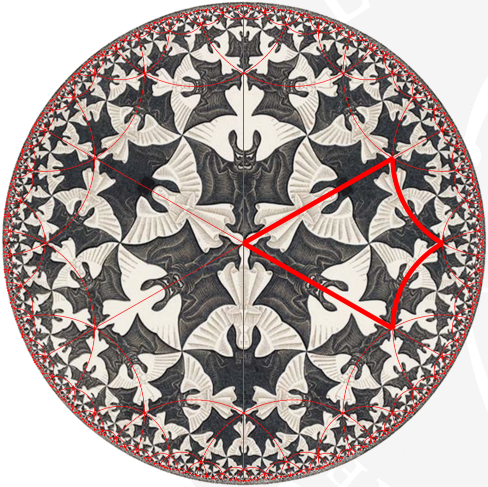
333 (or $p3$)

Escher (1955)

Spaces and symmetries

Conway *et al.* (2008), Escher (1960)

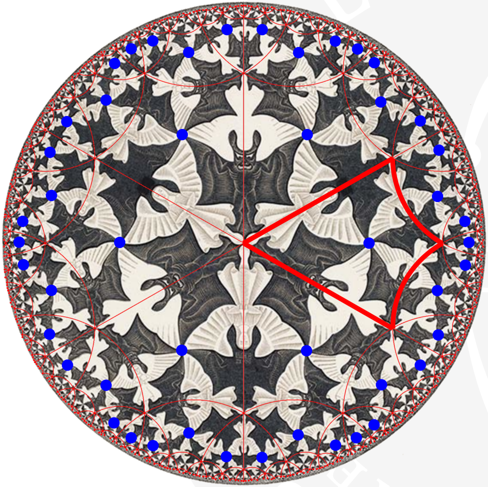
Spaces and symmetries



$*3333$

Conway *et al.* (2008), Escher (1960)

Spaces and symmetries

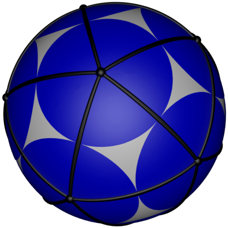


$*3333$ (not $4*3$)

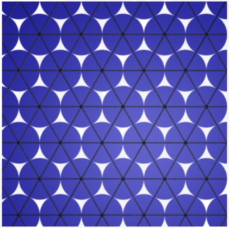
Conway *et al.* (2008), Escher (1960)

Spaces and symmetries

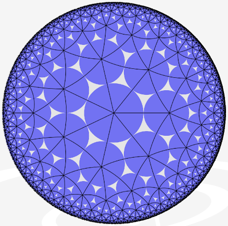
We can think of the embeddings from before as generalizations of the famous "penny packing" in $\mathbb{E}^2$ with $*23N$ symmetry:



$*235$



$*236$



$*237$

The density, ρ, of an N-contact packing is summed up beautifully in L. F. Tóth's formula:

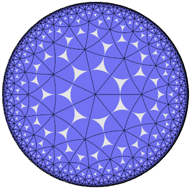
$$\rho(N) = \frac{3\csc\left(\frac{\pi}{N}\right) - 6}{N - 6}$$

The groups with orbifold symbols $*23N$ are called "honeycombs".

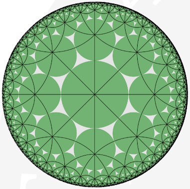
Tóth (1940)

Spaces and symmetries

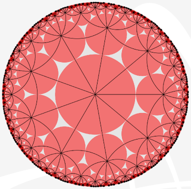
Most honeycombs are hyperbolic:



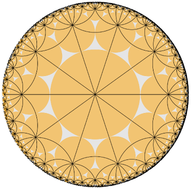
$\{3,7\}$, $*237$



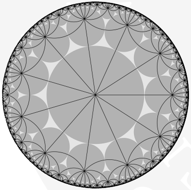
$\{3,8\}$, $*238$



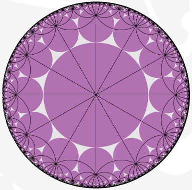
$\{3,9\}$, $*239$



$\{3,10\}$, $*23(10)$



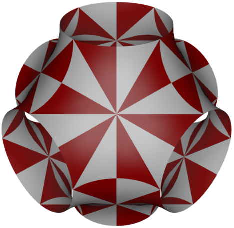
$\{3,11\}$, $*23(11)$



$\{3,12\}$, $*23(12)$

3-periodic minimal surfaces

Minimal surfaces are surfaces that locally minimize area (usually subjected to specific boundary conditions). The most symmetric 3-periodic minimal surfaces in $\mathbb{E}^3$ are known (for genus 3):



Primitive
$Im\bar{3}m$

Diamond
$Pn\bar{3}m$

Gyroid
$Ia\bar{3}d$

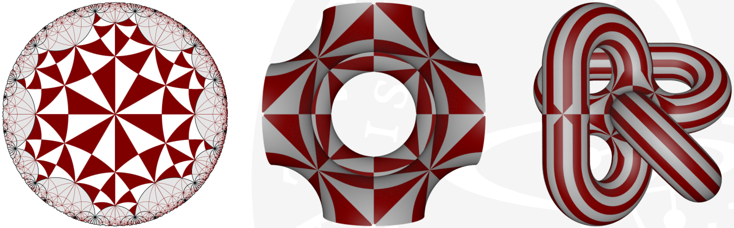
Their covering space is $\mathbb{H}^2$. And importantly, $*246$ symmetries in $\mathbb{H}^2$ can be lifted to symmetries in $\mathbb{E}^3$ via these surfaces.

Schwartz (1890), Schoen (1970), Bai *et al.* (2016)

3-periodic minimal surfaces

By identifying the opposing edges of a (hyperbolic) dodecagon, we can construct the TPMS.

Here, the P-surface:



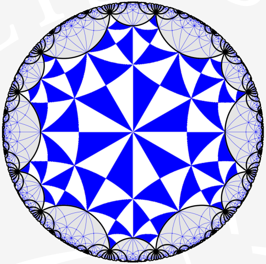
We have the unit cell of the P-surface:

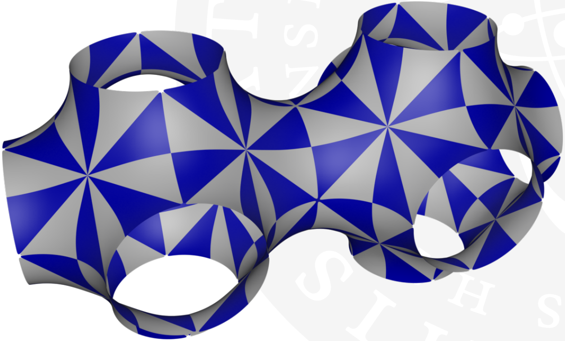
- in universal cover, $\mathbb{H}^2$, on the left (6 lattice vectors)
- as a 3-periodic surface in the middle (3 lattice vectors)
- as a 3-torus on the right (0 lattice vectors)

Sadoc & Charvolin (1989)

A walk on the P

Let us trace a "closed loop" on the P-surface by "jumping" from $*246$-domain to $*246$-domain





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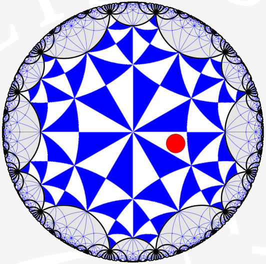

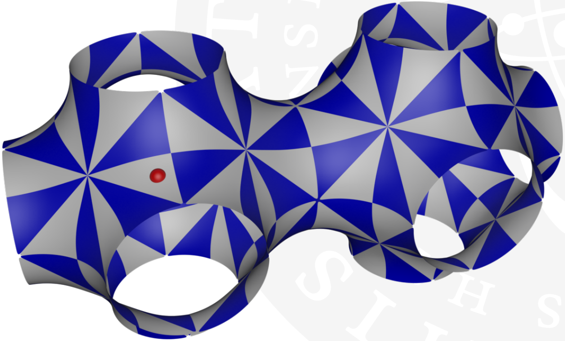

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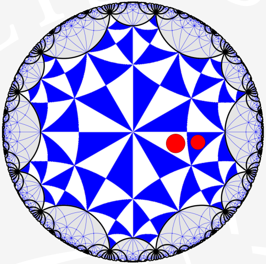

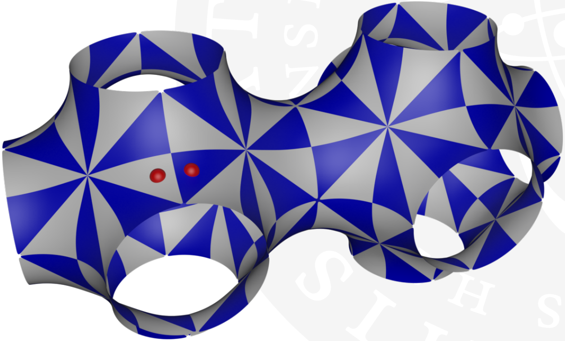

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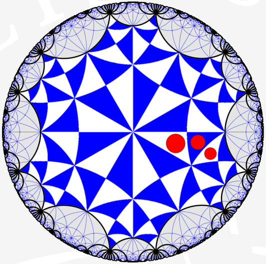

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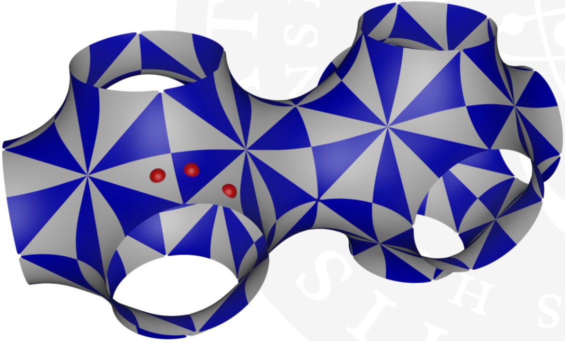

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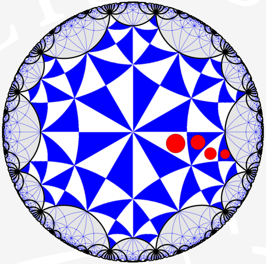

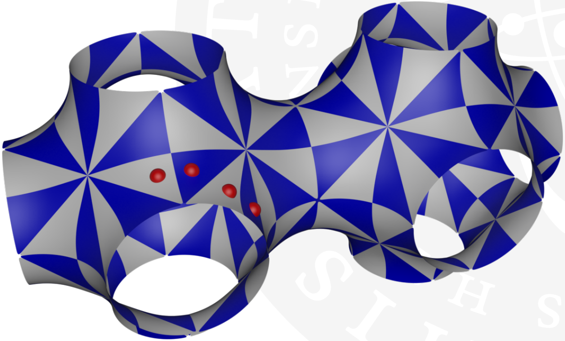

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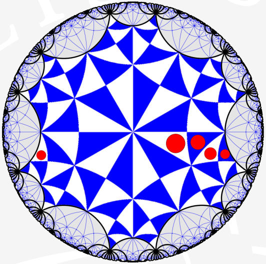

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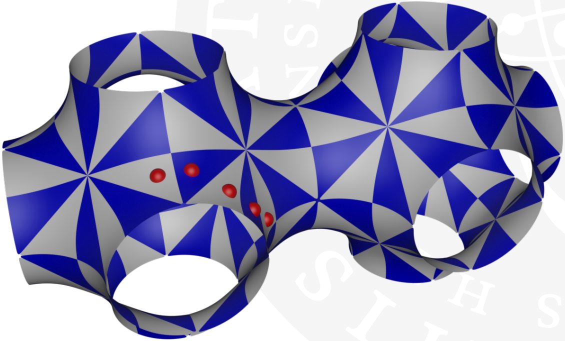

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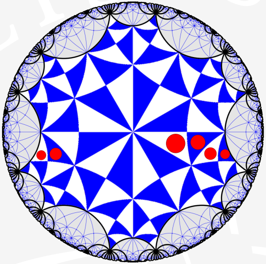

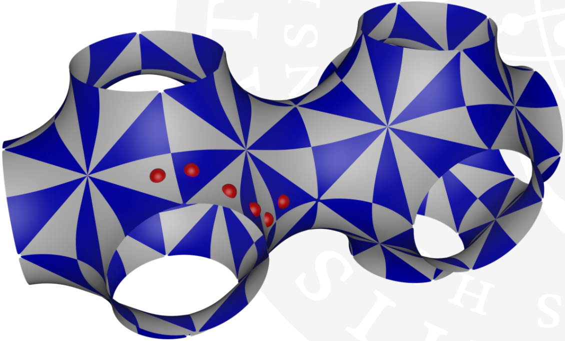

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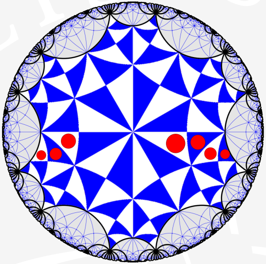

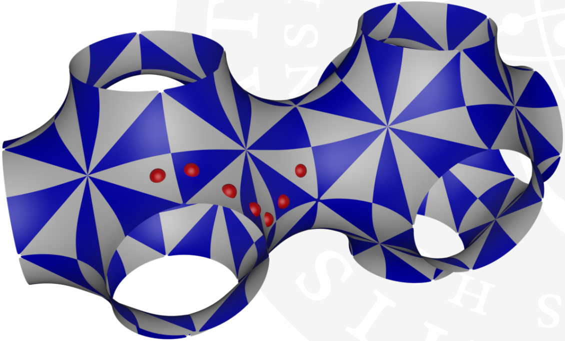

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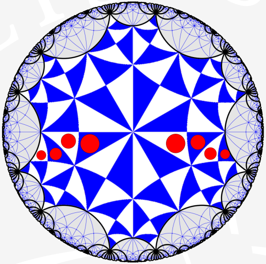

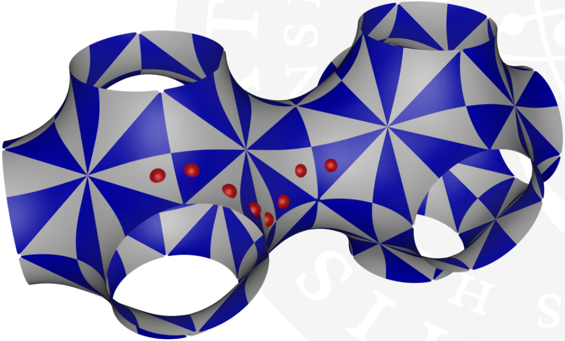

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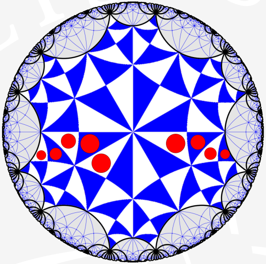

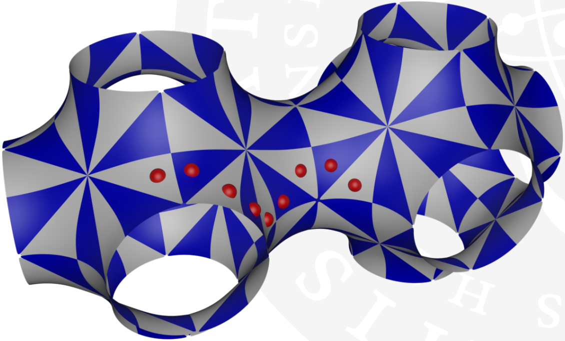

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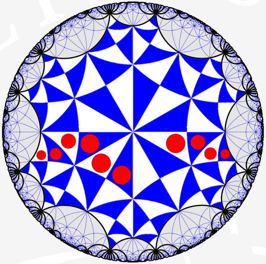

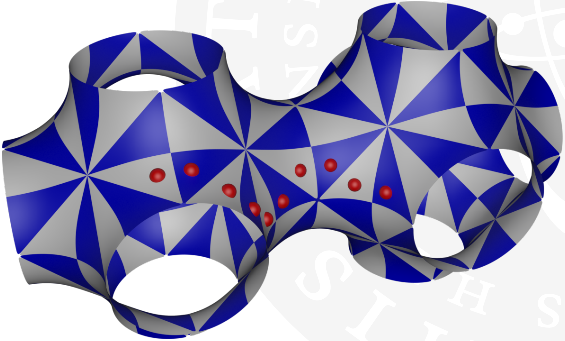

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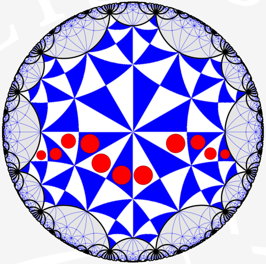

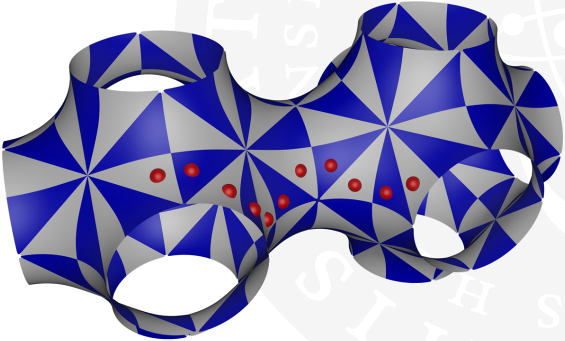

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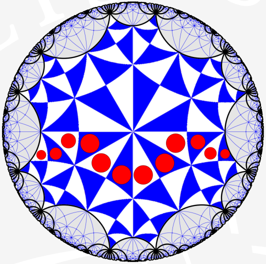

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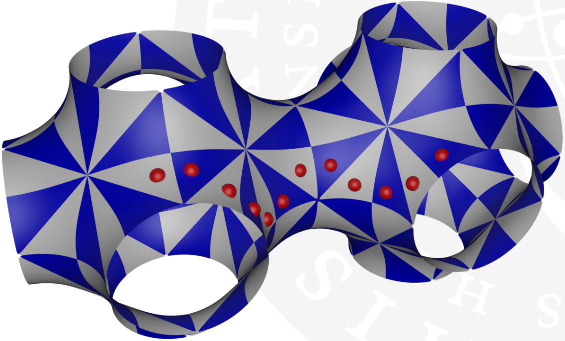

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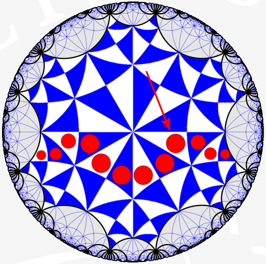
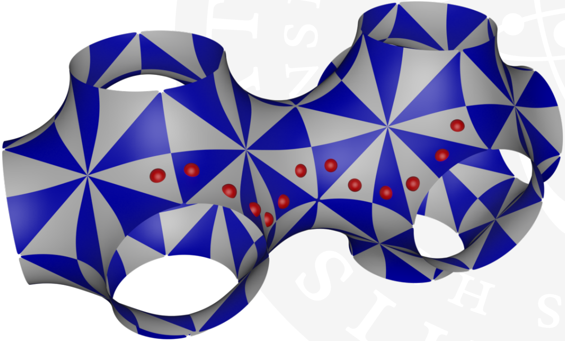
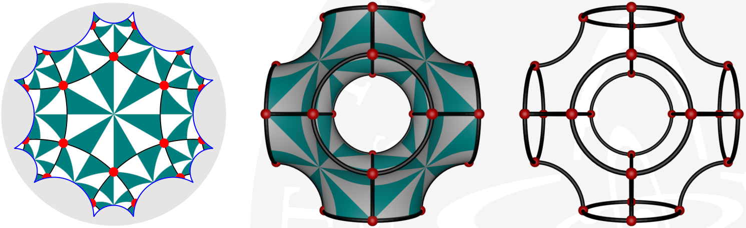




Lifting a net from $\mathbb{H}^2$ to $\mathbb{E}^3$

As a first example, we can decorate a $*246$ orbifold like this:



The pattern in $\mathbb{H}^2$ shares the symmetries of the P-surface, $Im\bar{3}m$ and $*246$ and is known as **sod** in databases. It outlines the cage-like structure of the mineral sodalite.

This works because the symmetries of the net in $\mathbb{H}^2$ are "commensurate" with the symmetries of the surface.

O'Keeffe *et al.* (2008), RCSR - http://rcsr.anu.edu.au

Lifting a net from $\mathbb{H}^2$ to $\mathbb{E}^3$

If we lift the same net from $\mathbb{H}^2$ onto the D-surface, we get a completely different net:

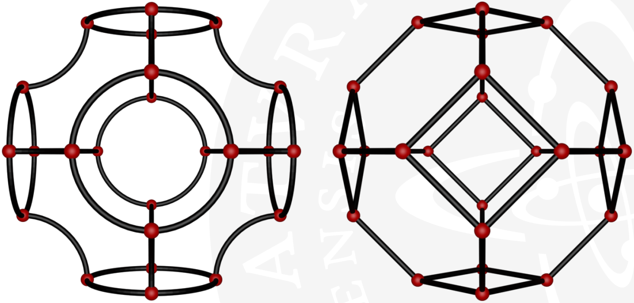
Here, we recover the well-known net **nbo** (named after the compound NbO) with extrinsic symmetry $Pn\bar{3}m$ and intrisic symmetry $*246$.

In general, for a commensurate decoration we get 4 nets in $\mathbb{E}^3$; 1 from the P- and $D-$ surface, and 1 from each of the 2 embeddings of the Gyroid.

Robins *et al.* (2005)

Relaxing a net

We usually distinguish between the emerging net and a relaxed version of the net (called the equilibrium placement).



In this case, the two structures are quite similar, but this process might change the structure quite a bit. Symmetries are preserved.

Delgado-Friedrichs & O'Keeffe (2005)

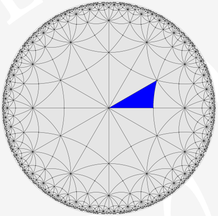
Commensurate symmetries

We know that the most symmetric periodic decoration of these surfaces we can construct will have symmetry $*246$.

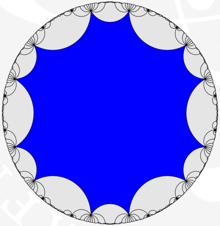
And the least symmetric one will just have translational symmetry (i.e. space group $P1$ and $\circ\circ\circ$).

So we need to know all symmetry groups, G, for which:

$$*246 \leq G \leq \circ\circ\circ$$



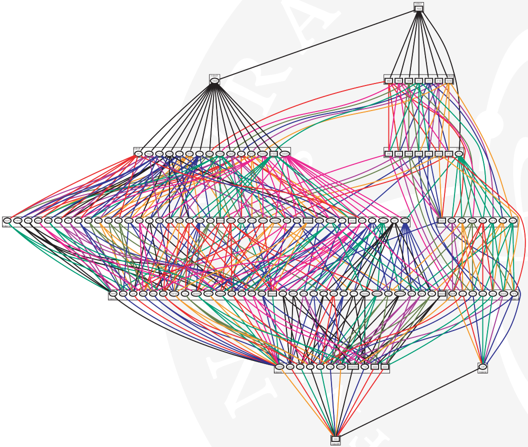
$*246$



$\circ\circ\circ$

Commensurate symmetries

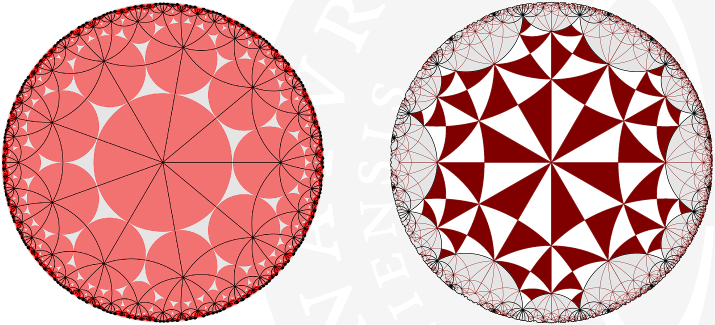
Robins and collaborators did this and ended up with 131 commensurate symmetry groups:



Robins *et al.* (2004)

Getting frustrated

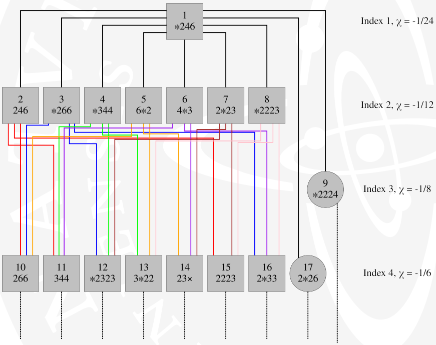
We now have the pieces needed to bring realise our hyperbolic disk packings in $\mathbb{E}^3$ via the TPMS.



All we need to do is deform the pattern on the left (with $*239$ symmetry) to be commensurate with the symmetries our TPMS (with $*246/\circ\circ\circ$ symmetry).

Getting frustrated

This can now be seen as a group theoretical problem, and GAP can help us find the shared subgroups of ∗23N and ∗246/∘∘∘:



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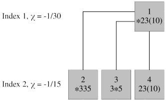

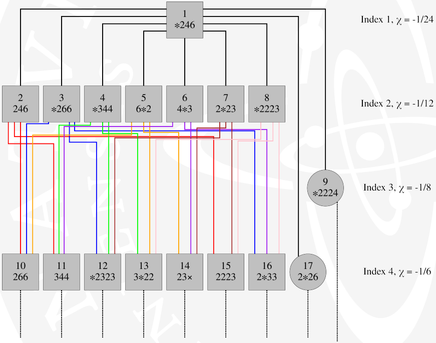

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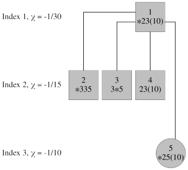

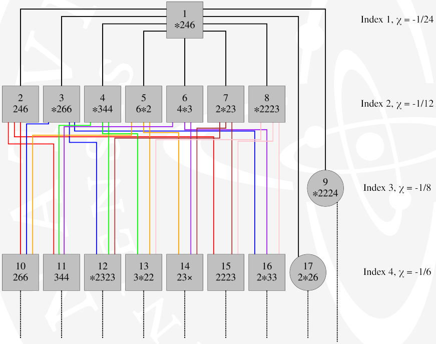

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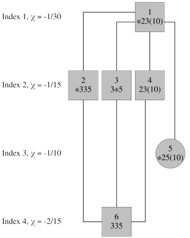

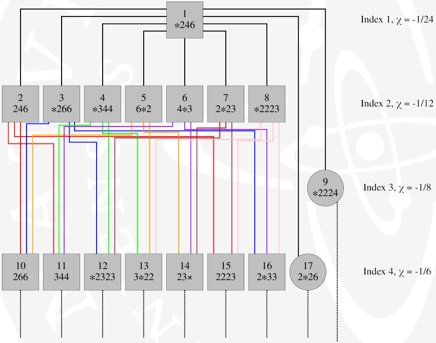

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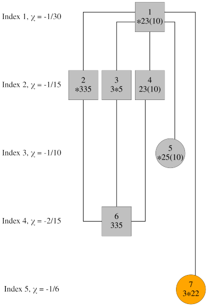
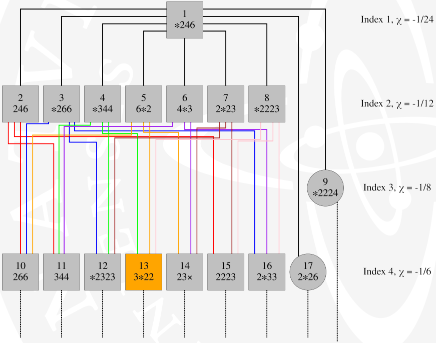
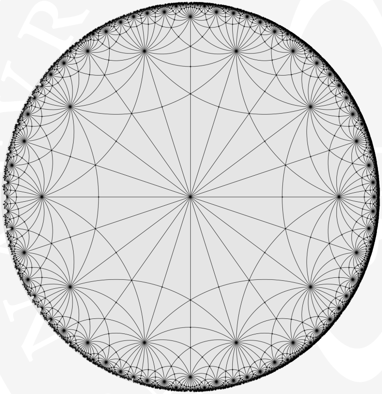




Getting frustrated

Visualizing the cosets of the subgroup can aid us a further:

- Starting with a tesselation of $\mathbb{H}^2$ of $*23(10)$ domains...



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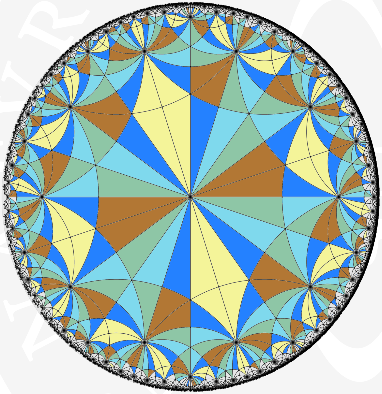
- Starting with a tesselation of $\mathbb{H}^2$ of $*23(10)$ domains...
- we can assign a color to each coset in the subgroup (using the subgroup generators)...



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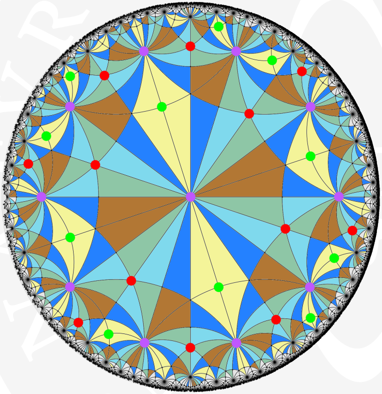

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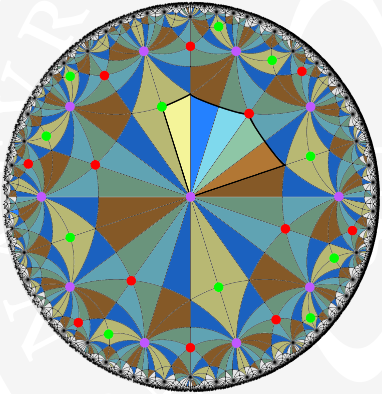

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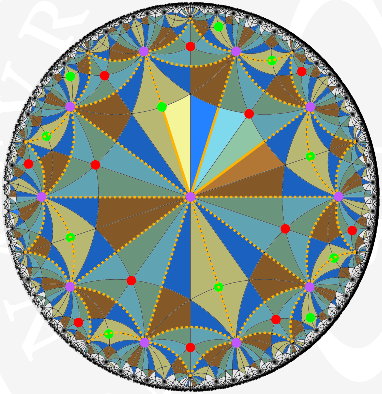
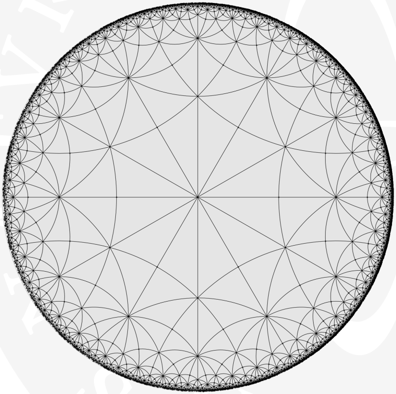
- and see how a single orbifold can be visualized...
- as well as the decoration needed to build the $[3,10]$-net.



Getting frustrated

Now we know how to deform the $\{3, 10\}$ net into a net we can lift to $\mathbb{E}^3$ via the TMPS and $*246$:

- Starting from a tesselation of $\mathbb{H}^2$ of $*246$ domains...



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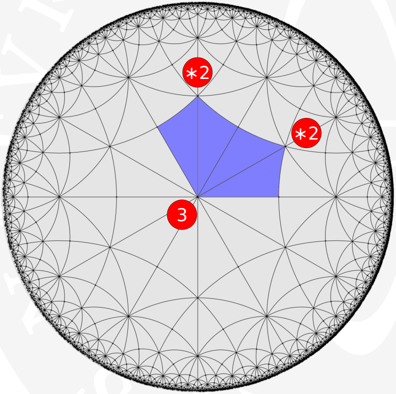

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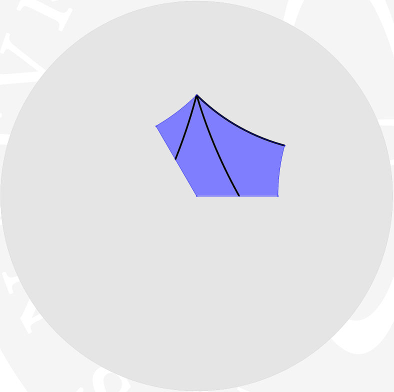

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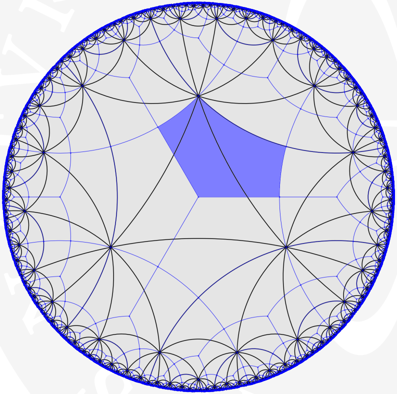

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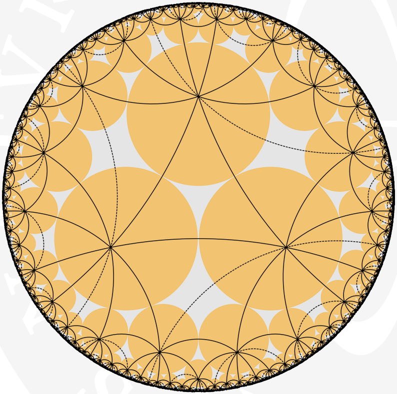
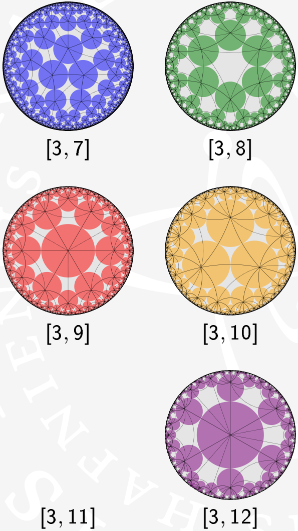
- we can construct a tesselation of commensurate $3*22$ domains...
- and decorate the tesselation with the motif we just found...
- and get a $[3, 10]$ commensurate with the symmetries of our TMPS.
- We can think of the pattern as an 8-connected packing of disks.



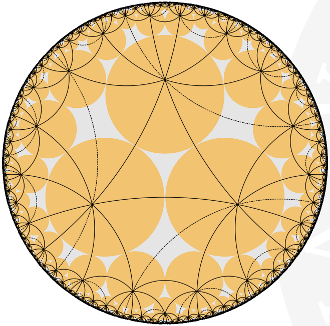
Getting frustrated

Net	Orbifold
$\{3,7\}$	$23\times$ 2223
$\{3,8\}$	$2{*}23$
$\{3,9\}$	$2{*}23$
$\{3,10\}$	$3{*}22$
$\{3,12\}$	$2{*}33$



From $\mathbb{H}^2$ to $\mathbb{E}^3$

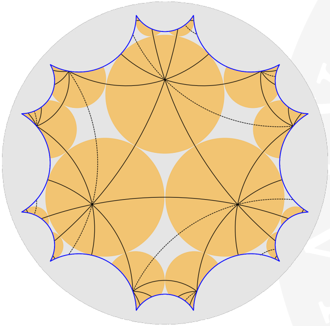
Due to the commensurate symmetry, we can visualize this as a net on the TMPS:



Delgado-Friedrichs & O'Keeffe (2005)

From $\mathbb{H}^2$ to $\mathbb{E}^3$

Due to the commensurate symmetry, we can visualize this as a net on the TMPS:



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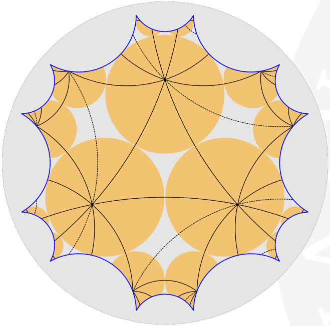

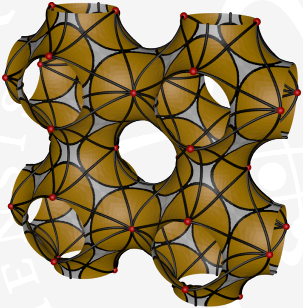

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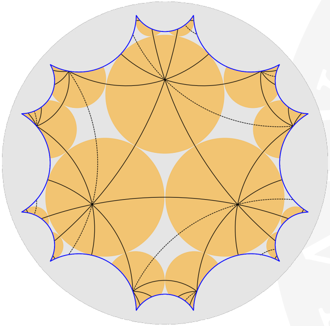
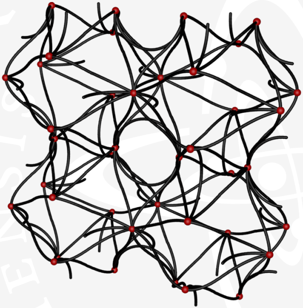




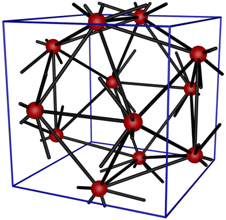
Using Systre, we compute equilibrium placements for all of our nets generated using the different surfaces.

Delgado-Friedrichs & O'Keeffe (2005)

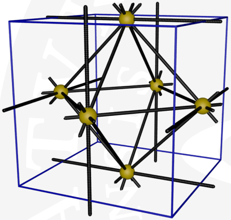
From $\mathbb{H}^2$ to $\mathbb{E}^3$

Generally, we get 4 different nets from each pattern in $\mathbb{H}^2$.

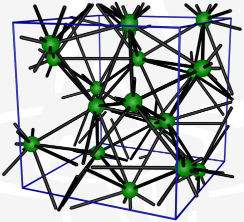
However, in this case, our mirror symmetry does not lift to the Gyroid, so the resulting nets from the two embeddings via the Gyroid are identical.



[3, 10] on $3{*}22$ via P
$Pm\bar{3}$



[3, 10] on $3{*}22$ via D
$P23$



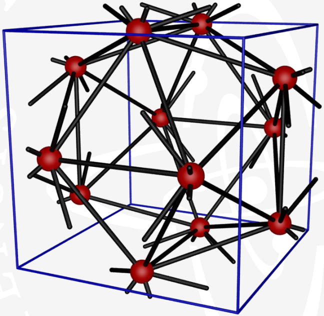
[3, 10] on $3{*}22$ via G
$I2_13$

Note that the net on the P is 9-valent. Edges may merge when computing canonical embeddings.

Hyde *et al.* (2014)

A note on enumeration

- We can "name" out honeycomb nets by their parent hyperbolic net, their (subgroup) orbifold symbol, and the surface on with which we lift them to $\mathbb{E}^3$.

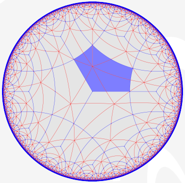


$[3, 10]$ on $3{*}22$ via P

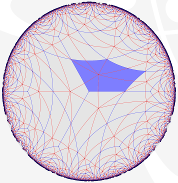
Ramsden *et al.* (2009), Evans *et al.* (2015), Kolbe & Evans (2018)

A note on enumeration

- We can "name" out honeycomb nets by their parent hyperbolic net, their (subgroup) orbifold symbol, and the surface on with which we lift them to $\mathbb{E}^3$.
- However:
  - Two subgroups can have the same orbifold symbol
  - Some subgroups "sit" in the $*246$ lattice in infinitely many ways
  - Some orbifolds have non-trivial symmetries (i.e. symmetries not in the $*246$ lattice)



$[3, 7]$ on 2223_{01}



$[3, 7]$ on 2223_{11}

Ramsden *et al.* (2009), Evans *et al.* (2015), Kolbe & Evans (2018)

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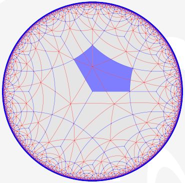
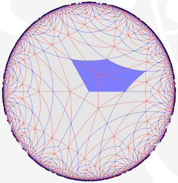
  - Some subgroups "sit" in the $*246$ lattice in infinitely many ways
  - Some orbifolds have non-trivial symmetries (i.e. symmetries not in the $*246$ lattice)
- Enumeration is hard. Computations get hard for very elongated orbifolds.



$[3, 7]$ on 2223_{01}



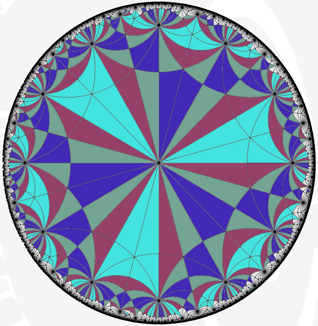
$[3, 7]$ on 2223_{11}

Ramsden *et al.* (2009), Evans *et al.* (2015), Kolbe & Evans (2018)

From $\mathbb{H}^2$ to $\mathbb{E}^3$

For $N \in \{7, 8, 9, 10, 12\}$, we:

- Determined all subgroups of 23N down to and including those that could be found as index-4 subgroups in $*246$

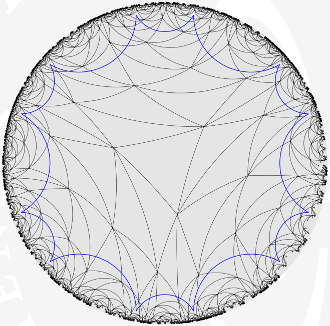


$2{*}33$ in $*23(12)$

From $\mathbb{H}^2$ to $\mathbb{E}^3$

For $N \in \{7, 8, 9, 10, 12\}$, we:

- Determined all subgroups of 23N down to and including those that could be found as index-4 subgroups in $*246$
- Built 35+ nets in $\mathbb{H}^2$ using these groups



$[3, 7]$ on $23\times$ (A)

From $\mathbb{H}^2$ to $\mathbb{E}^3$

For $N \in \{7, 8, 9, 10, 12\}$, we:

- Determined all subgroups of 23N down to and including those that could be found as index-4 subgroups in $*246$
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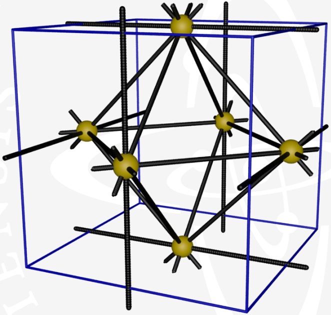
- These nets yielded 100+ nets in $\mathbb{E}^3$ from the outlined pipeline



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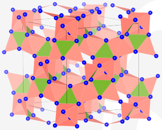
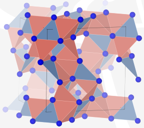
- We found 10 infinite deltahedra in our search. 5 are not reported elsewhere. Maybe?



$[3, 7]$ on 2223_{13} via G
pyc

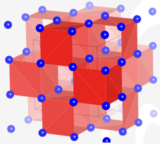


$[3, 7]$ on 2223_{11} via P
nca

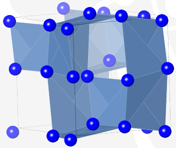
From $\mathbb{H}^2$ to $\mathbb{E}^3$

For $N \in \{7, 8, 9, 10, 12\}$, we:

- Determined all subgroups of 23N down to and including those that could be found as index-4 subgroups in $*246$
- Built 35+ nets in $\mathbb{H}^2$ using these groups
- These nets yielded 100+ nets in $\mathbb{E}^3$ from the outlined pipeline
- We found 10 infinite deltahedra in our search. 5 are not reported elsewhere. Maybe?
- As well as many other interesting structures

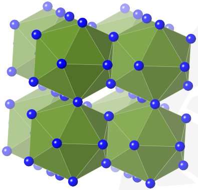


$[3, 9]$ on 2223_{31} via P

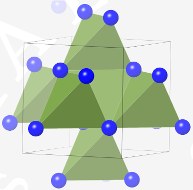


$[3, 12]$ on $2{*}33$ via G

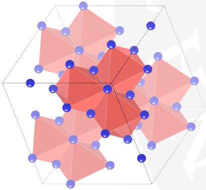
From $\mathbb{H}^2$ to $\mathbb{E}^3$



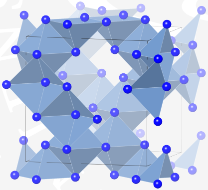
$[3,10]$ on $3{*}22$ via P
shy



$[3,12]$ on $2{*}33$ via P
xay

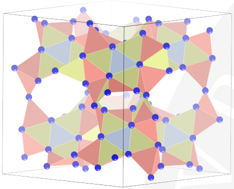


$[3,9]$ on $2{*}23$ via D
uty

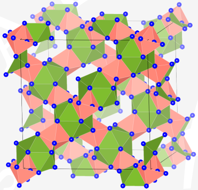


$[3,8]$ on 2223_{01} via G
tes

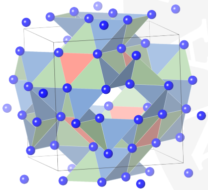
From $\mathbb{H}^2$ to $\mathbb{E}^3$



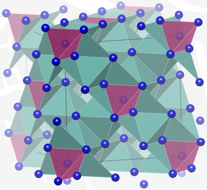
[3, 7] on 23× (A) via D
svm



[3, 7] on 23× (B) via D
svu



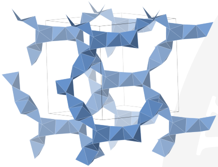
[3, 7] on 23× (B) via P



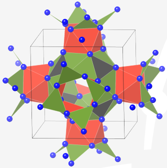
[3, 9] on 2223_{21} via D

From $\mathbb{H}^2$ to $\mathbb{E}^3$

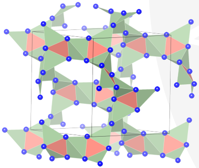
Other structures are nearly deltahedra:



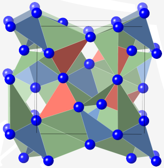
11%
[3, 7] on 2223_{13} via G



5%
[3, 7] on 2223_{11} via P (**svn-x**)

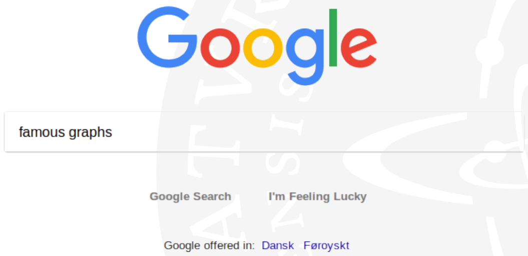


9%
[3, 8] on 2223_{12} via G (**lcv-e**)



13%
[3, 10] on $3{*}22$ via G

Looking for fame



Looking for fame

The 7-valent Klein graph [edit]

This graph is a 7-regular graph with 24 vertices and 84 edges, named after Felix Klein.

It is a Hamiltonian graph. It has chromatic number 4, chromatic index 7, radius 3, diameter 3 and girth 3.

It can be embedded in the genus-3 orientable surface, where it forms the dual of the "Klein map", with 56 triangular faces, Schläfli symbol $\{3,7\}_8$.[4]

It is the unique distance-regular graph with intersection array $\{7,4,1;1,2,7\}$; however, it is not a distance-transitive graph.[5]

Algebraic properties [edit]

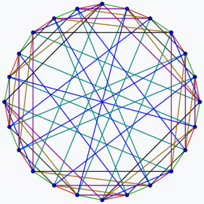
The automorphism group of the 7-valent Klein graph is the same group of order 336 as for the cubic Klein map, likewise acting transitively on its half-edges.

The characteristic polynomial of this Klein graph is equal to $(x-7)(x+1)^7(x^2-7)^8$.[6]

References [edit]

1. ^ Wolz, Jessica; *Engineering Linear Layouts*

The (7-valent) Klein graph



The 24-Klein graph

Named after	Felix Klein
Vertices	24
Edges	84
Radius	3
Diameter	3
Girth	3
Automorphisms	336
Chromatic number	4
Chromatic index	7
Properties	Symmetric Hamiltonian

Table of graphs and parameters

Looking for fame

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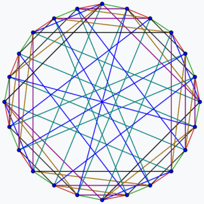

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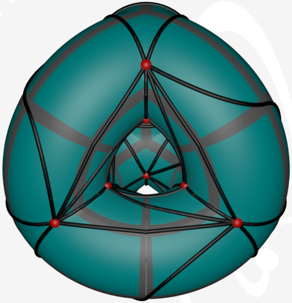
Table of graphs and parameters

Looking for fame

Due to computational constraints, we generated these [3, 7]-nets on the three-torus as well the Klein graph:

- 2223_{01}
- 2223_{41}
- 2223_{11}
- 2223_{12}
- 2223_{13}
- 2223_{32}
- 2223_{14}
- 2223_{21}
- 2223_{23}
- 2223_{31}
- Klein
- $23\times$ (A)
- $23\times$ (B)



[3, 7] via 2223_{01}

Dress (1987)

Looking for fame

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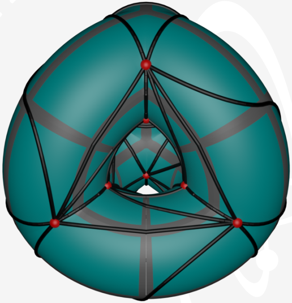


$[3, 7]$ via 2223_{01}

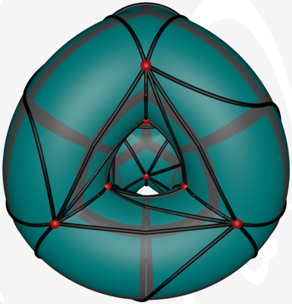
Using an invariant representation of the surface embedding (and double-checking with Delaney-Dress symbols), these embeddings are equilavent.

Dress (1987)

Looking for fame

Nauty and GAP can establish the group of automorphisms for each of our nets. Nauty classifies our [3, 7] nets as:

Orbifold symbol	$\|Aut(G)\|$
2223_{01} 2223_{41}	24
2223_{11}	24
2223_{12}	24
2223_{13} 2223_{32}	24
2223_{14}	24
2223_{21}	24
2223_{23}	24
2223_{31} Klein	336
$23\times$ (A)	24
$23\times$ (B)	24



[3, 7] via 2223_{01}

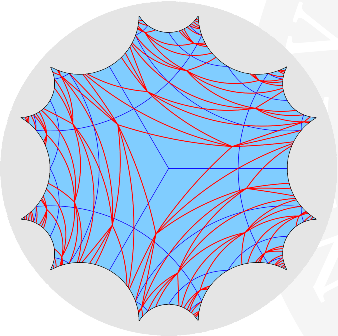
As before, we see that the [3, 7] net built via 2223_{31} is indeed the Klein graph.

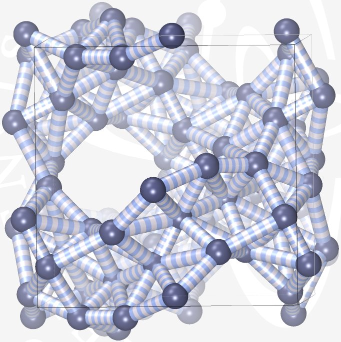
McKay & Piperno (2013)

Looking for fame

And in conclusion, we have produced four 3-periodic embeddings of the 7-valent Klein graph.

The prettiest ones is built via one of the embeddings of the Gyroid:

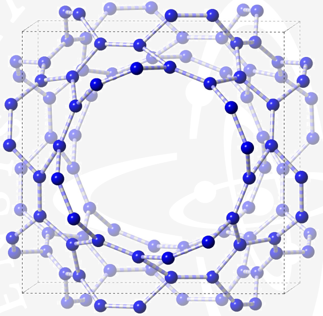




Outlook

Next steps:

- Very symmetric Schwarzites - i.e. the surface duals of the triangulations

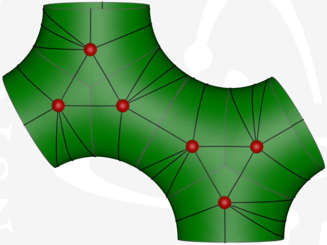


RCSR, http://rcsr.anu.edu.au, EPINET, http://epinet.anu.edu.au/

Outlook

Next steps:

- Very symmetric Schwarzites - i.e. the surface duals of the triangulations
- Embeddings via sub-periodic surfaces (and other surfaces) - e.g. the [3, 7] on 2223 on an HCB-surface on the right



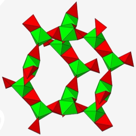
RCSR, http://rcsr.anu.edu.au, **EPINET**, http://epinet.anu.edu.au/

Outlook

Next steps:

- Very symmetric Schwarzites - i.e. the surface duals of the triangulations
- Embeddings via sub-periodic surfaces (and other surfaces) - e.g. the [3, 7] on 2223 on an HCB-surface on the right
- Curating and comparing our nets to existing databases such as RCSR and EPINET

pmc



RCSR reference: http://rcsr.net/nets/pmc

names: 3^7 G
key words: chiral

embed type	space group	volume	density	genus	td10	deg freedom
3	I4(1)32	144.1416	0.3330	49	416	4

a	b	c	alpha	beta	gamma
5.2432	5.2432	5.2432	90.0	90.0	90.0

vertices: 1

vertex	cn	x	y	z	symbolic	Wyckoff	symmetry	order
V1	6	0.2725	0.2998	0.4162	x, y, z	48 i	1	1

vertex	cs_1	cs_2	cs_3	cs_4	cs_5	cs_6	cs_7	cs_8	cs_9	cs_{10}	cum_{10}	vertex symbol
V1	6	9	12	16	24	32	48	64	96	108	416	3.3.3.3.3.3.3.4.4.20(45927).20(45927).20(45927).20(45927).20(52488).20(524

edges: 5

edge	x	y	z	symbolic	Wyckoff	symmetry
E1	0.3443	0.2862	0.3580	x, y, z	48 i	1
E2	0.3032	0.3750	0.4468	x, 3/8, 3/4-x	24 h	2

RCSR, `http://rcsr.anu.edu.au`, **EPINET**, `http://epinet.anu.edu.au/`

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Pedersen & Hyde
Proc. Natl. Acad. Sci. 115, 6905-6910 (2018)

Surface embeddings of the Klein and the Möbius-Kantor graphs
Pedersen, Delgado-Friedrichs, & Hyde
Acta Crystallogr. Sect. A 74, 223-232 (2018)

Hyperbolic crystallography of two-periodic surfaces and associated structures
Pedersen & Hyde
Acta Crystallogr. Sect. A 73, 124–134 (2017)

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Houdini 3D ANIMATION TOOLS

GAP 4

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- Vanessa Robins
- Olaf Delgado-Friedrichs

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- Myfanwy Evans
- Benedikt Kolbe