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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Malgorzata Marciniak **Email/Phone:** mmarciniak@lagcc.cuny.edu 5734620411

Speaker's Name: Timo de Wolff

Talk Title: An Introduction to Computational Algebraic Geometry and Polynomial Optimization

Date: 10 / 02 /2018 **Time:** 2 : 00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences:

Polynomial optimization problems (POP) minimize a multivariate real polynomial given finitely polynomial inequalities as constraints. Semidefinite programming using the classical sums of squares (SOS) certificates is compared with relative entropy programming using sums of nonnegative circuit polynomials (SONC) certificates.

CHECK LIST

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- ☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
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Speaker: Timo de Wolff

Title: **An Introduction to Computational Algebraic Geometry and Polynomial Optimization**

Note Taker: Malgorzata Marciniak

Let $f_1, \ldots, f_r \in \mathbb{C}[\boldsymbol{z}] = \mathbb{C}[z_1, \ldots, z_n]$ be polynomials. The associated ideal is defined as:

$$\langle f_1, \ldots, f_r \rangle := \left\{ \sum_{j=1}^{r} s_j f_j : s_j \in \mathbb{C}[\boldsymbol{z}] \right\} \subseteq \mathbb{C}[\boldsymbol{z}]$$

The associated variety is the Zariski-closure of the joint solution set:

$$V(f_1, \ldots, f_r) := \overline{\{\boldsymbol{z} \in \mathbb{C}^n : f_1(z) = \cdots = f_r = 0\}}$$

Two methods for solving: symbolic with Grobner basis and numerical with homotopy continuation

<u>Symbolic method</u>

Hilbert Nullstellensatz.

If ideals are generated by a single polynomial, then the membership can be tested polynomial division

If ideals are generated by more than one polynomial, then a similar procedure to polynomial division can be performed but not all sets of generators are suitable for the polynomial division.

Example: Let $f_1 = xy - 1$ and $f_2 = yz - 1$. Consider $f_3 = x - z$ and lexicographic order. We have $f_3 = zf_1 - xf_2 \in \langle f_1, f_2 \rangle$ however, a multivariate version of polynomial division always leads to a reminder.

Solution: use Grobner basis introduced by Buchberger in 1965 and independently by Hironaka in 1964

Theorem (Buchberger): Grobner bases always exist and are computable in finite time.

The algorithm is very expensive and requires exponential space and time to compute

Software: SINGULAR, Macaulay2, Magma, Maple

<u>Homotopy continuation</u>

Morgan, Sommese, Wampler 1990

Let us assume that the supports (exponents) are fixed but the coefficients are not. The system may be solved for some coefficients and then deformed to the desired system (keep track of the solutions numerically).

This is a very fast method, but problems need to be well conditioned: endpoints may run into infinity or singularity, over the real numbers ill conditioned area may not be avoidable. Complx and real dimensions of solutions do not agree.

Software: Bertini, Hom4PS, PHCpack, HomotopyContinuation.jl

Examples of polynomial optimization: image reconstruction, portfolio optimization, Max-cut (find a subgraph S I graph V to maximize the number of edges between S and V\S).

Motivation: Let $f, g_1, \ldots, g_s \in \mathbb{R}[\boldsymbol{x}] = \mathbb{R}[x_1, \ldots, x_n]$. Consider the constrained polynomial optimization problem (CPOP)

$$\min_{g_1,\ldots,g_s \geq 0} f(\boldsymbol{x})$$

Which is a non-convex optimization problem.

Solving CPOP is equivalent to computing

$$f_K^* = sup\{\gamma \in \mathbb{R} | f(\boldsymbol{x}) - \gamma \geq 0 \; for \; all \; \boldsymbol{x} \in K\}$$

With

$$K := \{\boldsymbol{x} \in \mathbb{R}^n | g_1(\boldsymbol{x}), \ldots, g_s(\boldsymbol{x}) \geq 0\}$$

This is an algebraic problem: *f* in nonnegative on semialgebraic set K (classic problem in algebraic geometry), which is NP-hard.

Idea: find certificates of non-negativity, for example sum of squares (SOS), which is a convex optimization problem.

Consider a vector space that consists of polynomials of *n* variables and degree at most *d:* $\mathbb{R}[\boldsymbol{x}]_{n,d}$ and define a cone of nonnegative polynomials as

$$P_{n,2d} = \left\{f \in \mathbb{R}[\boldsymbol{x}]_{n,2d} : f(\boldsymbol{x}) \geq 0 \; for \; all \; \boldsymbol{x} \in \mathbb{R}^n \right\}$$

Inside of $P_{n,2d}$ there is $\Sigma_{n,2d}$ the cone of polynomials that are sum of squares.

Theorem (Hilbert 1888): $P_{n,2d}$=$\Sigma_{n,2d}$ if and only if *n=1* or *2d=2* or *(n,2d)=(2,4)*.

Example: The Motzkin Polynomial (1965) is contained in $P_{2,6}$ but not in $\Sigma_{2,6}$:

$$M(x, y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1$$

Theorem (Blekherman: 2006, very rough version): For fixed degree $2d \geq 4$ almost every nonnegative polynomial is not a sum of square as *n* tends to infinity.

How to tackle applications?

Procedure: Model application as CPOP. Choose a certificate of nonnegativity (such as SOS) Translate the problem into a convex optimization problem. Attack convex optimization problem with suitable solver.

Warning: most applications cannot be solved straightforwardly

Sums of Squares in Practice

Advantages: SOS certificates provide a hierarchy of lower bounds for CPOPs under mild assumptions. There exist several preprocessing techniques. There exists software to translate the CPOP into a corresponding SDP. There exists various software to SDPs (SeDuMi, SDPT3, Mosek).

Issues: Numerical issues can occur, Convex optimization problems are, given a starting point, solvable in polynomial time in the input size. For SDPs corresponding to SOS certificates of n-variate degree d polynomials these are matrices of degree $\binom{n+d}{d}$.

Possible improvements: Create better solvers, find better representations of problems (make computations faster and more stable). Relax SOS certificates further, find new ways to certify negativity.

Circuit Polynomials

Definition: Define the set of circuit polynomials

$$f = \sum_{j=0}^{n} b_j \boldsymbol{x}^{\alpha(j)} + c\boldsymbol{x}^{\beta} \; \in \mathbb{R}[x_1, \ldots, x_n]$$

With the following properties:

- $\Delta \coloneqq New(f) = conx\{\alpha(0, \ldots, \alpha(n), \beta\} \subset \mathbb{R}^n$ is a simplex ($New(f)$ Newton polytope is the convex hull of all exponents)

- $\beta = \sum_{i=0}^{n} \lambda_i \alpha(i)$ with $\sum_{i=0}^{n} \lambda_i = 1$ and $\lambda_i > 0$

- for all j: $b_j > 0$ and $\alpha(j) \in (2\mathbb{N})^n$

Deciding nonnegativity of the circuit polynomials is very easy.

First define circuit number:

$$\Theta_f \coloneqq \prod_{j=0}^{n} \left(\frac{b_j}{\lambda_j}\right)^{\lambda_j}$$

Theorem (Iliman, dW 2014) known for special cases before: The following statements are equivalent for circuit polynomials

- F is nonnegative
- $|c| \leq \Theta_f$ or f is a sum of monomial squares

Definition: Let the set of sums of nonnegative circuits polynomials (SONC) be:

$$C_{n,2d} := \left\{ f \in \mathbb{R}[\boldsymbol{x}]_{n,2d} : f = \sum_{i=0}^{k} g_i \ \textit{for all } i \ g_i \textit{ is a nonnegative circuit polynomial} \right\}$$

Theorem (Iliman, dW 2014)

$C_{n,2d}$ is a convex cone in $P_{n,2d}$ which satisfies: $C_{n,2d} \subseteq \Sigma_{n,2d}$ if and only if $\Sigma_{n,2d} = \mathrm{P}_{n,2d}$.

Theorem (Dresser, Iliman, dW 2016)

For every n, d the cone $\mathrm{C}_{n,2d}$ is full dimension in $\mathrm{P}_{n,2d}$.

Problem: how one can check efficiently whether a polynomial has a SONC decomposition

POEM: a software for SONC Certificates (Effective Methods in Polynomial Optimization) written in Python.

Experimental comparison of SONC and SOS for unconstrained polynomial optimization.