Invariants of actions on Artin-Schelter regular algebras

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January 23, 2020

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Classical Invariant Theory:
Group G acting linearly on the algebra $\mathbb{k}[x_1, \ldots, x_n]$ and study $\mathbb{k}[x_1, \ldots, x_n]^G$.

Noncommutative Invariant Theory:
Replace:
$\mathbb{k}[x_1, \ldots, x_n]$ with appropriate noncommutative algebra A

G with a group (or Hopf algebra) that acts on A

to extend classical results.

Question: What results extend to this context?

Replace $\mathbb{k}[x_1, \cdots, x_n]$ by A, a noetherian Artin-Schelter domain generated in degree 1 ($\mathbb{k} = \mathbb{C}$).
Examples:

1. $\mathbb{k}_q[x_1 \ldots, x_n]$ with $x_j x_i = q x_i x_j$ for $j > i$.
2. $R[x; \sigma, \delta]$
3. Sklyanin algebras, Down-up algebras, etc.

Why A AS regular?

- Defined homologically.
- Commutative AS regular $\cong$ to polynomial ring.
- There is a growing body of results.
- There are lots of interesting open problems.

Problem: Consider A filtered or (skew) Calabi-Yau

H is a Hopf algebra acting on A:

- H is semi-simple Hopf algebra
- H preserves the grading on A
- A is an H-module algebra
- The action of H on A is inner-faithful.

Problem: Consider H not finite dimensional, or not semisimple and/or char k = p, or the grading not preserved.

Plan: Generalize

I. Bounds on the degrees of minimal generators of A^G

II. Reflection groups

III. Subgroups of $\mathrm{SL}_n(\mathbb{C})$

I. Bounds on degrees of minimal generators of A^H

$\beta(A^H) = \min\{k : A^H$ generated by elements of degree $\leq k\}$

Example: $\mathbb{k}[x_1, \ldots, x_n]^{S_n}$ generated by

$\sigma_m := \sum x_{i_1} x_{i_2} \ldots x_{i_m}$ for $m = 1, \ldots, n$

$$\beta(\mathbb{k}[x_1, \ldots, x_n]^{S_n}) = n.$$

Noether's bound

Theorem (Noether (1915)) $\beta(A^G) \leq |G|$
if $\mathbb{k}$ has char zero or $|G| <$char $\mathbb{k}$.

Fleischmann (2000) and Fogarty (2001)
extended to non-modular case ($|G|^{-1} \in \mathbb{k}$).

Theorem (Symonds (2011))
If G is a finite group of order $|G| > 1$ acting linearly on $A := \mathbb{k}[x_1, \ldots, x_n]$ with $n \geq 2$ then

$$\beta(A^G) \leq n(|G| - 1).$$

Relations among generators in degrees

$$\leq 2n(|G| - 1).$$

Noether bound can be sharp: $\epsilon = e^{2\pi i/n}$
$A = \mathbb{k}[x]$, $G = (\epsilon)$, $A^G = \mathbb{k}[x^n]$, $|G| = n = \beta(A^G)$.

Theorem (Domokos and Hegedüs (2000))
If G non-cyclic then A^G can be generated by polynomials of total degree:

$\leq 3|G|/4$ if $|G|$ is even, and
$\leq 5|G|/8$ if $|G|$ is odd.

Sezer (2002): Extended to non-modular case.

Göbel's bound – involves the dimension of the representation

Theorem (Göbel (1995)) In any characteristic, G a group of permutations of x_i for $i = 1, \ldots, n$:

$$\beta(A^G) \leq \max\{n, \binom{n}{2}\}.$$

Noncommutative Example

Example: $A = \mathbb{k}_{-1}[x, y], yx = -xy$ and $g : x \leftrightarrow y$ acts on A. $\mathcal{O}(x) = x + y$

$$\mathcal{O}(x^2) = x^2 + y^2 = (x + y)^2$$

$$\mathcal{O}(xy) = xy + yx = 0$$

A set of generators of A^G is $x + y$ and $x^3 + y^3$ (or $x^2y + xy^2$). $\beta(A^G) = 3 > |G| = 2$.

The Noether bound does not hold.

Skew polynomial example

[FKMP]: arXiv: 1907.06761

$A = \Bbbk_{-1}[x, y]$, $\epsilon = e^{2\pi i/n}$, and $G = \langle g \rangle$

$$g = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}, \quad g^2 = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \text{ so } |G| = 2n.$$

For n odd $\beta(A^G) = 3n$

$\beta(A^G) - |G| = n$

(in a 2-dimensional representation of G)

Problem: Find $\beta(\Bbbk_{-1}[x_1, \ldots x_n]^G)$.

Down-up algebra example

$$A = \mathbb{k}\langle x, y\rangle / (y^2x = xy^2, yx^2 = x^2y)$$

Basis of the form $x^i(yx)^jy^k$

g : $x \leftrightarrow y$ acts on A.

Invariants include: $x + y$,

$xy + yx$ (or $x^2 + y^2$),

$x^3 + y^3, x^2y + xy^2, (xyx + yxy)$

$$\beta(A^G) = 3 > |G| = 2$$

Down-up example continued

$$A = \Bbbk\langle x, y\rangle / (y^2x = xy^2, yx^2 = x^2y)$$

$$G = \langle g \rangle \quad g = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix} \quad |G| = 2n$$

n odd $: \beta(A^G) = 3n$

Noether's bound fails when n is odd.

Problem: Find $\beta(A^G)$ for A down-up.
$A(\alpha, \beta)$ generated by x, y, with relations:

$$y^2x = \alpha yxy + \beta xy^2, \quad yx^2 = \alpha xyx + \beta x^2y.$$

For $A(0,-1) = \mathbb{k}\langle x, y\rangle/(y^2x = -xy^2, yx^2 = -x^2y)$
computer computation suggest

$$\beta(A^G) = 4n \text{ for } n \text{ odd.}$$

For $A(0,1)$ and $A(2,-1)$, any element of $\mathrm{GL}_2(\mathbb{k})$
acts as an automorphism on A.

$$A = \Bbbk[x_1, \ldots, x_n]$$

$\Bbbk[f_1, \ldots, f_n]$ a subring of primary invariants.

$$\beta(A^G) \leq \sum_{j=1}^{n} \deg(f_j) - n$$

Theorem [KKZ](2014)

A quantum polynomial algebra of dimension n,
H semisimple Hopf algebra, and $C \subset A^H \subset A$,
C a graded iterated Ore extension
$C = \Bbbk[f_1][f_2 : \tau_2, \delta_2] \ldots [f_n : \tau_n, \delta_n]$,
A_C finitely generated, $\deg f_i > 1$ for at least 2 distinct i's then $\beta(A^H) \leq \sum \deg f_i - n$.

Theorem [KKZ](2014): $A = \mathbb{k}_{-1}[x_1, \ldots, x_n]$

- $\beta(A^{S_n}) = 2n - 1$ (generators: $s_k = \mathcal{O}(x_1^2 \cdots x_{k-1}^2 x_k)$, $k = 1, \ldots, n$)
- $\beta(A^{A_n}) = 2n - 3$ (generators: $s_k, k = 1, \ldots, n-1$, and $\mathcal{O}(x_1 x_2 \cdots x_n)$)
- $\beta(A^G) \leq 3n^2/4$ for G permutations.
- $\beta(A^G) \leq n^2$ for G a subgroup of $S_n \rtimes \{\pm 1\}$

Questions:

- Is $\beta(A^G)$ biggest for cyclic groups?
- Is $\beta(A^G)$ a function of just $|G|$, or does it also depend on the dimension of the representation?
- Is $\beta(A^G)$ a linear function of $|G|$, of the dimension of the representation?
- Does the dim H play the same role as $|G|$?
- Can $\beta(A^H)$ be larger when H is not semisimple? What happens in char p?
- What about $\beta(Q(A)^H)$?

II. Reflection Groups

Let $\Bbbk$ be a field of characteristic zero.

Theorem (1954). The ring of invariants $\Bbbk[x_1, \cdots, x_n]^G$ under a finite group G is a polynomial ring if and only if G is generated by reflections.

A linear map g on V is called a reflection of V if all but one of the eigenvalues of g are 1,
i.e. dim $V^g =$ dim $V - 1$.

Example: Transposition permutation matrices are reflections, and S_n is generated by reflections.

Question:

What groups (Hopf algebras) are "reflection groups (Hopf algebras)" for A?

Definition: H is a reflection Hopf algebra for A if A^H is AS regular.

Reflection Groups: S_2?

Invariants under a symmetric group

Example: $S_2 = \langle g \rangle$, for $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, acts on $A = \Bbbk_{-1}[u, v]$ and A^{S_2} is generated by

$$P_1 = u + v \text{ and } P_2 = u^3 + v^3$$

The generators are NOT algebraically independent.
A^{S_2} is NOT AS-regular
(but it is a hyperplane in an AS regular algebra).

The transposition $(1, 2)$ is NOT a “reflection".

Definition of "reflection": Want A^g AS regular

All but one eigenvalue of g is 1 $\rightsquigarrow$

[Jing-Zhang (1997)] The trace function of g acting on A of dimension n has a pole of order $n-1$ at $t=1$, where

$$Tr_A(g,t) = \sum_{k=0}^{\infty} \text{trace}(g|A_k)t^k$$

$$= \frac{1}{(1-t)^{n-1}q(t)} \text{ for } q(1) \neq 0.$$

Reflection Groups: Examples illustrating the definition of reflection.

$\mathbf{Examples}$:$G =< g >$ on $A = \mathbb{k}_{-1}[u, v]$ $(vu = -uv)$:

(a) $g = \begin{bmatrix} \epsilon_n & 0 \\ 0 & 1 \end{bmatrix}, Tr_A(g, t) = \dfrac{1}{(1-t)(1-\epsilon_n t)}$,
$A^g = \mathbb{k}\langle u^n, v \rangle$ is AS regular (classical reflection).

(b) $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Tr_A(g, t) = \dfrac{1}{1+t^2}$,
A^g is NOT AS regular.

(c) $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, Tr_A(g, t) = \dfrac{1}{(1-t)(1+t)}$,
$A^g = \mathbb{k}[u^2 + v^2, uv]$ is AS regular (“mystic reflection").

Example: The binary dihedral groups of order 4ℓ generated by

$$g_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for λ a primitive 2ℓth root of unity, acts on $A = \Bbbk_{-1}[x, y]$.

$$A^G = \Bbbk[xy,\ x^{2\ell} + y^{2\ell}].$$

Shephard-Todd-Chevalley Theorem: H=kG A = skew polynomials

[KKZ] (2010)
G is a reflection group for $A = \Bbbk_{q_{ij}}[x_1, \cdots, x_n]$ iff G is generated by "reflections".

[Bazlov and Berenstein](2014) ("Demystification")
If G is a reflection group for $A = \Bbbk_{q_{ij}}[x_1, \cdots, x_n]$, then there is a classical reflection group G' with $\Bbbk G \cong \Bbbk G'$ as algebras.

Questions:

- When is the following generalization of the Shephard-Todd-Chevalley Theorem true:
  A^G is AS regular if and only if G is generated by "reflections"?
- Do some AS regular algebras have other kinds of reflections?
- What if char $\mathbb{k}$ = p?

Example: Jordan Plane $A = \mathbb{k}\langle x, y\rangle/(xy - yx - y^2)$ with char $\mathbb{k} = 2$. The transvection g with $g.x = x + y$ and $g.y = y$ has $A^{\langle g\rangle} = \mathbb{k}\langle x^2, y\rangle$ so g so is a reflection for A.

[FKMW] arXiv 1810.12935, 1907.06763:

Hopf algebras H that act on a quadratic AS regular algebra of dimension 2 or 3 as a Hopf Reflection Algebra:
Examples:

- (Masuoka) $\mathcal{A}_{4m}$
- (Masuoka) $\mathcal{B}_{4m}$
- (Pansera) H_{2n^2}
- (Kashina) Some of the 16 dimensional Hopf algebras

Problem: Find all reflection Hopf algebras for $A = \mathbb{k}_{-1}[u, v]$.

Dual reflection groups: [KKZ](2017), [CKZ] (2017), [CKZ] arXiv 1801.09020

$H = \Bbbk G^\circ = \Bbbk^G$ is a commutative algebra.

H acts inner faithfully on A if and only if $A_1 = S_{g_1} \oplus \cdots \oplus S_{g_n}$ where $\{g_i\}$ generate G.

A is graded by G and $A^H = A_e$.

When A_e is AS regular we call G a *dual reflection group for* A.

Dual reflection groups: Example

Example: $G = D_8$ is a dual reflection group for $A = \Bbbk_{\pm 1}[u, w][v; \sigma]$, where $\sigma(u) = aw$, $\sigma(w) = bu$, and D_8 grading $u \mapsto r, v \mapsto r\rho, w \mapsto r\rho^2$.
Relations in A:

$$wu = \pm uw \quad \text{grade } \rho^2$$

$$vu = awv \quad \text{grade } \rho^3$$

$$vw = buv; \quad \text{grade } \rho$$

The invariant subring is

$$A^H = A_e \cong \Bbbk[u^2, w^2][v^2; \sigma']$$

an AS regular algebra.

Necessary conditions for G to be a dual reflection group:

$$A = A^H \oplus \oplus_{g \neq e} u_g A^H$$

Let $p(t) := 1 + \sum_{g \neq e} t^{\ell(g)}$ (the Poincaré polynomial)

$$p(1) = |G|$$

$$H_{A^H}(t) = \frac{H_A(t)}{p(t)} \text{ so } p(t)$$

is a product of cyclotomic polynomial.,

Dual reflection groups: Recall D_8 example

For $H = k^{D_8}$ acting on $A = k\langle u, \quad v, \quad w\rangle$

(group grades) $\qquad r \quad r\rho \quad r\rho^2$

(A has PBW basis of the form $u^i w^j v^k$, $B := A^H$)

$$A = B \oplus uB \oplus vB \oplus wB \oplus uvB \oplus uwB \oplus wvB \oplus uwvB$$

$$e \quad r \quad r\rho \quad r\rho^2 \quad \rho \quad \rho^2 \quad \rho^3 \quad r\rho^3$$

$$p(t) = 1 + 3t + 3t^2 + t^3 = (1+t)^3$$

$$p(1) = 8$$

$$H_B(t) = \frac{1}{(1-t)^3(1+t)^3} = \frac{1}{(1-t^2)^3}$$

Work with Kent Vashaw:
Consider G of order 16.

- Look at all minimal generating sets of G.
- Find the Poincaré polynomial of the generating set.
- See the relations that would be imposed on A.
- Do any relations give A AS regular?
- Is A_e regular?

For G the semidihedral group of order 16 with generating set with $p(t) = (1+t)^4$

$A := \Bbbk\langle x_1, x_2, x_3, x_4\rangle/$(relations below)

$$x_3^2 = x_1x_2 \text{ (grade c)}, \quad x_4^2 = x_2x_1 \text{ (grade cd)}$$

$$x_1x_3 = x_2x_4 \text{ (grade acd)}, \quad x_3x_1 = x_2x_3 \text{ (grade a)}$$

$$x_1x_4 = x_4x_2 \text{ (grade ad)}, \quad x_4x_1 = x_3x_2 \text{ (grade ac)}.$$

Peter Goetz proved that A is AS regular.

$A_e = \Bbbk[x_1^2, x_2^2, x_3x_4, x_4x_3]$, commutative polynomial ring,

$$H_{A^H}(t) = \frac{1}{(1-t)^4(1+t)^4} = \frac{1}{(1-t^2)^4}.$$

Problem: Classify all dual reflection groups

III.Generalization of SL$_n(\mathbb{C})$

Homological Determinant (Jørgensen and Zhang (2000)): Scalar associated to a map in local cohomology (the determinant for polynomial rings).

$\mathrm{hdet} : H \to \Bbbk$ a homomorphism.

Generalized Watanabe's Theorem[KKZ] (2009): A^H is AS-Gorenstein when H has $\mathrm{hdet} \circ S = \epsilon$ (trivial homological determinant).

Morally: Hopf actions on A with trivial homological determinant should behave like actions of subgroups of SL$_n(\mathbb{C})$ on $\Bbbk[x_1, \ldots, x_n]$.

Permutation Actions on $\Bbbk_{-1}[x_1 \ldots, x_n]$

If g is a 2-cycle and $A = \Bbbk_{-1}[x_1 \ldots, x_n]$ then $\mathrm{hdet}(g) = 1$.

For ALL groups G of $n \times n$ permutations, A^G is AS-Gorenstein.

Not true for commutative polynomial ring – e.g.

$$\mathbb{C}[x_1, x_2, x_3, x_4]^{\langle (1,2,3,4) \rangle}$$

is not Gorenstein while

$$\mathbb{C}_{-1}[x_1, x_2, x_3, x_4]^{\langle (1,2,3,4) \rangle}$$

is AS-Gorenstein.

When A has dimension 2 and hdet trivial:

CKWZ H have been classified
("Quantum binary polyhedral groups")

CKWZ A^H is a hypersurface in an AS regular algebra of dimension 3
("Quantum Kleinian singularities")

CKWZ There is a McKay cooresppondence

CKMW Matrix factorizations corresponding to MCM modules

Auslander's Theorem

Let G be a "small" subgroup of $\mathrm{GL}_n(\Bbbk)$ (finite with no reflections), and let $A = \Bbbk[x_1, \ldots, x_n]$. Then the skew-group ring $A\#G$ is isomorphic to $\mathrm{End}_{A^G}(A)$ as algebras ($(a\#g)(x) = ag(x)$).

Pertinency – Bao, He, Zhang (2016)

grade($_A M$) = $j(_A M) := \min\{i : \mathrm{Ext}^i_A(M, A) \neq 0\}$
Assume that A is GK-Cohen-Macaulay
$j(M) + \mathrm{GKdim}(M) = \mathrm{GKdim}(A)$

Def. The *pertinency of the H-action on A* is

$$\mathrm{p}(A, H) := \mathrm{GKdim}\, A - \mathrm{GKdim}((A\#H)/(1_A\# \textstyle\int)).$$

Theorem: TFAE for GKdim$A \geq 2$:

- Auslander's Theorem: $A\#H \cong \mathrm{End}_{A^H}(A)$ naturally.
- $\mathrm{p}(A, H) \geq 2$

Auslander's Theorem holds for:

- CKWZ H semisimple acting on A (AS regular dim 2) with trivial hdet
- BHZ Group actions on $U(\mathfrak{g})$ with trivial hdet for certain $\mathfrak{g}$
- GKMW Permutation actions on $A = \mathbb{k}_{-1}[x_1, \ldots, x_n]$
- GKMW Permutation of x, y, z in generic 3-dim Sklyanin algebra
- GHZ Group actions on most Noetherian graded down-up algebras with trivial homological determinant
- CKZ Group coactions on Noetherian graded down-up algebras with trivial homological determinant
- C Group actions by "small groups" acting on AS regular algebras of dim 2

Question: Does Auslander’s Theorem hold for Hopf actions with trivial homological determinant acting on an AS regular algebra?

$R = \Bbbk[x_1, \dots, x_n]/I$ is a graded isolated singularity if R_p is regular for all homogeneous prime ideals $p \neq (x_1, \dots, x_n)$.

Theorem (Iyama-Yoshino): For $G \subseteq \mathrm{SL}_n(\Bbbk)$. Then $\Bbbk[x_1, \dots, x_n]^G$ is a graded isolated singularity if and only if for all $e \neq g \in G$ have no eigenvalues $= 1$.

Examples: $A = \Bbbk[x_1, x_2, x_3]$

$G = \langle (1\ 2\ 3) \rangle$: A^G NOT an isolated singularity.

'

$G = \langle \begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{bmatrix} \rangle$: A^G IS an isolated singularity.

$$\text{tails}B = \text{gr}B/\text{gr}_0 B.$$

Theorem [BHZ]: When p$(A, H) =$GKdim(A), then tailsA^H has finite gldim, and hence is a graded isolated singularity (Mori-Ueyama (2016)).

Let $V_n = \Bbbk_{-1}[x_1, \ldots, x_n]$.
The following A^G have isolated singularities:

- GKMW Permutation of x, y, z on generic 3-dim Sklyanin algebra
- GKMW $\langle (1\ 2)(3\ 4), (1\ 3)(2\ 4) \rangle$ acting on V_4,
- GKMW $\langle (1\ 2)(3\ 4) \cdots (2n-1\ 2n) \rangle$ acting on V_{2n}
- CYZ Cyclic permutation for V_n when $n = 2^a p^b$ for $p \geq 7$.

ALL groups have an element with eigenvalue 1, in contrast to the commutative case.

[CYZ] Conjecture: Cyclic permutation for V_n is isolated singularity if and only if n not divisible by 3 or 5. True for $n < 77$.

n	2	3	4	5	6	7	8	9	10	11	12	13	14
p	2	2	4	4	5	7	8	8	9	11	[8,11]	13	14

2, 2, 4, 4, 5, 7, 8, 8, 9, 11 is not in Online Encyclopedia of Integer Sequences!

Question:

When is A^H a graded isolated singularity?

THANKS !