Open Problems in subfactors and unitary $\otimes$-cats

Def: A UTC is a TC $\mathcal{C}$ [rigid, Cauchy-complete, simple 1] w/ $*: \mathcal{C}(a\to b) \to \mathcal{C}(b \to a)$ $\forall a,b \in \mathcal{C}$ satisfying $f^{**}=f$, $(f\circ g)^* = g^* \circ f^*$, $(f\otimes g)^* = f^* \otimes g^*$, and s.t. all associators are unitary ($u^*u=\mathrm{id}$ and $uu^*=\mathrm{id}$)

Thm (Longo-Roberts): $\forall c \in \mathcal{C}$, $\mathcal{C}(c \to c)$ is finite dim'l.

Cor: All UTC's are semisimple.

Examples:

$1 \xrightarrow{\upsilon} H \boxtimes K$ and $K \boxtimes H \xrightarrow{\Lambda} 1$ s.t. $\eta_H = 1_H$ etc.

① $\mathrm{Bim}(R)$, dualizable R-R bimods where R is the hyperfinite II_1 factor: $R := \bigotimes^{\infty} M_2(\mathbb{C})$ 'completed' wrt its unique trace.

② Std invariant of a finite index II_1 subfactor.] also a UTC

Subfactor: $R \subset A$ unital inclusion of factors.

- Assume ${}_RA$ is finitely gen proj. $\iff$ $\mathrm{vNdim}_R L^2A < \infty$

Standard invariant: $\mathcal{C}(R\subset A)$ = R-R bimods generated by ${}_RL^2A_R$ under $\oplus$, $\boxtimes_R$, $\le$, $\overline{\cdot}$, w/ bdd intertwiners, together w/ Q-system [std C* Frobenius alg] $L^2A \boxtimes_R L^2A \xrightarrow{m} L^2A$

Remark: $\mathcal{C}(R\subset A)$ comes equipped w/ fully faithful embedding $\mathcal{C}(R\subset A) \hookrightarrow \mathrm{Bim}(R)$. Hence the Q-system L^2A completely remembers SF $R\subset A$. Really must forget this embedding to get std invariant.

Dictionary: finite index subfactor $R \subset A$ $\iff$ Some data. Embedding $\mathcal{C}(R\subset A) \hookrightarrow \mathrm{Bim}(R)$ + Q-system L^2A.

I. Basic structure theory open qus.

hard FA   automatic

I.1: Given a UTC $\mathcal{C}$, when does there exist a fully faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Bim}(R)$?

- $\mathcal{C}$ amenable [Yamagami]

*** - ***** * If $\mathcal{C}$ fd? auto adj. ?

I.1.1: Does $\exists$ a single $\mathcal{C}$ which does not embed?

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I.1.2: Given a UTC $\mathcal{C}$, how many ways can $\mathcal{C} \hookrightarrow \mathrm{Bim}(R)$?

- Amenable $\mathcal{C}$ should be unique [Tomatsu 18]

[- non-Amenable could be wild / non-classifiable

Evidence: $\exists$ unclassifiably many hyperfinite subfactors w/ std invariant $A_3 * D_4$. [Brothier - Vaes] + Ioana

Ioana Γ non-amenable. Free, mp, mixing actions of Γ

I.2: Given a UTC $\mathcal{C}$ and a separable alg obj $A \in \mathcal{C}$ can A be endowed w/ the structure of a Q-system?

- Open for A connected: $\mathcal{C}(1 \to A) \cong \mathbb{C}$ so that $\mathrm{Hom}_{A-A}(A \otimes A \to A) \cong \mathbb{C}$ giving ! multiplication up to scalar

Equivalently: Is m^* an A-A bimodule map?

Equivalently: Is any indecomp. $\mathcal{C}$-mod cat unitarizable?

Thm [Reutter 2019-20]: unitary structure on a semisimple $\otimes$-cat is unique, as is any unitary structure on a $\mathcal{C}$-mod cat.

This theorem is actually much stronger and suggests an approach to unitary extension theory:

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I.3: Given a unitary fusion cat $\mathcal{C}$, construct equivalence of cat grps $\underline{\mathrm{BrPic}^*(\mathcal{C})} \cong \underline{\mathrm{Aut}^{br}_{\otimes,*}(Z(\mathcal{C}))} \cong \underline{\mathrm{Aut}^{br}_{\otimes}(Z(\mathcal{C}))} \cong \underline{\mathrm{BrPic}(\mathcal{C})}$

? Reutter ENO

II. Connections to (low dim'l) higher categories (3 cats

~~ENO/DSPS~~ / JFS (S, Haiyang): There is a sym. monoidal:

- 3-cat of (multi) fusion cats
- 4-cat of braided fusion cats.

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II.1: understand basic operations on (braided) fusion cats in terms of higher cats. Ex: $\mathcal{C}\mathcal{C}$ ⇝ End($\mathcal{C}$) mon 2-cat ⇝ End($id_{\mathcal{C}}$) BTC.

II.1.1: $\underline{\underline{G}} \to \underline{\mathrm{BrPic}}(\mathcal{C}) \longleftrightarrow$ G-extensions ← [ENO]

$\underline{\underline{G}} \to \underline{\mathrm{Pic}}(\mathcal{C}) \longleftrightarrow$ G-crossed braided extensions

II.1.2: Witt Equivalence of nondeg BFC $\longleftrightarrow$ ∃ invertible 1-mor $A \xrightarrow{\mathcal{C}} B$ [JMPP

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II.1.2': Is every non-deg BFC Witt equivalent to $\mathcal{C}(\mathfrak{g}, k)$ (up to Galois conj.) ?

II.1.3: Study $A \xrightarrow{\mathcal{D}} B \xrightarrow{\mathcal{E}} \mathcal{C}$; $\mathcal{D} \boxtimes_{\mathcal{B}} \mathcal{E}$ 1-composition in 4-cat.

eg: Equivariantization: $\mathrm{Vec} \xrightarrow{\mathcal{C}} \mathrm{Rep}\,G \xrightarrow{\mathrm{Rep}\,G} \mathrm{Vec}$ ⇝ $\mathcal{C}_G$ [claimed MP Pf ___]

II.1.4: Given any monoidal 2-cat $\mathcal{C}$ w/ G-action $\underline{\underline{G}} \to \mathcal{C}$, get a $\underline{G\text{-crossed braided category}}$ [JPR*]

Mantra: Braided ⊗-cats "come from" 3-cats $\mathcal{C}$

G-crossed braided ⊗-cats come from $B\underline{\underline{G}} \to \mathcal{C}$.

*** Does MTC come from a 2CFT ?

- BDH ∃ sym mon 3-cat of conf nets
- Conj: [G-JF] ∃ sym mon 4-cat of (...) TPMS

II.1.5: Given an object A in a 3-cat $\mathcal{C}$, when is the braided TC End(id_A) non-degenerate?

- A "fully dualizable" is not quite enough, but is sufficient for FusCat and Conf Net

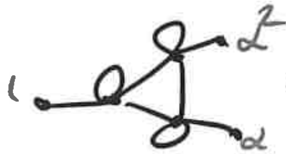
II.2: Where does (extended) Haagerup come from?

The Haagerup subfactor / fusion cat has been constructed by

- AH '99 using Ocneanu's connections
- Izumi '01 using Cuntz alg endos
- Peters '09 using PAs

] all brute force techniques

Found via small index subfactor classification [H '94]
and twisted cat classification [MPS]

Example:  ⟷  $\mathbb{Z}/3 \times \mathbb{Z}/3$ near gp.

Izumi: $\langle 1, \alpha_{pl}^{2}\rangle$ is $\mathrm{rep}(\mathbb{Z}/3)$, lifts to $Z(\mathcal{H})$

- not quite a $\mathbb{Z}/2$ crossed braided extension.

 ⟷ ] $\mathbb{Z}/2$ crossed braided ext of $\underbrace{\mathbb{Z}/3}_{\mathcal{C}_A^{loc}}$

$A \in SU(2)_4 = \mathcal{C}$ $\qquad$ $\mathcal{C}_A$

II.2.1: Is $\mathbb{Z}/3 \times \mathbb{Z}/3$ near gp some other kind of extension of $\mathbb{Z}/3 \times \mathbb{Z}/3$? maybe by a monoidal 2-cat $\mathcal{M}_{\mathcal{C}} \to \underline{\underline{\mathrm{Pic}}}(\mathcal{C})$?

II.2.2: More examples!

II.3: What is the correct notion of unitarity for higher cats?

Ansatz: Unitary FC's and (coord free) conformal nets form unitary 3-cats.

↳ finite μ-index