# Topology and geometry of quantum invariants (open problems)

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# Settings and talk theme

*3-manifolds:* $M$=compact, orientable, with empty or tori boundary.
*Links:* Smooth embedding $K : \coprod S^1 \to M$.
*Link complements:* $\overline{M \setminus n(K)}$; toroidal boundary

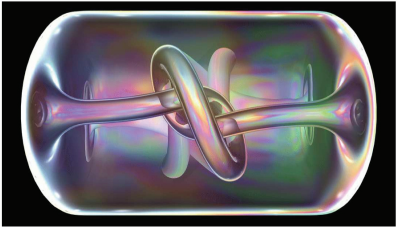


Figure credit: J. Cantarella, UGA

**Talk:** Relations between perspectives with focus on open problems.

## 3-manifold topology/geometry:

- Invariants arising from geometric structures:( e.g. hyperbolic volume)
- $\pi_1$-injective embedded surfaces

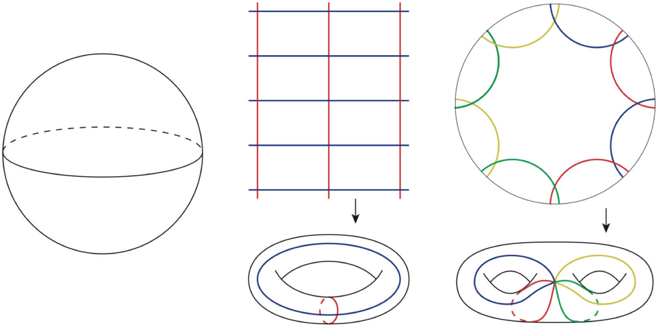
## Quantum topology invariants:

- Jones type knot polynomials
- Witten -Reshetikhin-Turaev invariants of 3-manifolds and knots
- TQFTs

# Warm up: Geometrization in 2-d:

Every (closed, orientable) surface can be written as $S = X/G$, where $X$ is a model geometry and $G$ is a discrete group of isometries.

$X = \mathbf{S}^2$ $\qquad$ $X = \mathbb{E}^2$ $\qquad$ $X = \mathbb{H}^2$



- *Curvature:* $k = 1, 0, -1$

# Geometrization in 3-d:

In dimension 3, there are eight model geometries:

$$X = \mathbf{S}^3 \quad \mathbb{E}^3 \quad \mathbb{H}^3, \ \mathbf{S}^2 \times \mathbb{R}, \ \ \mathbb{H}^2 \times \mathbb{R}, \ \ Sol, \ \ Nil, \ \ \widetilde{SL_2(\mathbb{R})}$$

Recall $M$= compact, oriented, $\partial M$=empty or tori

**Theorem (Thurston 1980 + Perelman 2003)**

*For every 3-manifold M, there is a canonical way to cut M along spheres and tori into pieces $M_1, \ldots, M_n$, such that each piece is $M_i = X_i/G_i$, where $G_i$ is a discrete group of isometries of the model geometry $X_i$.*

- *Canonical* : "Unique" collection of spheres and tori.
- Poincare conjecture: $\mathbf{S}^3$ is the only compact mode.
- Hyperbolic 3-manifolds form a rich and very interesting class.
- Cutting along tori, manifolds with toroidal boundary will naturally arise. Knot complements fit in this class.

# Geometric decomposition picture for this talk:

**Theorem (Knesser, Milnor 60's, Jaco-Shalen, Johanson 1970, Thurston 1980 + Perelman 2003)**

*M=oriented, compact, with empty or toroidal boundary.*

1. *There is a unique collection of 2-spheres that decompose M*

$$M = M_1 \# M_2 \# \dots \# M_p \# (\# S^2 \times S^1)^k,$$

   *where $M_1, \dots, M_p$ are compact orientable irreducible 3-manifolds.*

2. *For M=irreducible, there is a unique collection of disjointly embedded essential tori $\mathcal{T}$ such that all the connected components of the manifold obtained by cutting M along $\mathcal{T}$, are either Seifert fibered manifolds or hyperbolic.*

- *Seifert fibered manifolds:* For this talk, think of it as

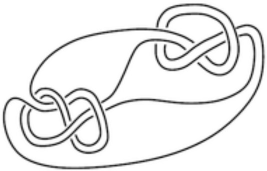
$$S^1 \times \text{surface with boundary} + \text{union of solid tori.}$$

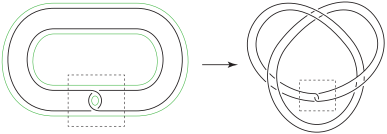
  Complete topological classification [Seifert, 60']

- *Hyperbolic:* Interior admits complete, hyperbolic metric of finite volume.

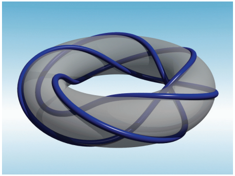
# Thee types of knots:

*Satellite Knots:* Complement contains embedded “essential” tori; There is a *canonical* (finite) collection of such tori.





*Torus knots:* Knot embeds on standard torus in T in $S^3$ and is determined by its class in $H_1(T)$. Complement is SFM.



*Hyperbolic knots:* Rest of them.

# Rigidity for hyperbolic 3-manifolds:

## Theorem (Mostow, Prasad 1973)

*Suppose $M$ is compact, oriented, and $\partial M$ is a possibly empty union of tori. If $M$ is hyperbolic (that is: $M \setminus \partial M = \mathbb{H}^3/G$), then $G$ is unique up to conjugation by hyperbolic isometries. In other words, a hyperbolic metric on $M$ is essentially unique.*

- By rigidity, every geometric measurement (e.g. *volume*) of a hyperbolic $M$ is a *topological invariant*
- *Gromov norm of $M$:* $v_{\text{tet}}||M|| = \text{Vol}\,(H)$, where
- $\text{Vol}\,(H)$ = sum of the hyperbolic volumes of hyperbolic parts in geometric decomposition
- $v_{\text{tet}}$ = volume of the regular hyperbolic tetrahedron.
- If $M$ Seifert fibered then $||M|| = 0$
- *Cutting along tori*: If $M'$ is obtained from $M$ by cutting along an embedded torus $T$ then

$$||M|| \leqslant ||M'||,$$

with equality if $T$ is incompressible.

# Jones Polynomials–Quantum invariants

1980's: Jones polynomial (subfactor theory)— Ideas originated in physics and in representation theory led to vast families invariants of knots and 3-manifolds. (*Quantum invariants*)

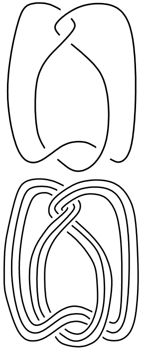
For this talk we discuss:

- The *Colored Jones Polynomials*: Infinite sequence of Laurent polynomials $\{J_{K,n}(t)\}_n$ encoding the *Jones polynomial* of $K$ and these of the links $K^s$ that are the parallels of $K$.
- Formulae for $J_{K,n}(t)$ come from representation theory of $SU(2)$ (decomposition of tensor products of representations).
  They look like

$$J_K^1(t) = 1, \quad J_K^2(t) = J_K(t)\text{- Original JP,}$$

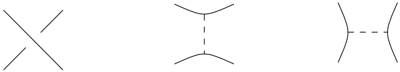
$$J_K^3(t) = J_{K^2}(t) - 1, \quad J_K^4(t) = J_{K^3}(t) - 2J_K(t), \ \ldots$$

- $J_{K,n}(t)$ can be calculated from any knot diagram via processes such as *Skein Theory*, *State sums*, *R-matrices*, *Fusion rules*....



# The skein theory approach

- $A$ or $B$ resolutions, $D_A, D_B$, of a crossing of $D = D(K)$.



- Kauffman bracket: polynomial $\langle D \rangle \in \mathbb{Z}[t^{\pm 1/4}]$, regular isotopy invariant:
  - $\langle L \coprod \bigcirc \rangle = -(t^{1/2} + t^{-1/2})\langle L \rangle := \delta \langle L \rangle$
  - $\langle L \rangle = t^{-1/4}\langle D_A \rangle + t^{1/4}\langle D_B \rangle$
  - $\langle \bigcirc \rangle = -t^{1/2} - t^{-1/2}$
- Chebyshev polynomials:

$$S_{n+2}(x) = xS_{n+1}(x) - S_n(x), \quad S_1(x) = x, \quad S_0(x) = 1.$$

- $D^m$ diagram obtained from $D$ by taking $m$ parallels copies.
- For $n > 0$, we define (where $w = w(D)$=*writhe*):

$$J_K^n(t) := ((-1)^{n-1} t^{(n^2-1)/4})^w (-1)^{n-1} \langle S_{n-1}(D) \rangle$$

- $\langle S_{n-1}(D) \rangle$ is linear extension on combinations of diagrams.

# The CJP predicts Volume?

- **Question:** How do the *CJP* relate to geometry/topology of knot complements?
- *Renormalized CJP.*

$$J'_{K,n}(t) := \frac{J^n_K(t)}{J^n_{\bigcirc}(t)}.$$

**Volume conjecture.** [ Kashaev+ H. Murakami - J. Murakami] Suppose $K$ is a knot in $S^3$. Then

$$2\pi \cdot \lim_{n\to\infty} \frac{\log |J'_{K,n}(e^{2\pi i/n})|}{n} = \mathrm{Vol}\,(S^3 \setminus K)$$

- The conjecture is wide open:
- $4_1$ (by Ekholm), knots up to 7 crossings (by Ohtsuki)
- torus knots (Kashaev-Tirkkonen); satellites of torus knots (Zheng).

. **Some difficulties:**

- For families of links we have $J^n_K(e^{2\pi i/n}) = 0$, for all $n$.
- "State sum" for $J^n_K(e^{2\pi i/n})$ has oscillation/cancelation.
- No good behavior of $J^n_K(e^{2\pi i/n})$ with respect to geometric decompositions.

# Colored Jones polynomials are q-homonomic

- (Garoufalidis-Le, 04):The sequence $\{J_{K,n}(t)\}_n$ has a *recursive relation.* For a fixed knot $K$ the colored Jones function $J_K(n)$ satisfies a non-trivial linear recurrence relation of the form

$$a_d(t^{2n}, t)J_K(n+d) + \cdots + a_0(t^{2n}, t)J_K(n) = 0$$

for all $n$, where $a_j(u, v) \in \mathbb{C}[u, v]$.

- **Example:** For $K$=the trefoil knot

$$J_K^n = t^{-6(n^2-1)} \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} t^{24j^2+12j} \frac{t^{8j+2} - t^{-(8j+2)}}{t^2 - t^{-2}}.$$

- The relation is

$$(t^{8n+12} - 1)J_K^{n+2} + (t^{-4n-6} - t^{-12n-10} - t^{8n+10} + t^{-2})J_K^{n+1}$$
$$-(t^{-4n+4} - t^{-12n-8})J_K^n = 0.$$

# Coarse relations: Colored Jones polynomial

For a knot $K$, and $n = 1, 2, \dots,$ we write its *n-colored Jones polynomial*:

$$J_K^n(t) := \alpha_n t^{m_n} + \beta_n t^{m_n - 1} + \cdots + \beta'_n t^{k_n + 1} + \alpha'_n t^{l_n} \in \mathbb{Z}[t, t^{-1}]$$

- (Garoufalidis-Le, 04): Each of $\alpha'_n, \beta'_n \dots$ satisfies a *linear recursive relation* in $n$, with integer coefficients .

  $$(\text{ e. g. } \quad \alpha'_{n+1} + (-1)^n \alpha'_n = 0).$$

- Given a knot $K$ any diagram $D(K)$, there exist explicitly given functions $M(n, D)$ and $L(n, D)$ $m_n \leq M(n, D)$ and $l_n \geq L(n, D)$. For "nice" knots, where we have $m_n = M(n, D)$ and $l_n = L(n, D)$, we have *stable coefficients*
- (Dasbach-Lin, Armond) If $m_n = M(n, D)$, then

  $$\beta'_K := |\beta'_n| = |\beta'_2|, \quad \text{and} \quad \beta_K := |\beta_n| = |\beta_2|,$$

  for every $n > 1$.
- **Remark.** "nice" turns out to be what Lickorish-Thistlethwaite call *adequate knots*.

# A Coarse Volume Conjecture?

**Theorem (Lackenby, Dasbach-Lin, Futer-K.-Purcell, Giambrone, 05-'15')**

*There universal constants $A, B > 0$ such that for any hyperbolic link that is "nice' we have*

$$A\,(\beta'_K + \beta_K) \;\leq\; Vol\,(S^3 \setminus K) \;<\; B\,(\beta'_K + \beta_K).$$

**Question.** Does there exist function $B(K)$ of the coefficients of the colored Jones polynomials of a knot $K$, that is easy to calculate from a "nice" knot diagram such that for hyperbolic knots, $B(K)$ is coarsely related to hyperbolic volume $\mathrm{Vol}\,(S^3 \setminus K)$ ?
Are there constants $C_1 \geq 1$ and $C_2 \geq 0$ such that

$$C_1^{-1} B(K) - C_2 \;\leq\; \mathrm{Vol}\,(S^3 \setminus K) \;\leq\; C_1 B(K) + C_2,$$

for all hyperbolic $K$?

- C. Lee, Proved CVC for classes of links that don't satisfy the standard "nice" hypothesis ( 2017)

# How Strong is the Degree of CJP?

- $d_+[J^n_K]$ =maximum degree of CJP and $d_+[J^n_K]$ =minimum degree
- **Problem.** The degree $d_+[J^n_K]$ detects the unknot: If $d_+[J^n_K] = d_+[J^n_{\bigcirc}] = 0.5n$, for all $n$, then $K = \bigcirc$.
- **Problem.** The degrees $d_+[J^n_K],\ d_-[J^n_K]$ detect all torus knots $T_{p,q}$: If

$$d_+[J^n_K] = d_+[J^n_{T_{p,q}}] \text{ and } d_-[J^n_K] = d_-[J^n_{T_{p,q}}],$$

then, $K = T_{p,q}$ (or $K = T_{q,p}$)

- **Problem.** Does the span $d_+[J^n_K] - d_-[J^n_K]$ "characterize" alternating knots? Is the knot $K$ is alternating if and only if there are $a, b \in \mathbb{Z}$ (depending only on $K$) such that

$$a + b = 1 \quad \text{and} \quad d_+[J^n_K] - d_-[J^n_K] = an^2 + bn - (b + a)?$$

- **Problem.** Do $d_+[J^n_K],\ d_-[J^n_K]$ detect the figure-8 knot? If $d_\pm[J^n_K] = d_\pm[J^n_{4_1}]$, then is $K = 4_1$?
- All are implications of variations of *Slopes Conjectures*

# The topology of the degree of CJP

- $d_+[J^n_K]$ =maximum degree of CJP
- The $q$-holonomicity property of CJP implies:
- Given $K$ there is $N_K > 0$, such that, for $n \geq N_K$,

$$d_+[J^n_K] = a_K(n)\, n^2 + b_K(n)n + c_K(n),$$

- where $a_K(n), b_K(n), c_K(n) : \mathbf{N} \to \mathbb{Q}$ are periodic functions.
- Similarly, $d_-[J^n_K]$ =maximum degree of CJP:

$$d_-[J^n_K] = a^*_K(n)\, n^2 + b^*_K(n)n + c^*_K(n),$$

- where $a^*_K(n), b^*_K(n), c^*_K(n) : \mathbf{N} \to \mathbb{Q}$ are periodic functions.
- We have finitely many cluster points

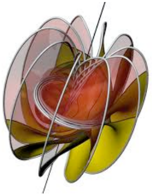
$$js_K = \{4a_K(n)\}' \quad \text{and} \quad js^*_K = \{4a^*_K(n)\}',$$

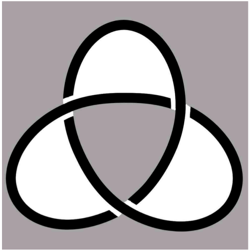
- (called Jones Slopes) and finitely many cluster points

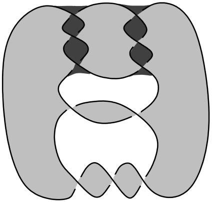
$$xs_K = \{2b_K(n)\}',\ xs^*_K = \{4b^*_K(n)\}'.$$

# Surfaces in knot complements

- There are several properly embedded surfaces in knot complements– some non-orientable.







- **Definition.** A surface $S$, properly embedded in $\mathbf{S}^3 \setminus K$ is called is *essential* if inclusion induces injection

$$\pi_1(S, \partial S) \longrightarrow \pi_1(\mathbf{S}^3 \setminus K, \partial(\mathbf{S}^3 \setminus K)).$$

- **Definition.** A (primitive) class in $H_1(\partial(S^3 \setminus K)) \cong \mathbb{Z} \times \mathbb{Z}$, determined by an element in $s \in \mathbf{Q} \cup \{\infty\}$, is called *a boundary slope of K* if there is an *essential* suface $S$ such that each component of $\partial S$ represents $s$.

# Slopes Conjectures

- **Definition.** A Jones surface of $K$ is an essential surface $S \subset M_K = S^3 \smallsetminus K$
- $\partial S$ represents a Jones slope $4a(n) = a/b \in js_K$, with $b > 0$, $\gcd(a, b) = 1$, and
$$\frac{\chi(S)}{|\partial S| b} = 2b_K(n).$$
- Similarly, for a a Jones slope $4a^(n) = a^*/b^* \in js_K$, with $b^* > 0$, $\gcd(a^*, b^*) = 1$, and
$$\frac{\chi(S)}{|\partial S| b^*} = -2b^*_K(n).$$
- ***Strong* Slope Conjecture.**
- (Garoufalidis) All Jones slopes are boundary slopes.
- ( *K+Tran*) *All Jones slopes are realized by Jones surfaces*.
- **Remark/Problem.** No knots with more than two Jones slopes are known.

# Figure-8/alternating

- **Question.** Are there knots with total number of Jones slopes $> 2$?
- **Definition.** A Jones surface $S$ of a knot $K$ is called *characteristic* if the number of sheets $b|\partial S|$ divides the *Jones period* of $K$.
- For all the knots the SSC is known, the Jones surfaces can be taken to be characteristic!
- **Question.** Is every Jones slope realized by a characteristic Jones surface?
- Howie and Greene gave a characterization of alternating knots in terms of their (spanning) surfaces. Their result, together with positive answer to last question, imply that If $K$ satisfies the SSC and $d_{\pm}[J_K^n] = d_{\pm}[J_{4_1,n}]$, then $K$ is isotopic to $4_1$.
- A positive answer to last question, together with Howie's theorem, imply the CJP characterization of alternating knots:

$$K \text{ is alternating iff } d_+[J_K^n] - d_-[J_K^n] = an^2 + (2-a)n - 2?$$

# Status

- **SSC known for:**
- Alternating knots (Garoufalidis)
- Adequate knots (Futer-K-Purcell)
- Knots up to nine crossings (Garoufalidis, Tran-K., Howie)
- Montesinos knots (Lee-van der Veen, Garoufalidis-Lee-van der Veen, Leng-Yang-Liu)
- Iterated torus knots
- Graph knots (Motegi-Takata, Baker-Motegi-Takata)
- families of non-Montesinos knots, non-adequate knots (Howie-Do, Lee)
- **SSC is closed under:**
- Connect sums (Motegi-Takata)
- Cabling operations (Tran-K.)
- Whiterhead doubling (Baker-Motegi-Takata)

# A Volume Conjecture for all 3-manifolds

- (Turaev-Viro, 1990): For odd integer $r$ and $q = e^{\frac{2\pi i}{r}}$

$$TV_r(M) := TV_r(M, q),$$

  a real valued invariant of compact oriented 3-manifolds $M$
- $TV_r(M, q)$ are combinatorially defined invariants and can be computed from triangulations of $M$ by a *state sum* formula. Sums involve *quantum* 6$j$*-sympols*. Terms are highly "oscillating" and there is term cancellation. Combinatorics have roots in representation theory of quantum groups.
- We work with the $SO(3)$ quantum group.
- (Q. Chen- T. Yang, 2015): compelling experimental evidence supporting
- **Volume Conjecture :** For $M$ compact, orientable

$$2\pi \cdot \lim_{r\to\infty} \frac{\log(TV_r(M, e^{\frac{2\pi i}{r}}))}{r} = v_{\mathrm{tet}}||M||,$$

  where $r$ runs over odd integers.

# What we know:

The Conjecture is verified for the following.

- *(Detcherrry-K.-Yang, 2016)* (First examples) of hyperbolic links in $S^3$: The complement of $4_1$ knot and of the Borromean rings.
- *(Ohtsuki, 2017)* Infinite family of closed hyperbolic 3-manifolds: Manifolds obtained by *Dehn filling* along the $4_1$ knot complement.
- *(Belletti-Detcherry-K- Yang, 2018)* Infinite family of cusped hyperbolic 3-manifolds that are universal: They produce all $M$ by Dehn filling!
- *(Kumar 2019)* Infinite families of octahedral hyperbolic links in $S^3$.
- *(Detcherry-K, 2017)* All links zero Gromov norm links in $S^3$ and in connected sums of copies of $S^1 \times S^2$.
- *(Detcherry, Detcherry-K, 2017)* Several families of 3-manifolds with non-zero Gromov, with or with or without boundary.

- For links in $S^3$ Turaev-Viro invariants relate to colored Jones polynomials (Next)

# Link complements in $S^3$:

$TV_r(S^3 \setminus K, e^{\frac{2\pi i}{r}})$ is expressed via (multi)-colored JP. For knots we have:

**Theorem (Detcherry-K.-Yang, 2016)**

*For $K \subset S^3$ and $r = 2m+1$ there is a constant $\eta_r$ independent of $K$ so that*

$$TV_r(S^3 \setminus K, e^{\frac{2\pi i}{r}}) = \eta_r^2 \sum_{n=1}^{m} |J_K^n(e^{\frac{4\pi i}{r}})|^2.$$

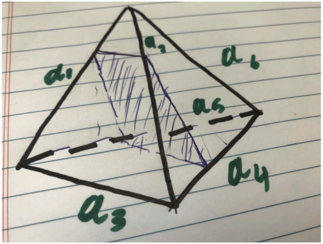
- Theorem implies $TV_r(S^3 \setminus K, e^{\frac{2\pi i}{r}}) \neq 0$ for any link in $S^3$! The quantity $\log(TV_r(S^3 \setminus K))$ is well defined.
- The values of CJP in Theorem are different that these in "original" volume conjecture. Calculations of Detcherry-K.-Yang and K. H. Wong (*Whitehead chains*) prompt the following.
- **Question.** Is it true that for any hyperbolic link $L$ in $S^3$,

$$2\pi \cdot \lim_{m \to \infty} \frac{\log |J_L^{\mathbf{m}}(e^{\frac{4\pi i}{2m+1}})|}{m} = \mathrm{Vol}(S^3 \setminus L)?$$

- where $J_L^{\mathbf{m}}$ denotes the unicolored CJP of $L$.

# Building blocks of TV invariants relate to volumes

- Color the edges of a triangulation with certain "quantum " data



- Colored tetrahedra get "6*j*-symbol" $\mathbf{Q} := Q(a_1, a_2, a_3, a_4, a_5, a_6)$= function of the $a_i$ and $r$. $TV_r(M)$ is a weighted sum over all tetrahedra of triangulation (*State sum*).
- *(BDKY)* Asympotics of **Q** relate to volumes of geometric polyhedra:

$$\frac{2\pi}{r} \log\left(|\mathbf{Q}|\right) \leqslant v_{\mathrm{oct}} + O\left(\frac{\log r}{r}\right).$$

- Proved VC for "octahedral" 3-manifolds, where $TV_r$ have "nice" forms. In general, hard to control term cancellation in state sum.

# A more Robust statement?:

$$LTV(M) = \limsup_{r \to \infty} \frac{2\pi}{r} \log(TV_r(M)), \text{ and } ITV(M) = \liminf_{r \to \infty} \frac{2\pi}{r} \log(TV_r(M))$$

**Conjecture:** There exists universal constants $B, C > 0$ and $E \geq 0$ such that for any compact orientable 3-manifold $M$ with empty or toroidal boundary we have

$$B\,||M|| - E \leqslant ITV(M) \leqslant LTV(M) \leqslant C\,||M||.$$

- Half is done:

## Theorem (Detcherry-K., 2017)

*There exists a universal constant $C > 0$ such that for any compact orientable 3-manifold $M$ with empty or toroidal boundary we have*

$$ITV(M) leq LTV(M) \leqslant C||M||,$$

*In particular, if $ITV(M) > 0$ then $||M > 0$.*

# Outline of last theorem:

1. Use asymptotics of the quantum 6$j$-symbols to show
$$ITV(M) \leq LTV(M) < v_8(\#\text{ of tetrahedra needed to triangulate } M).$$
2. View TV invariants as part of the $SO(3)$-Witten- Reshetikhin-Turaev and TQFT (*Roberts*). Use approach of Blanchet-Habegger-Masbaum-Vogel.
3. Use TQFT properties to show that if $M$ is a Seifert fibered manifold, then
$$LTV(M) = ||M|| = 0.$$
4. Show that If $M$ contains an embedded torus $T$ and $M'$ is obtained from $M$ by cutting along $T$ then
$$LTV(M) \leqslant LTV(M').$$
5. Use a theorem of Thurston to show that there is $C > 0$ such that for any hyperbolic 3-manifold $M$
$$LTV(M) \leq C||M||.$$
6. Use parallel behavior of $LTV(M)$ and $||M||$ under geometric decomposition.

# Exponential growth and AMU conjecture:

- $\Sigma$=compact, oriented surface and $\mathrm{Mod}(\Sigma)$ its mapping class group.
- For $f \in \mathrm{Mod}(\Sigma)$ let $M_f$=mapping torus of $f$. The Invariants $TV_r(M)$ grow exponetiallly in $r$, iff

$$lTV(M_f) := \liminf_{r\to\infty} \frac{2\pi}{r} \log(TV_r(M)) > 0.$$

- **Exponential growth conjecture.** For $f \in \mathrm{Mod}(\Sigma)$, we have $lTV(M_f) > 0$ if and only if $||M_f|| > 0$.
- For odd integer $r$, and a primitive $2r$-th root of unity there is quantum representation

$$\rho_r : \mathrm{Mod}(\Sigma) \to \mathrm{PGL}_{d_r}(\mathbb{C}).$$

- **AMU conjecture** (*Aderesen-Masbaoum-Ueno*) A mapping class $\phi \in \mathrm{Mod}(\Sigma)$ has pseudo-Anosov parts if and only if, for any big enough $r$, we have $\rho_{r,c}(\phi)$ has infinite order.
- Exponential growth conjecture implies AMU conjecture. Detcerry-K. gave many constructions of manifolds with $lTV(M) > 0$ and used these constructions to build substantial evidence for AMU conjecture.

# Some References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