Open Problems in the theory of (symmetric) tensor categories

k alg. closed

Symmetric tensor category over k (STC)

- k-linear, artinian (abelian & finite length & Hom are f. dim'l)
- monoidal ($\otimes$)
- rigid
- symmetric $c: X\otimes Y \to Y\otimes X$ $c^2 \overset{?}{=} 1$
- $\otimes$ bilinear on morphisms
- End$(\mathbb{1}) = k$

Example ① G affine gp. scheme $/k$
($\mathcal{O}(G)$ comm. Hopf algebra $/k$)

$\mathcal{C} = \mathrm{Rep}(G)$ $F: \mathcal{C} \to \mathrm{Vec}$ fiber functor $G = \mathrm{Aut}_{\otimes}(F)$
& if $\mathcal{C}$ is any STC with $F: \mathcal{C} \to \mathrm{Vec}$ then
$\mathcal{C} = \mathrm{Rep}(G)$, $G = \mathrm{Aut}_{\otimes}(F)$ (and F is unique).
Such $\mathcal{C}$ are called Tannakian

② $\mathrm{SVec} = \mathrm{Vec}^{\mathbb{Z}/2} \ni V = V_0 \oplus V_1$ $v \in V_0$ $|v| = 0$ char k $\neq 2$
$v \in V_1$ $|v| = 1$

$c(v\otimes w) = (-1)^{|v|\cdot|w|} w\otimes v$ v, w are homog.
G affine supergroup scheme ($\mathcal{O}(G)$ comm. Hopf alg in SVec)

$z \in G_0 \in G$, $z^2=1$, z acts by parity on $\mathcal{O}(G)$.

$\mathcal{C} = \mathrm{Rep}(G,z)$ – f.d. reps of G on superspaces s.t. z acts by parity

$F: \mathcal{C} \to \mathrm{SVec}$, $G = \mathrm{Aut}_{\otimes}(F)$

Also, if $\mathcal{C}$ is any STC w/ such F, then $\mathcal{C} = \mathrm{Rep}(G, z)$

$G = \mathrm{Aut}_{\otimes}(F)$ and z is parity autom.

Q: Are there others?

Yes! Deligne categories $\mathrm{Rep}\, S_t$, $\mathrm{Rep}\, GL_t$ – interpolations of $\mathrm{Rep}\, S_n$, $\mathrm{Rep}\, GL_n$ to non-integer n

char $k=0$
$k=\mathbb{C}$
$t \in \mathbb{C}$

If $t \notin \mathbb{Z}$ then these are semisimple STC

$X \in \mathcal{C}$ length$(X^{\otimes n})$

DEF: $\mathcal{C}$ is of <u>moderate growth</u> if length$(X^{\otimes n}) \leq C_X^n$

If $\exists\, F: \mathcal{C} \to \mathrm{SVec}$ then $C_X = \dim_{\mathbb{C}} F(X)$

But $\mathrm{Rep}\, S_t$ etc don't have moderate growth: length$(X^{\otimes n}) \geq c\sqrt{n!}$

Thm. (Deligne) Any STC of moderate growth in char 0 is supertannakian.

Question 1

① moderate: $\log \overset{\text{length}}{\ell}(X^{\otimes n}) \sim \lambda n$ (these are all in char 0)

② $\lambda n \log n$ $(\mathrm{Rep}\, S_t, GL_t, O_t)$

③ $\lambda \cdot n^2$ $\mathrm{Rep}\, GL_t(\mathbb{F}_q)$ (Knop)

↳ Are any other rates of growth possible?

Q: $\exists$? cat. of non-moderate growth in char p?

Deligne, 2015 $\boxed{\text{Yes}}$ $\mathrm{Rep}\, S_t$, $t \in \mathbb{Z}_p$ $t = \ldots t_2 t_1 t_0$

V taut object $t_i = \dim \Lambda^{p^i} V$ $t_i \in \mathbb{F}_p$

Lemma: In char p, $\dim X \in \mathbb{F}_p$

t = p-adic dim. of $X = \mathrm{Dim}(X)$

Not semisimple $\forall t$

Question 2.

Do there exist semisimple STC of non-moderate growth in char p?

Semisimplification

$\mathcal{C}$ Karoubian symm. rigid monoidal cat.
(Krull-Schmidt)

$\forall\, f: X \to X$ nilpotent, $\mathrm{tr}(f) = 0$

(this holds if $\mathcal{C} \in \mathcal{C}^{ab}$)

Def A morphism $f: X \to Y$ is negligible if $\forall\, g: Y \to X$ $\mathrm{tr}(fg) = 0$.

$\mathcal{N} = \{$neglig. morphisms$\}$ $\underbrace{\overline{\mathcal{C}} = \mathcal{C}/\mathcal{N}}$ semisimple STC

simple objects = indec. obj. of $\mathcal{C}$ of dim $\neq 0$

this is the $\boxed{\text{semisimplification}}$ of $\mathcal{C}$

Let $\mathcal{C} \subset \mathrm{Rep}\, S_t$, $t \in \mathbb{Z}_p$ be a Karoubian rigid mon. subcat. eg. $X \in \mathrm{Rep}\, S_t$, $\mathcal{C} = \langle$direct summands in direct sums of $X^{\otimes n} \otimes X^{* \otimes m}\rangle$

Question 3. Does this have moderate growth?

True if $\mathcal{C} = \langle V \oplus \Lambda^p V \oplus \Lambda^{p^2} V \oplus \dots \oplus \Lambda^{p^n} V \rangle$

Now moderate growth only? Still Deligne thm. false.

$\mathrm{Rep}_k(\mathbb{Z}/p) = \mathrm{Ver}_p$ Verlinde Category

$g^p = 1 \quad (g-1)^p = 0$

Jordan blocks $J_1, \dots, J_{p-1}, \cancel{J_p}$

$\downarrow$

simple objects $\underset{\mathbb{1}}{\underset{\|}{L_1}}, \dots, L_{p-1}$

$$L_m \otimes L_n = \bigoplus_{i=1}^{\min(m,n,p-m,p-n)} L_{(m-n)+2i-1}.$$

$\underline{p>2}$ $\quad \langle \underset{\mathbb{1}}{\underset{\shortparallel}{L_1}}, \underset{\psi}{\underset{\shortparallel}{L_{p-1}}} \rangle = SVec$

$Ver_p = Ver_p^t \boxtimes SVec$

$\quad \curvearrowleft$ odd dim. Jordan block

$\underline{p=5}$ $\quad Ver_5^t = \langle \mathbb{1}, \underset{L_3}{\underset{\shortparallel}{X}} \rangle \qquad X^{\otimes 2} = X \oplus \mathbb{1}$ Fibonacci Fusion Rule

$F: \mathcal{C} \to SVec$

$\dim F(X) = d$

$d^2 = d+1$

Thm (Ostrik, 2015) If $\mathcal{C}$ is a semisimple STC w/ finitely many simples then $\exists!\ F: \mathcal{C} \to Ver_p$ ($\Rightarrow$ Concrete realization of $\mathcal{C}$ as Reps of some finite gp. scheme in Ver_p) (e.g., $p=2,3$, then Deligne thm. holds for such $\mathcal{C}$

Conjecture 1 (Ostrik) This also holds for semisimple categories w/ infinitely many objects

Applications:

Benson's Conj. V f.d. rep. of a finite gp. G/k

$V\otimes V^*$ - $\underbrace{\text{how many summands of dim} \equiv 0 \bmod p?}_{\text{rank}(V)}$

Ex. $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ $p=2$

$\mathbb{Z}_3\times\mathbb{Z}_3$ $p=3$

Benson's Conj. If $p=2$ and V is odd dim. then rank$(V)=1$ ($V\otimes V^* = \mathbb{1} \oplus$ $\oplus$ of indec. reps of even dim.)

Tensor cats formulation: $\mathcal{C}_V \subset \operatorname{Rep} G$

$\overline{\mathcal{C}_V}$ - semisimplification
(throw away reps of even dim)

- sym. semisimple tensor cat.

Conj. 1. $\Rightarrow \exists\ \overline{\mathcal{C}_V} \to \text{Vec} \Rightarrow \overline{\mathcal{C}_V} = \operatorname{Rep} H$ ← affine gp scheme (finite ~~type~~) semisimple

Nagata Thm: $H = \underset{\uparrow}{\Gamma} \ltimes (\underset{\uparrow}{A^\vee} \times \underset{\uparrow}{\mathbb{T}})$

Γ: finite gp of order prime to p

$A^\vee$: dual of a finite ab. gp.

$\mathbb{T}$: torus

Benson Conj: H is abelian

Open problems on symmetric tensor categories

P. Etingof, MSRI, Jan 24, 2020

1.) k - alg. closed field.

Symmetric tensor category (STC) over k:

- k linear artinian (abelian, finite length, $\dim \mathrm{Hom}(X,Y) < \infty$).
- monoidal
- rigid
- symmetric.
- $\otimes$ bilinear
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2. The category $\mathrm{SuperVec} = \mathrm{Vec}^{\mathbb{Z}/2} \cong \{V = V_0 \oplus V_1\}$ with ($|v| = 0$, $v \in V_0$; $|v| = 1$, $v \in V_1$)

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Then have a fiber functor $F: \mathcal{C} \to \mathrm{Supervec}_k$

$G = \mathrm{Aut}_{\otimes}(F)$, z - parity autom. of F.

Such cat. (admitting $F: \mathcal{C} \longrightarrow \mathrm{Supervec}_k$) are called *supertannakian*.

Q: Is any STC supertannakian?

No: Deligne categories $\mathrm{Rep}\, S_t$, $\mathrm{Rep}\, GL_t$, $\mathrm{Rep}\, O_t$ etc in char 0 are a counterexample. They are "big": length$(X^{\otimes n})$ grows faster than exponential in n. They are semisimple for $t \notin \mathbb{Z}$.

Def. $\mathcal{C}$ is of *moderate growth* if $\forall X \in \mathcal{C}$ $\exists\, C_X \geq 0$ s.t. $\ell(X^{\otimes n}) \leq C_X^n \ \forall n$.

Thm (Deligne). Any STC of moderate growth over a field of char 0 is supertannakian.

(For unitary categories: moderate growth is automatic, and $\mathcal{C} = \mathrm{Rep}\, G$ for a compact top. gp, maybe with altered commutativity using central $z \in G$, $z^2 = 1$. This is due to Doplicher and Roberts).

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But they are non-semisimple for all t.

Question 2. Do there exist semisimple STC of non-moderate growth in char p?
Candidates could be provided by semisimplification. Let $\mathcal{D}$ be a Karoubian $\overset{\text{sym}}{}$ rigid monoidal k-lin category,
A morphism $f: X \to Y$ is negligible if $\forall g: Y \to X$, $tr(fg) = 0$.

$$\boxed{\begin{tikzcd} \mathbb{1} \arrow[r] \arrow[rrr, bend right, "tr\, f"'] & X \otimes X^* \arrow[r, "f \otimes 1"] & X \otimes X^* \arrow[r] & \mathbb{1} \end{tikzcd}}$$

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Such morphisms form a ensor ideal $\mathcal{N}$
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Simple objects = indec. of $\mathcal{D}$ of nonzero dimension.

Question 3. Let $\mathcal{C} \subset$ Rep S_t, Gl_t, O_t in char p e a Karoubian rigid mon. subcat. Does $\overline{\mathcal{C}}$ have moderate growth? (true if take $\mathcal{C}$ gen. by $\Lambda^{p^i} V$).

② In char p Deligne thm fails.

Ex. $\mathrm{Ver}_p = \overline{\mathrm{Rep}_k \mathbb{Z}/p}$ Verlinde category.

Simple objects $\underset{}{\overset{\mathbb{1}}{\overset{\shortparallel}{L_1}}}, \dots, L_{p-1}$ (Jordan blocks of size $1, \dots, p-1$; the block of size p dies as it has $\dim = 0$).

$$L_m \otimes L_n = \bigoplus_{i=1}^{\min(m,n,p-m,p-n)} L_{|m-n|+2i-1}.$$

Verlinde rule (which motivates the terminology).

$\Rightarrow \mathrm{Ver}_2 = \mathrm{Vec}$, $\mathrm{Ver}_3 = \mathrm{Supervec}$

$\underset{p>2}{\mathrm{Ver}_p} \supset \mathrm{Supervec} = \langle \underset{L_1}{\overset{\shortparallel}{\mathbb{1}}}, \underset{L_{p-1}}{\overset{\shortparallel}{\psi}} \rangle.$

$\mathrm{Ver}_p = \mathrm{Ver}_p^+ \boxtimes \mathrm{Supervec}$

$\uparrow$ odd sized Jordan blocks.

$\mathrm{Ver}_5^+ = \langle \mathbb{1}, \underset{L_3}{\overset{}{\underset{\shortparallel}{X}}} \rangle$, $X^{\otimes 2} = X \oplus \mathbb{1}$,

so if $F: \mathcal{C} \to \mathrm{Supervec}$ then $\Rightarrow\Leftarrow$

$\dim_k F(X) = d$ with $d^2 = d+1$ $(d \notin \mathbb{Z})$.

Same for $p > 5$. In fact, Ver_p is incompressible (any symm. tensor functor $E: \mathrm{Ver}_p \to \mathcal{D}$ is a fully faithful embedding). So perhaps to generalize Deligne's thm, we need to replace supervec with Ver_p.

Ostrik's thm: If $\mathcal{C}$ is fusion (semisimple, f. many simple objects) then $\exists!$ fiber functor $F:\mathcal{C}\to Ver_p$.

This gives an explicit realization of $\mathcal{C}$: $\mathcal{C}=Rep(G,\pi_1)$ where $G = Aut_{\otimes}F$ is an finite semisim. group scheme in Ver_p.

Conjectural classification of such schemes: Let G be an simple alg. group$/k$, h_G Cox. number, p prime $p>h_G$. Then can define $Ver_p(G)=Tilt_p(G)/\mathcal{N}=\overline{Tilt_p(G)}$ - semisimplification of the category of tilting modules.

Verlinde fiber f-r: Restr. to principal $SL_2\subset G$. or $PGL_2\subset G$

Ex: $Ver_p(sl_2)=Ver_p$.

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True for $p=2,3$. E.g. for $p=2$ this is Nagata's thm: a semisimple affine group scheme (of finite type) in ch. p is $\Gamma\ltimes(A^\vee\times\mathbb{T})$ where A is a finite abelian group (and $A^\vee$ is the dual group scheme) and $\mathbb{T}$ a torus. So $p=5$ is the first open case. In this case Ostrik's conj says that $\mathcal{C}\cong\Big(\mathcal{P}\boxtimes\big(\underset{\overset{\parallel}{Fib}}{Ver_5^+}\big)^{\otimes m}\Big)^\Gamma$.

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G a finite p-group ($\underset{p>2}{\mathbb{Z}_p^2}$, $\mathbb{Z}_2^3$ are already very interesting)

$V \in \mathrm{Rep}\,G$, Consider the Karoubian rigid mon. subcat gen. by V (all indecomposables of $\mathrm{Rep}(G)$ occuring in $V^{\otimes m} \otimes V^{*\otimes n}$), and consider $\overline{\mathcal{C}}_V$ (its semisimplification, i.e. we throw away indecomposables which have $\dim = 0 \mod p$).

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Elementary formulation: if V is an odd-dim indec. rep. of a finite 2-group in char 2 then $sl(V) \overset{def}{=} \mathrm{Ker}(V \otimes V^* \to \mathbb{1})$ has all indec. direct summands of even dim $\Leftrightarrow$ $V \otimes W$ (V, W odd dim) has a unique odd dim direct summand.

This would follow from OC2 and the statement that $\Gamma = 1$.

For any finite group OC2 implies that $\forall V\ \exists d_V$ s.t. $\forall W \subset V^{\otimes m} \otimes V^{*\otimes n}$ direct summand $W \otimes W^*$ has at most d_V odd summands.

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hm. (EO) If $f_V(z)$ is not a polyn then it has rad. of conv. 1.

Rem. If we take usual dim instead we get

$$\sum \dim S^iV \cdot z^i = (1-z)^{-\mathrm{Dim}_+ V}$$

p-adic dimension of V $(\in \mathbb{Z}_p)$.

For finite categories we expect it to be in $\mathbb{Z}$.

Conj. Let $\mathcal{C}$ be a finite sym. tens. cat. Then $\mathrm{Ext}^\bullet(1,1)$ is a finitely generated algebra.

For affine group schemes this is due to Friedlander - Suslin

for supergroup scheme ~~by~~ to Drupieski.
These proofs are based on the theory of polynomial functors.

Open problem: Develop the theory of polynomial functors for Ver_p.

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Rem. If we take usual dim instead we get $\sum \dim S^i V \cdot z^i =$ ~~$\mathrm{Dim}(V)$~~ $(1-z)^{-\mathrm{Dim}_+ V}$

p-adic dimension of V $(\in \mathbb{Z}_p)$.

For finite categories we expect it to be in $\mathbb{Z}$.

Conj. Let $\mathcal{C}$ be a finite sym. tens. cat. Then $\mathrm{Ext}^{\bullet}(1,1)$ is a finitely generated algebra.

For affine group schemes this is

for supergroup scheme ~~by~~ to Drupieski.
These proofs are based on the theory of polynomial functors.
Open problem: Develop the theory of polynomial functors for Ver_p.