Quantum symmetries and phases of matter: a physicist's perspective

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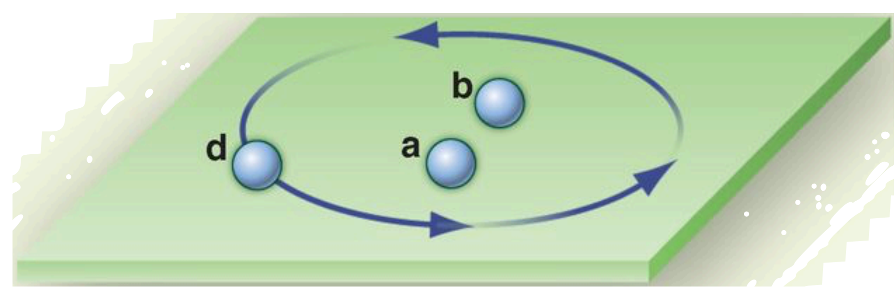
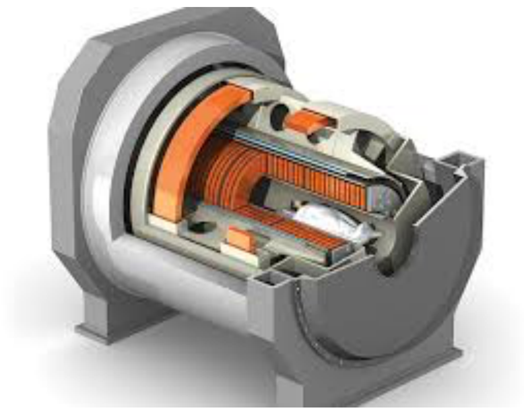
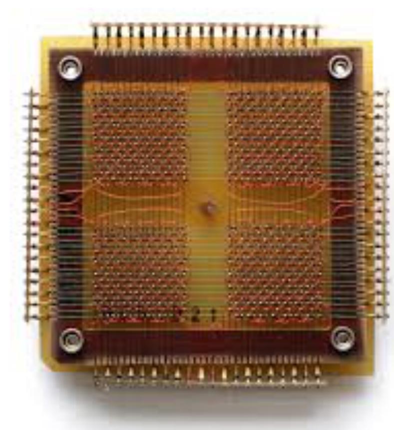


Condensed matter physics: what can materials do?

Superconductivity

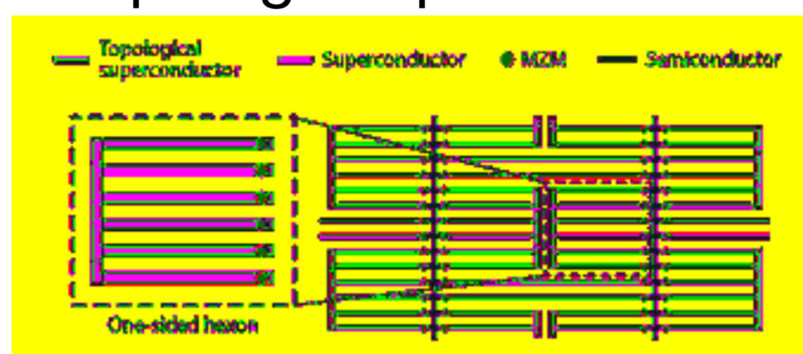


magnetism

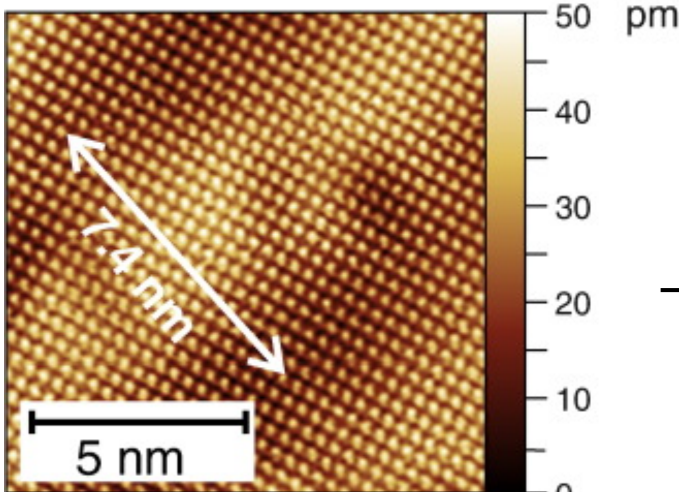


semiconductors

topological phases



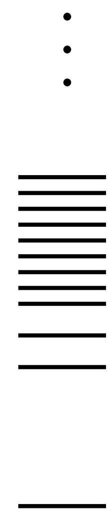
How to approach this problem?



Microscopically, a real material with a complicated Hilbert space

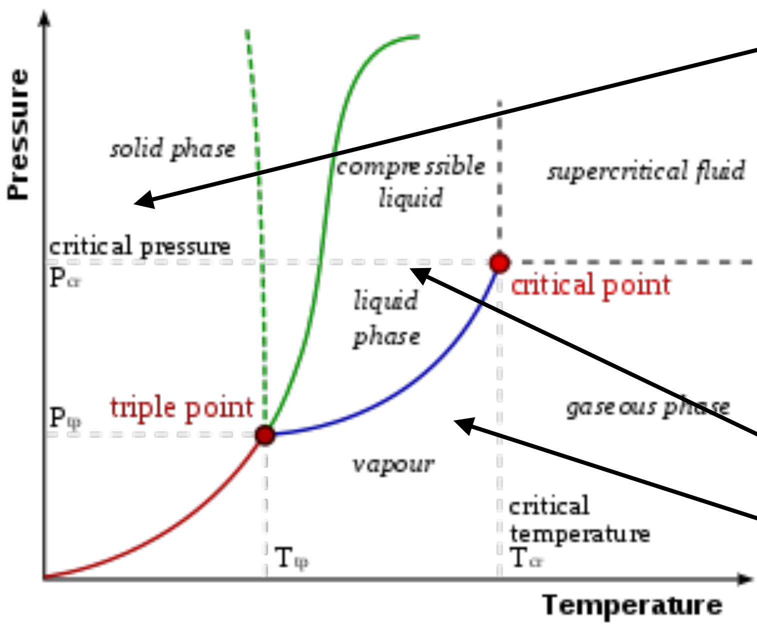
H

Lots of chemistry/ physics…



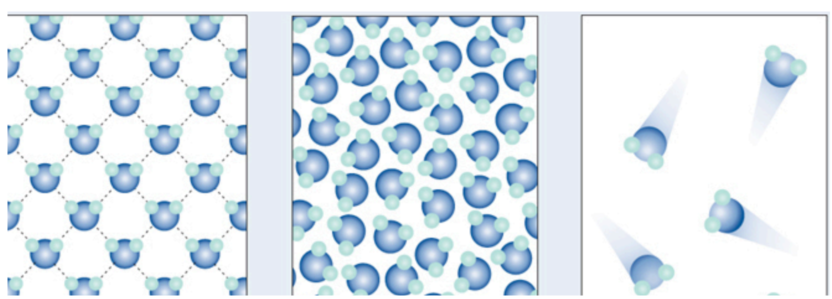
Spectrum of H

Phases of matter: a paradigm for classifying possible material behaviors



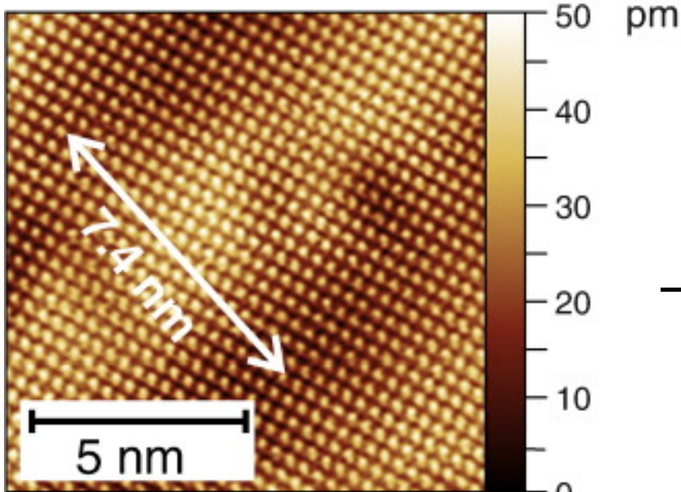
Solid region: crystalline order (broken translational symmetry)

Liquid and gas: no broken translational symmetry.



Many different microscopic systems in a phase, but all have the same low-temperature behavior

How to approach this problem?

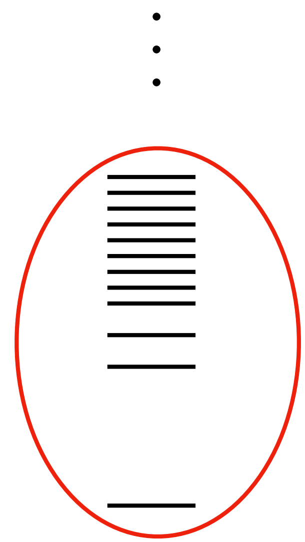


Microscopically, a real material with a complicated Hilbert space

H →

chemistry/ physics…

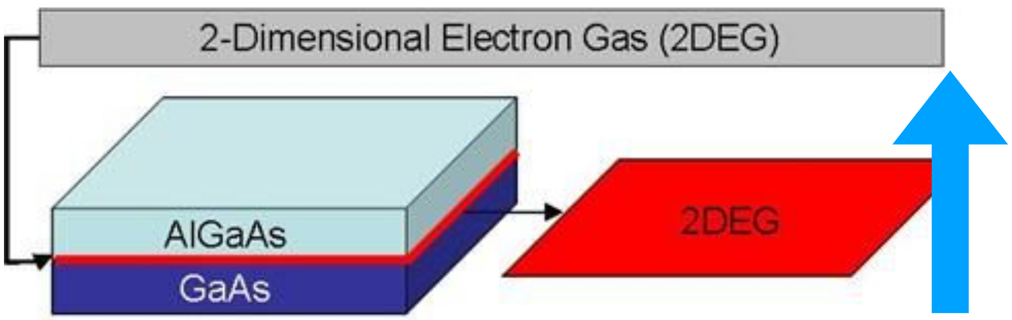
Math



Spectrum of H

- Phase of matter (and low-temperature properties): understand the ground state and the low-energy excited states of the simplest possible H in that phase.

What does this have to do with you? An example

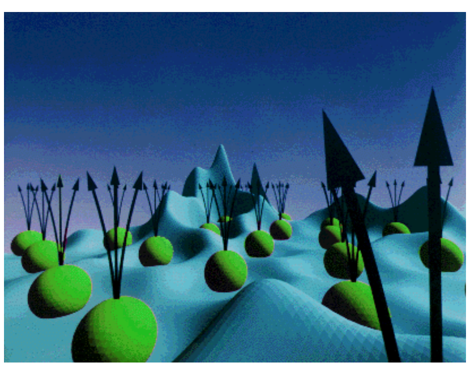


strong magnetic field

Microscopically, electrons *(charge 1)* in a strong magnetic field

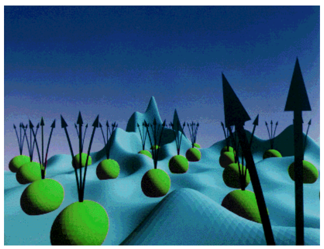
H

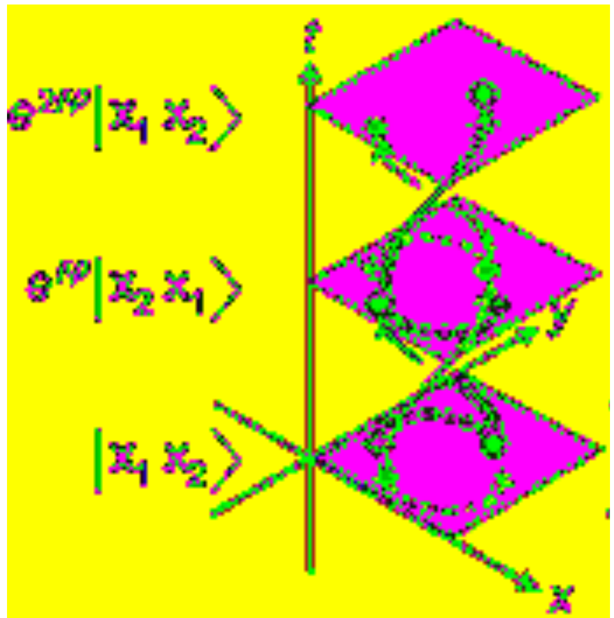
(Specifics not important)



Low-lying excited states: quasiparticles with fractional charge (e.g 1/(2l+1))

What does this have to do with you? An example

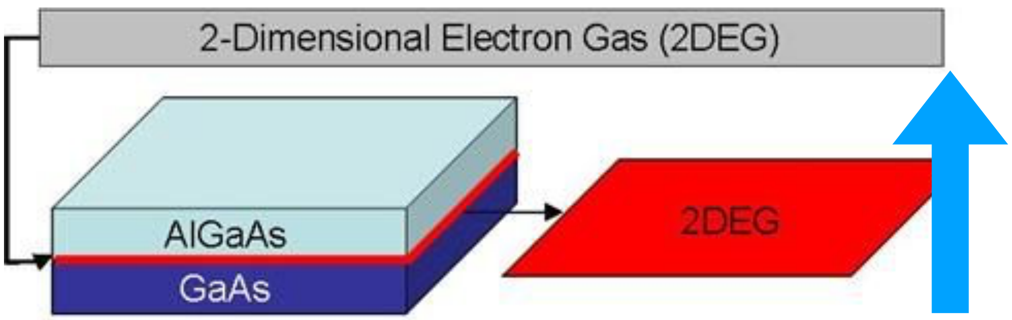




- Quasiparticles with fractional charge are “anyons”

- Exchange them twice and the wave function changes by a phase

What does this have to do with you? An example



strong magnetic field

Microscopically, electrons *(charge 1)* in a strong magnetic field

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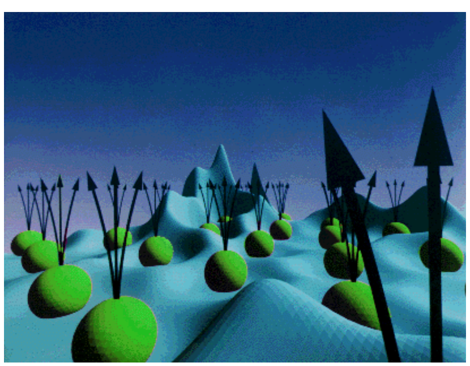


Mathematical description: anyons = simple objects in a UMTC (with projective U(1) symmetry action)

Summary: from materials to math

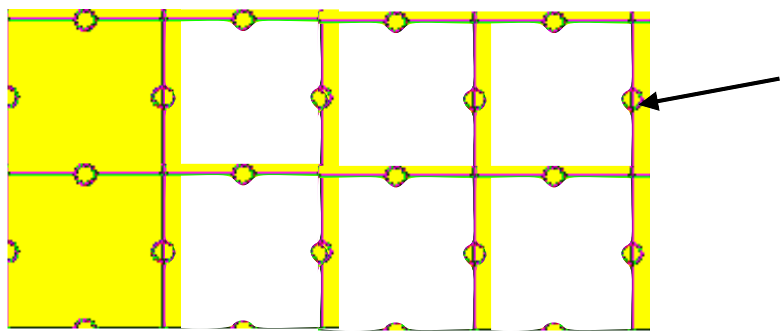
- Material properties (at 0 temperature and long lengthscales):
  - Determined by ground state and low-lying excited states (quasiparticles)
  - basically the same within a phase
- Effective description:
  - Simplified Hilbert space & Hamiltonian (maybe nothing to do with real material)
  - Can be used to deduce correct mathematical structure (e.g. UMTC)

Goals of the rest of the talk

- Introduce some (simple) Hamiltonians that describe “fracton” phases in 3+1 d.
- Tell you some things that are understood about their structure and how to generate more examples
- Describe some open questions about these types of systems which may be mathematically interesting

Simple Hamiltonian warm-up: the Toric code

Hilbert space: 2 states per edge

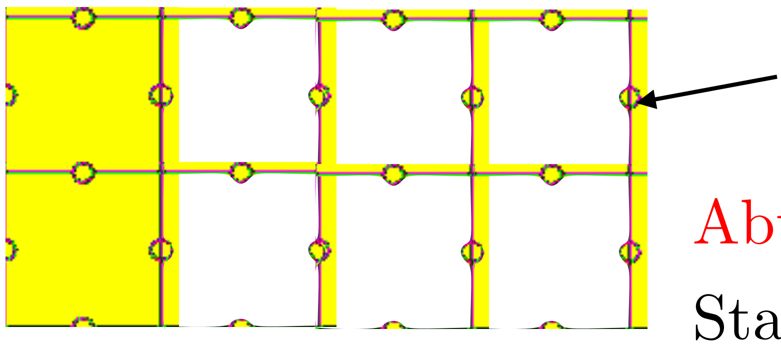


$\sigma_i^z \in \{-1, 1\}$

- Example of a "simple" Hamiltonian leading to a topological phase (Particles are simple objects in D($\mathbb{Z}_2$))

Simple Hamiltonian warm-up: the Toric code

Hilbert space: 2 states per edge



$$\sigma_i^z \in \{-1, 1\}$$

Abuse of notation!

State: σ^z = Eigenvalue

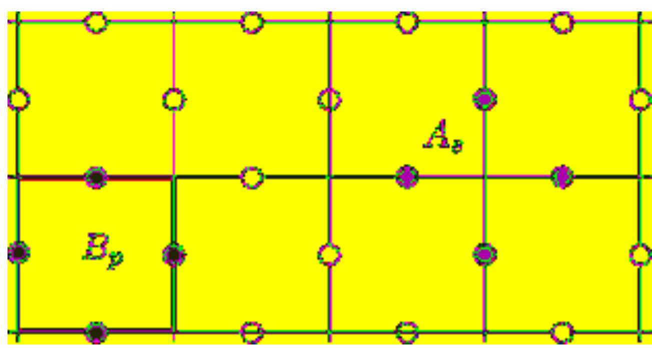
Operator: σ^z = Pauli matrix

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^x \sigma^z = -\sigma^z \sigma^x$$

Simple Hamiltonian warm-up: the Toric code

Hilbert space: 2 states per edge



$\sigma_i^z \in \{-1, 1\}$

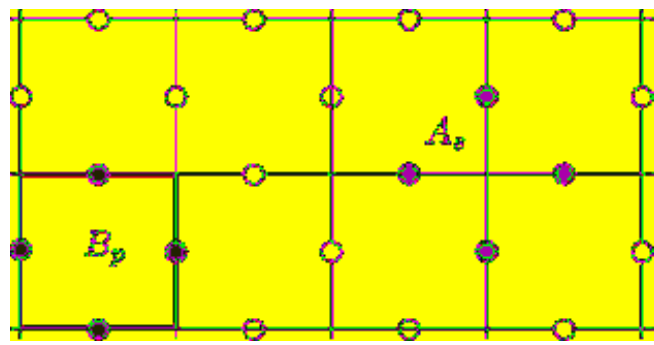
$$B_P = \prod_{i \in \partial p} \sigma_i^x \qquad A_v = \prod_{i \in {}^* v} \sigma_i^z$$

$$\sigma^x \sigma^z = -\sigma^z \sigma^x$$

$$[A_v, B_p] = 0$$

Simple Hamiltonian warm-up: the Toric code

Hilbert space: 2 states per edge



$$\sigma_i^z \in \{-1, 1\}$$

$$B_P = \prod_{i \in \partial p} \sigma_i^x \qquad A_v = \prod_{i \in^* v} \sigma_i^z$$

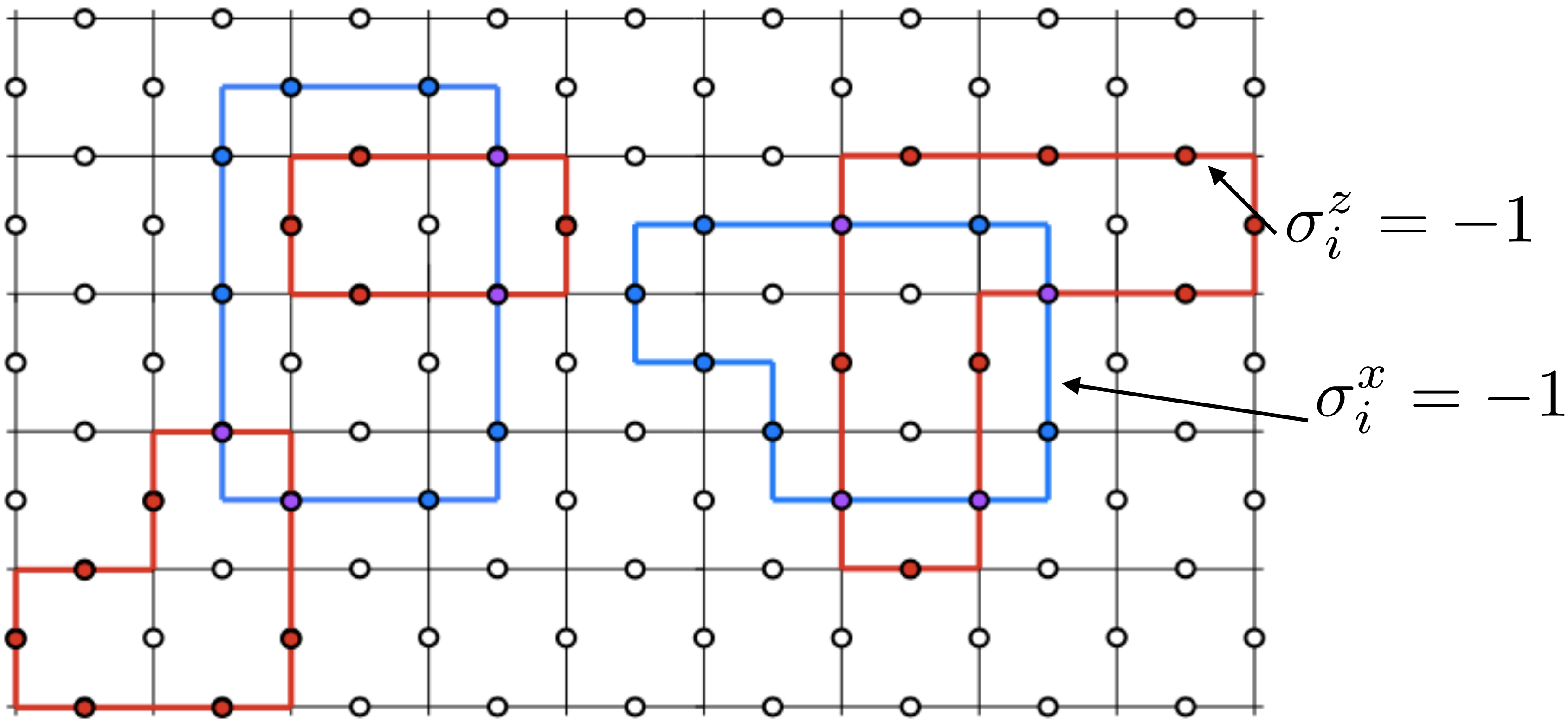
$$H = -\sum_p B_p - \sum_v A_v$$

Hamiltonian: commuting projectors

Ground state: loop gas /string net

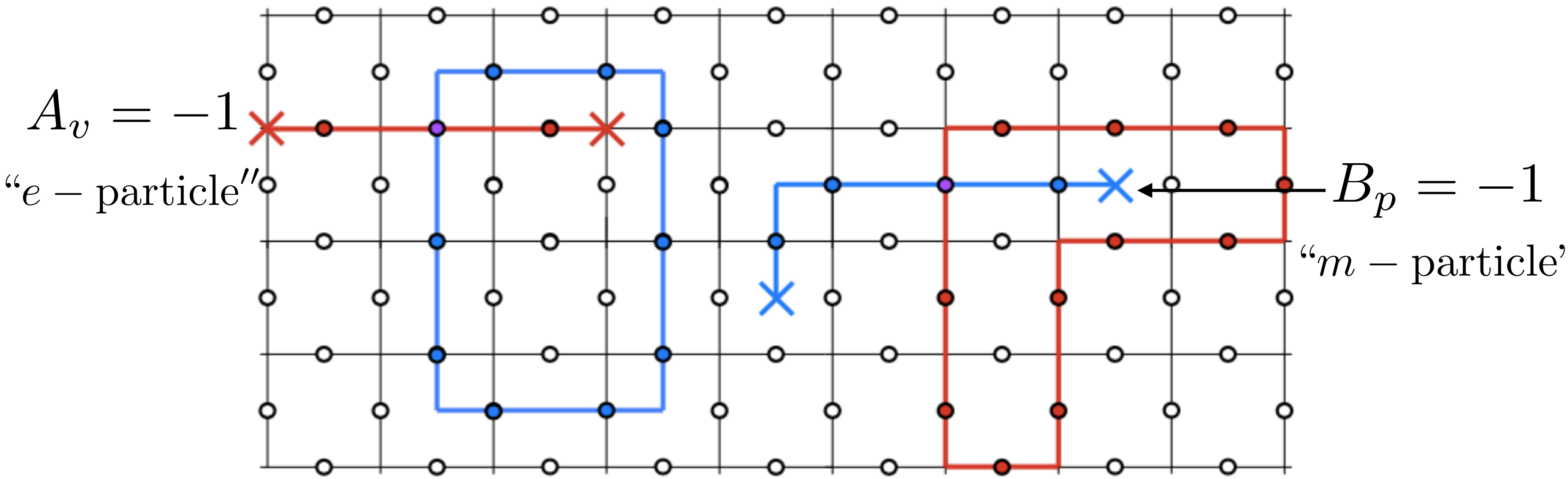
$$H = -\sum_p B_p - \sum_v A_v$$

$$B_p = \prod_{i \in \partial p} \sigma_i^x = 1 \qquad A_v = \prod_{i \in * v} \sigma_i^z = 1$$



Quasiparticles in the Toric code

$$H = -\sum_p B_p - \sum_v A_v$$



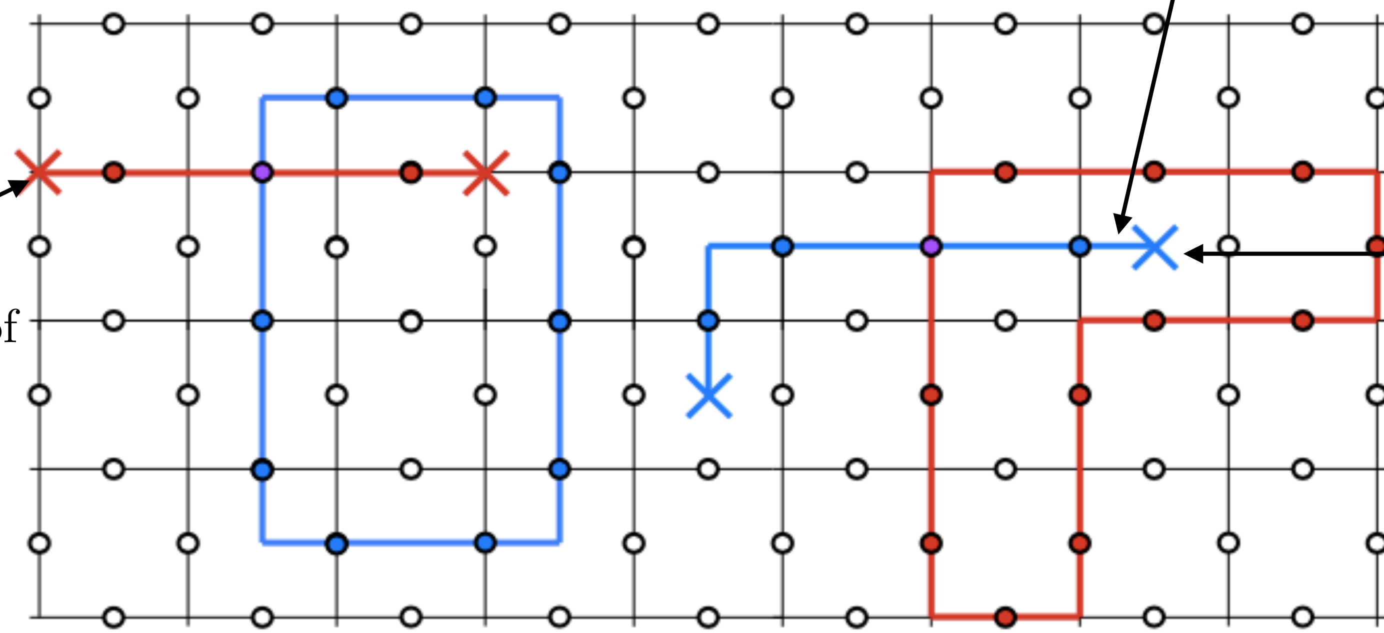
Quasiparticles in the Toric code

$$B_P = \prod_{i \in \partial p} \sigma_i^x \qquad A_v = \prod_{i \in^* v} \sigma_i^z$$

Operator: $S_v = \prod_i \sigma_i^x$

Operator: $S_p = \prod_i \sigma_i^z$

$A_v = -1$



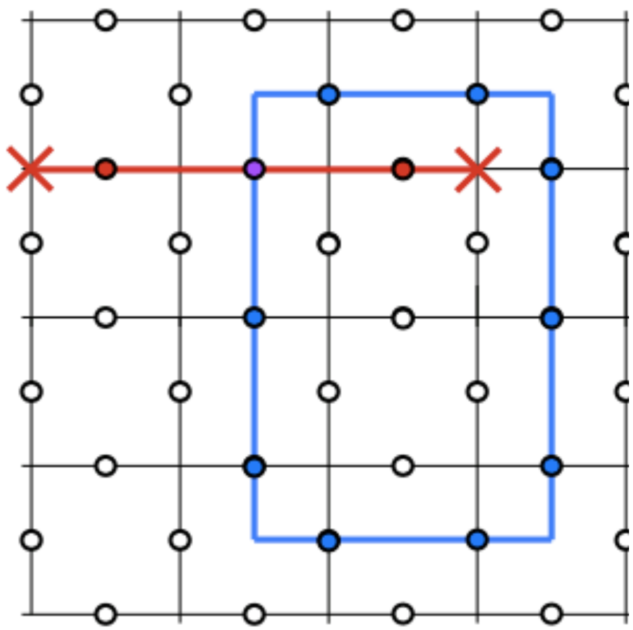
$B_p = -1$

Odd number of $\sigma_i^z i \in^* v$

Odd number of $\sigma_i^x i \in \partial p$

Statistics

$$\sigma_i^z \sigma_i^x = -\sigma_i^x \sigma_i^z$$

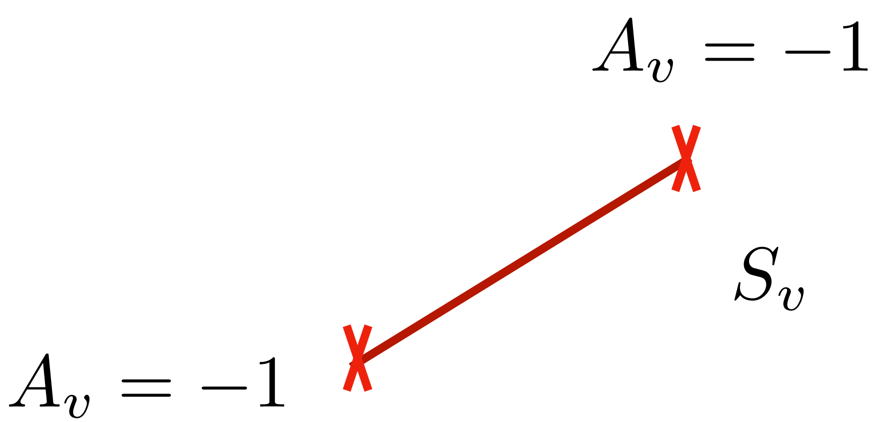


$$S_p^{-1} S_v^{-1} S_p S_v = -1$$

Statistics

$$S_p^{-1} S_v^{-1} S_p S_v = -1$$

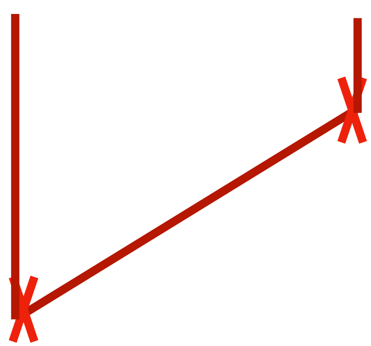
$$\text{time } t_0$$



Statistics

$$S_p^{-1} S_v^{-1} S_p S_v = -1$$

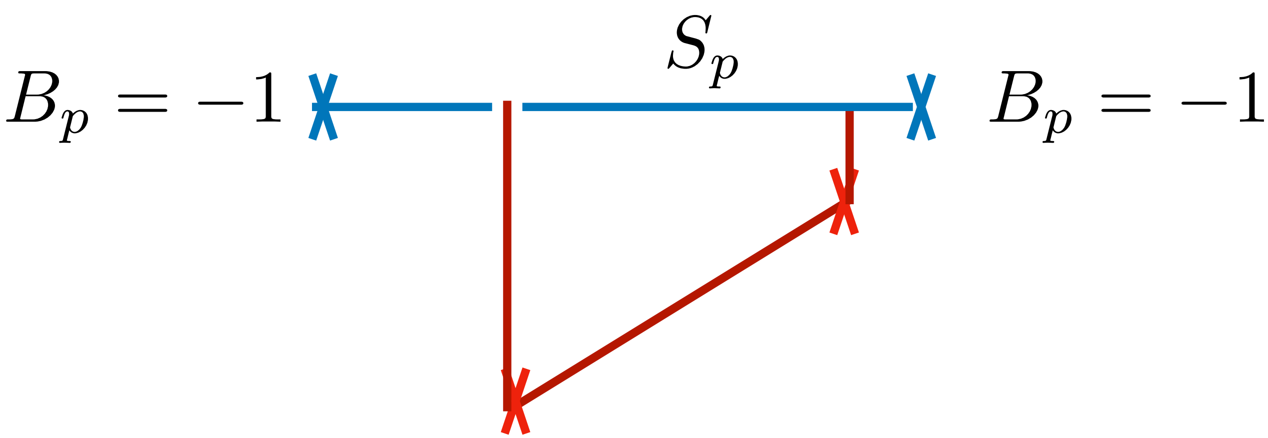
Evolve forward in time...



Statistics

$$S_p^{-1} S_v^{-1} S_p S_v = -1$$

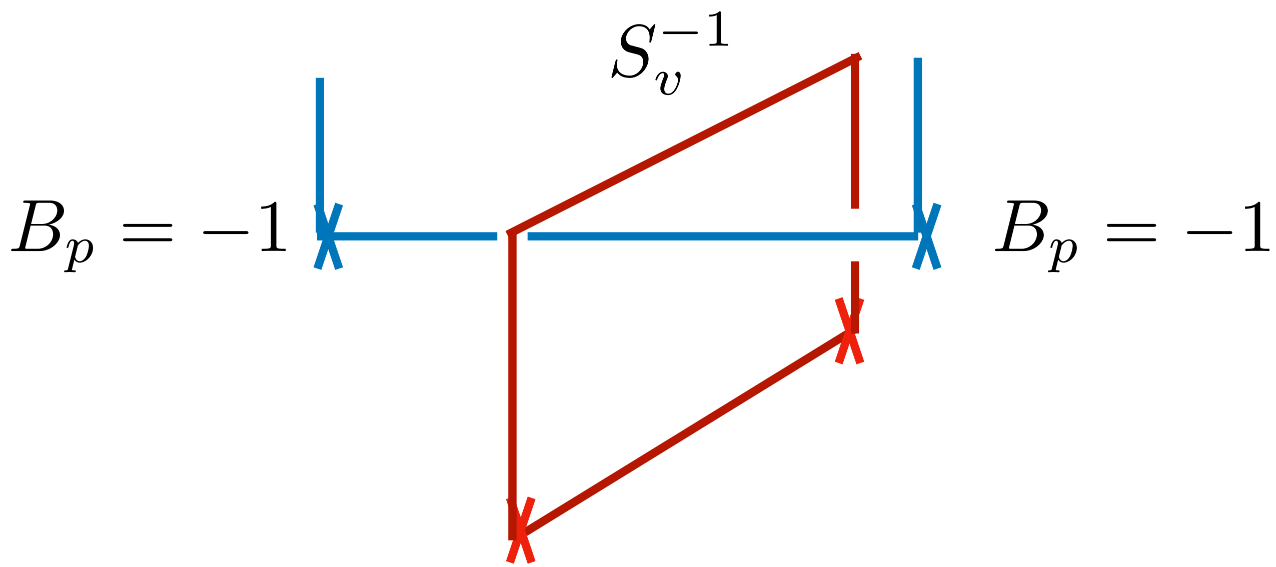
time t_1



Statistics

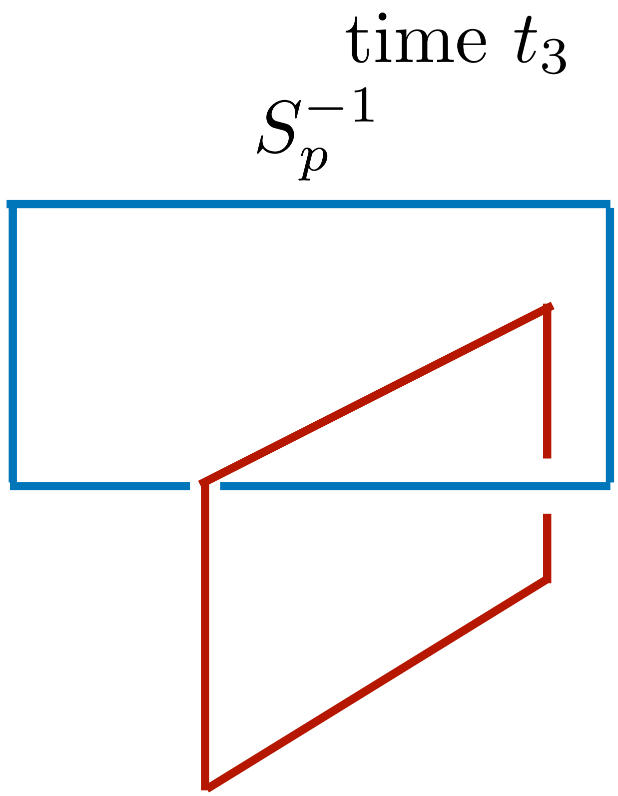
$$S_p^{-1} S_v^{-1} S_p S_v = -1$$

time t_2

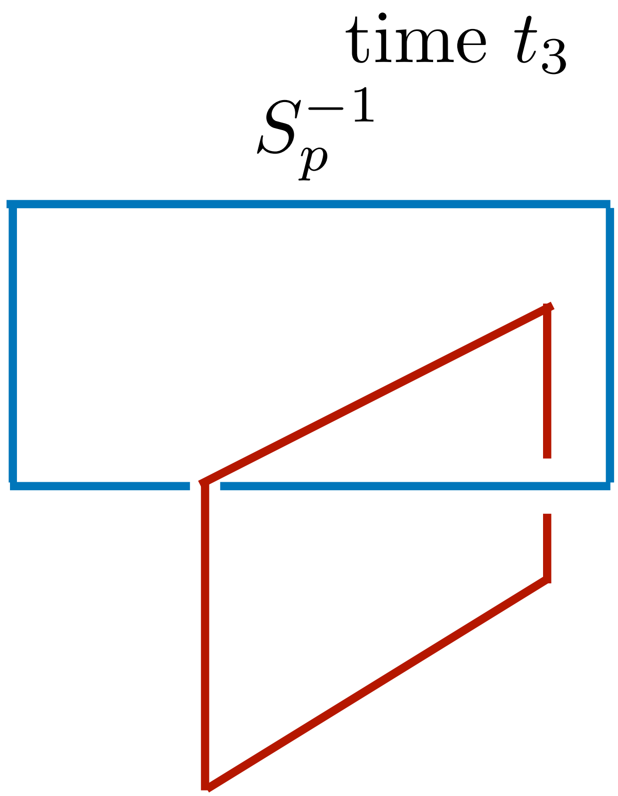


Statistics

$$S_p^{-1} S_v^{-1} S_p S_v = -1$$



Statistics



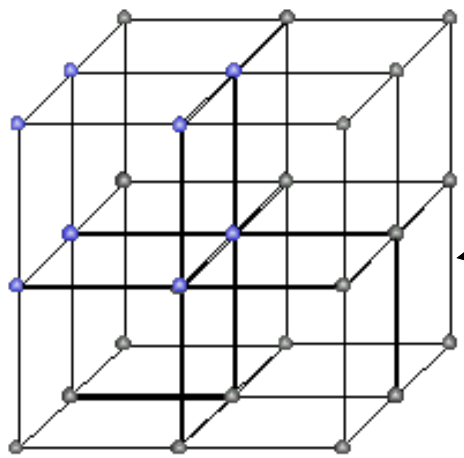
- Space-time process of "braiding" (S-matrix in UMTC) described by the operator product $S_p^{-1} S_v^{-1} S_p S_v = -1$

Simple Hamiltonians for Fracton phases

- Fracton phase: has some features reminiscent of topological order (UMTC), but many important differences, including
  - Explicit dependence on a lattice geometry (no smooth space-time)
  - Particles with restricted mobility
  - statistical (braiding-like) interactions in 3D, due to restricted mobility
- Goal: explore these properties and mathematical structure through simple models. (General framework not yet known)

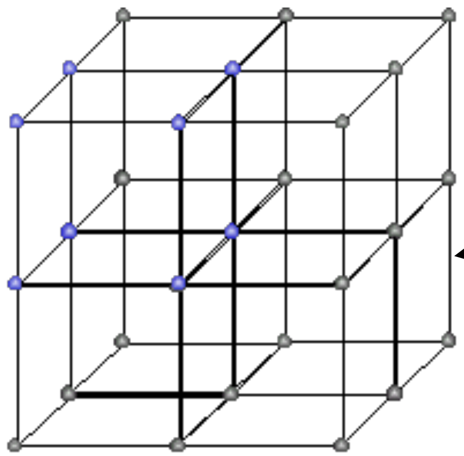
Vijay, Haah, Fu

Model 1: X-cube



$\sigma_i^z \in \{-1, 1\}$

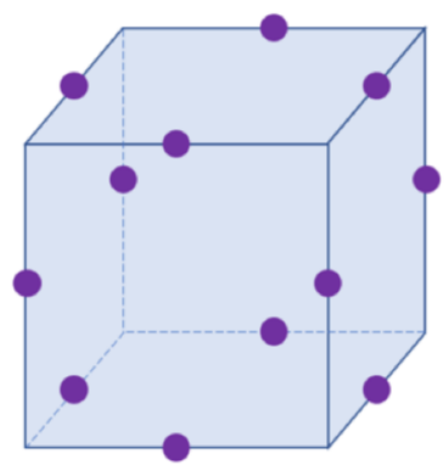
Model 1: X-cube

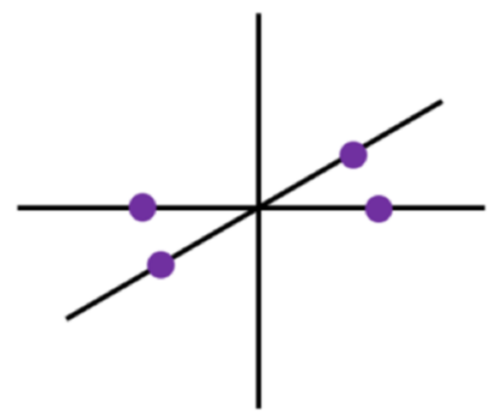


$$\sigma_i^z \in \{-1, 1\}$$

Commuting operators

$$A_v^i, \;\; B_c$$





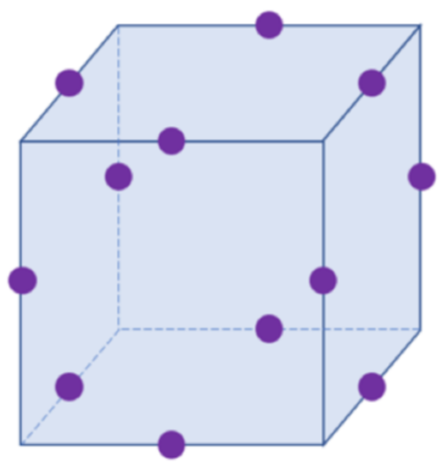
$$B_c = \prod_{i \in \partial c} \sigma_i^x$$

$$A_v^z = \prod_{i \in *^{xy} v} \sigma_i^z$$

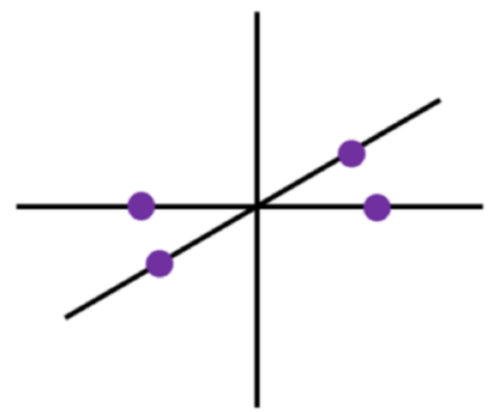
Model 1: X-cube

$$H = -\sum_v (A_v^x + A_v^y + A_v^z) - \sum_c B_c$$

Sum of Commuting projectors



$$B_c = \prod_{i \in \partial c} \sigma_i^x$$



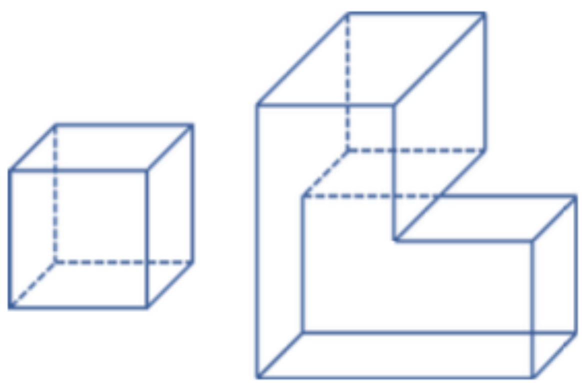
$$A_v^z = \prod_{i \in *_{xy} v} \sigma_i^z$$

Prem, Huang, Song, Hermele

Ground state: “cage net”

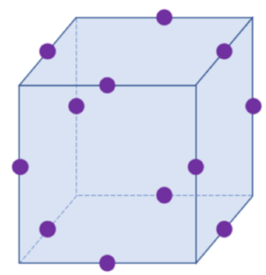
$$H = -\sum_v (A_v^x + A_v^y + A_v^z) - \sum_c B_c$$

$A_v^x = A_v^y = A_v^z = +1$ **: Even number of blue edges in each plane at each vertex**

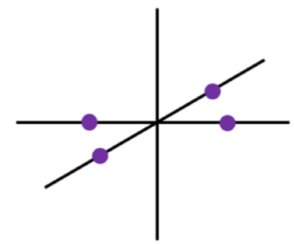


$B_c = +1$ **: equal amplitude superposition of all cages**

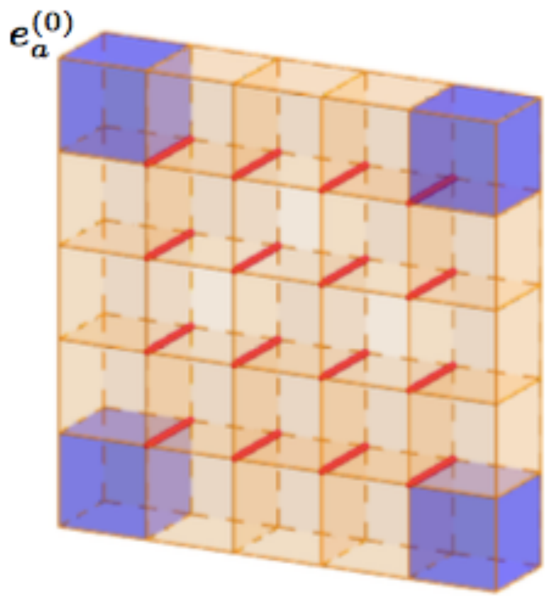
Excitations of X-cube



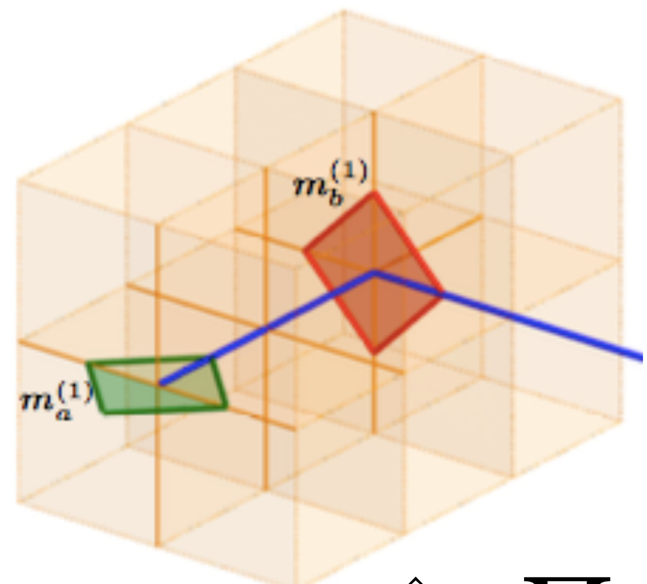
$$B_c = \prod_{i \in \partial c} \sigma_i^x$$



$$A_v^z = \prod_{i \in *^{xy} v} \sigma_i^z$$

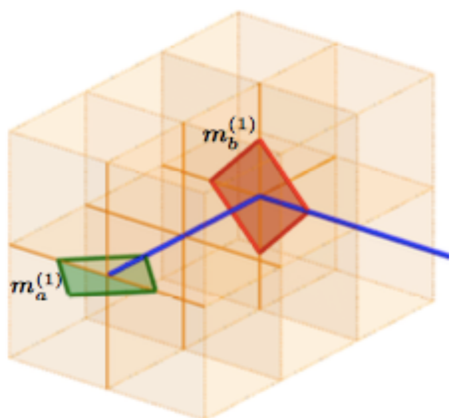


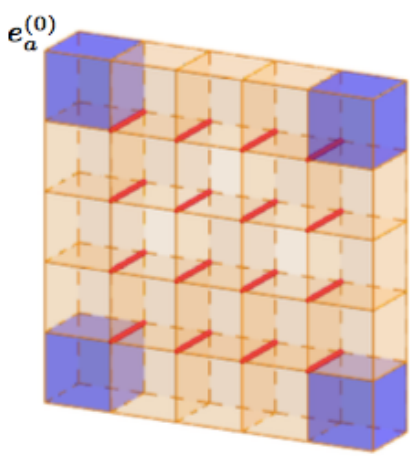
Membrane operator: $\hat{M} = \prod_i \sigma_i^z$



line operator: $\hat{l} = \prod_i \sigma_i^x$

Notions of fusion





Pai, Hermele

- Like in Toric code, we have

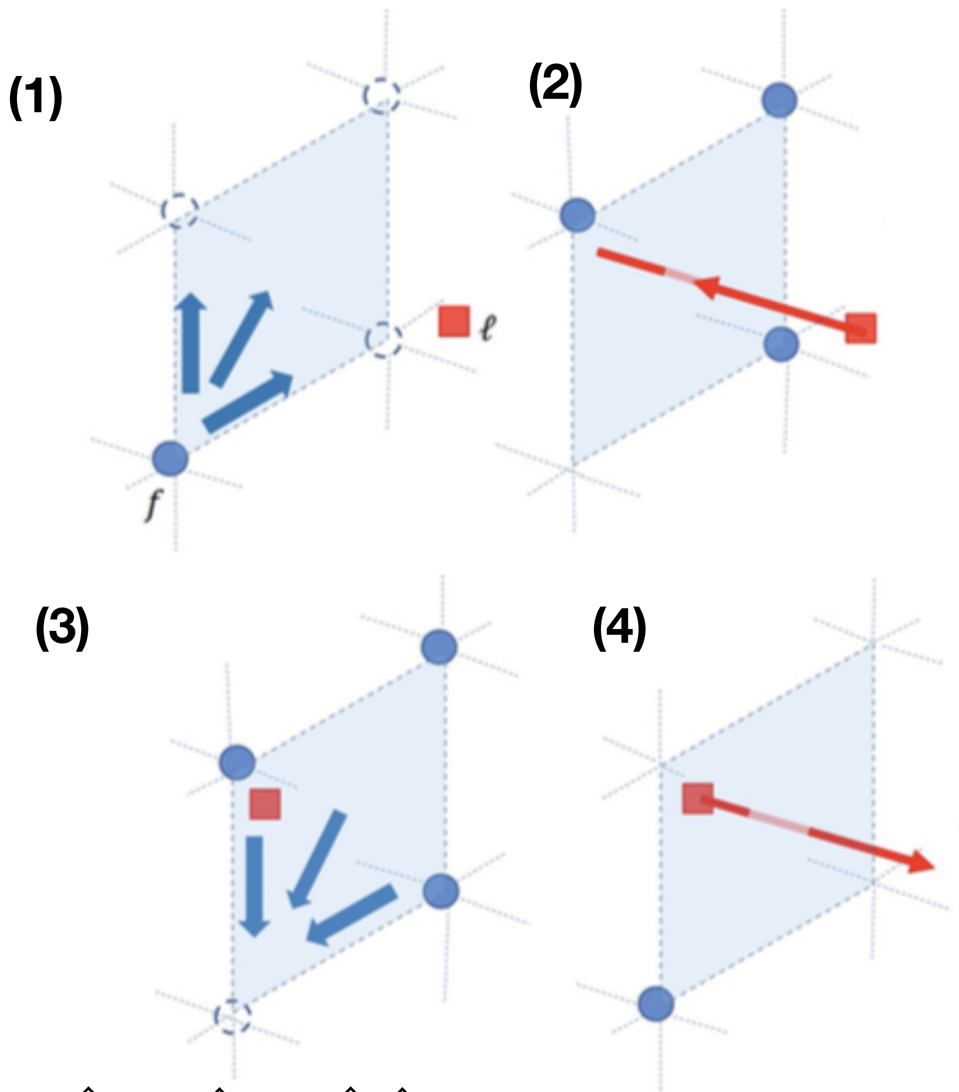
$$e_i \times e_i = 1$$
$$m_i^a \times m_i^a = 1$$

- But lines cannot turn corners, so there is a distinct m^x for each value of y and z

- Membranes must be square, so there's a different e for every site.

Pai, Hermele
You, Devakul, FJB, Sondhi

Statistical processes:

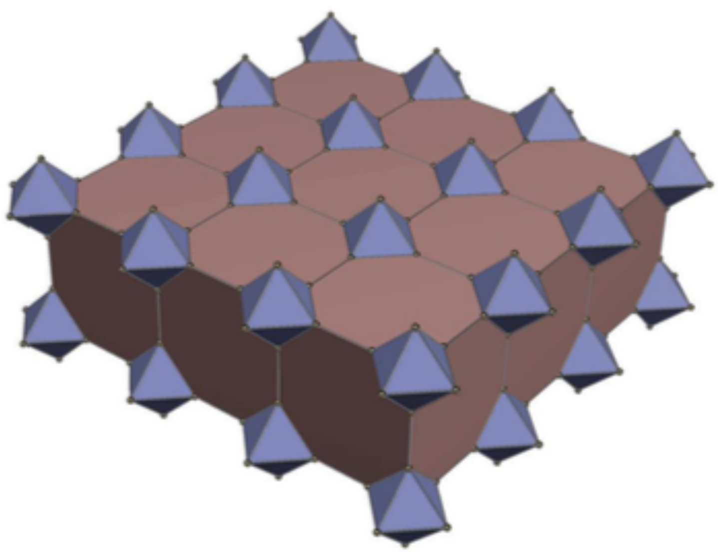


$$\hat{l}^{-1}\hat{M}^{-1}\hat{l}\hat{M} = -1$$

- This is different from statistics associated with 2D topological phases: extra particles are created at intermediate times
- But we call it statistics because it is invariant under a (restricted) family of geometric distortions
- The membrane cannot be pulled over the line if we require that the fractons are never close to the lineon

You, Devakul, FJB, Sondhi
Ma, Hermele

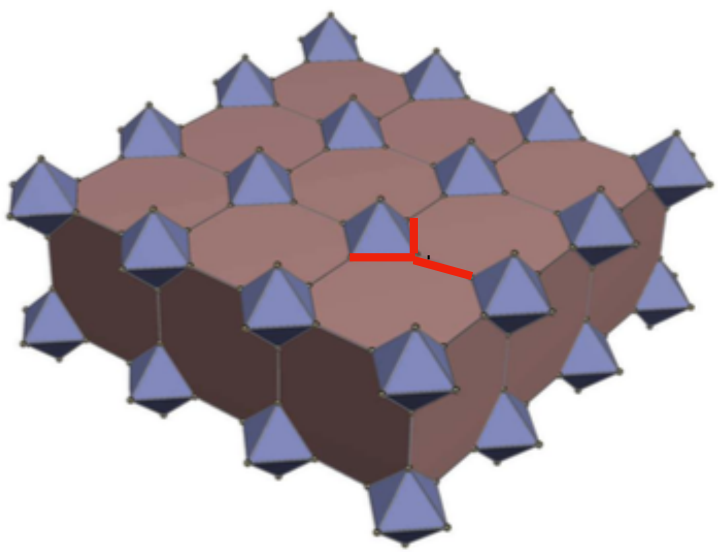
Model 2: Twisted X-cube



$$\sigma_i^z \in \{-1, 1\}$$

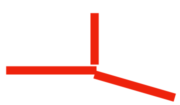
on each edge

Model 2: Twisted X-cube



Commuting operators

$$A_v^i, \; B_c, B_p^i$$



$$A_v^z = \prod_i \sigma_i^z$$

$$B_p^z = (i)^n \prod_i \sigma_i^x$$

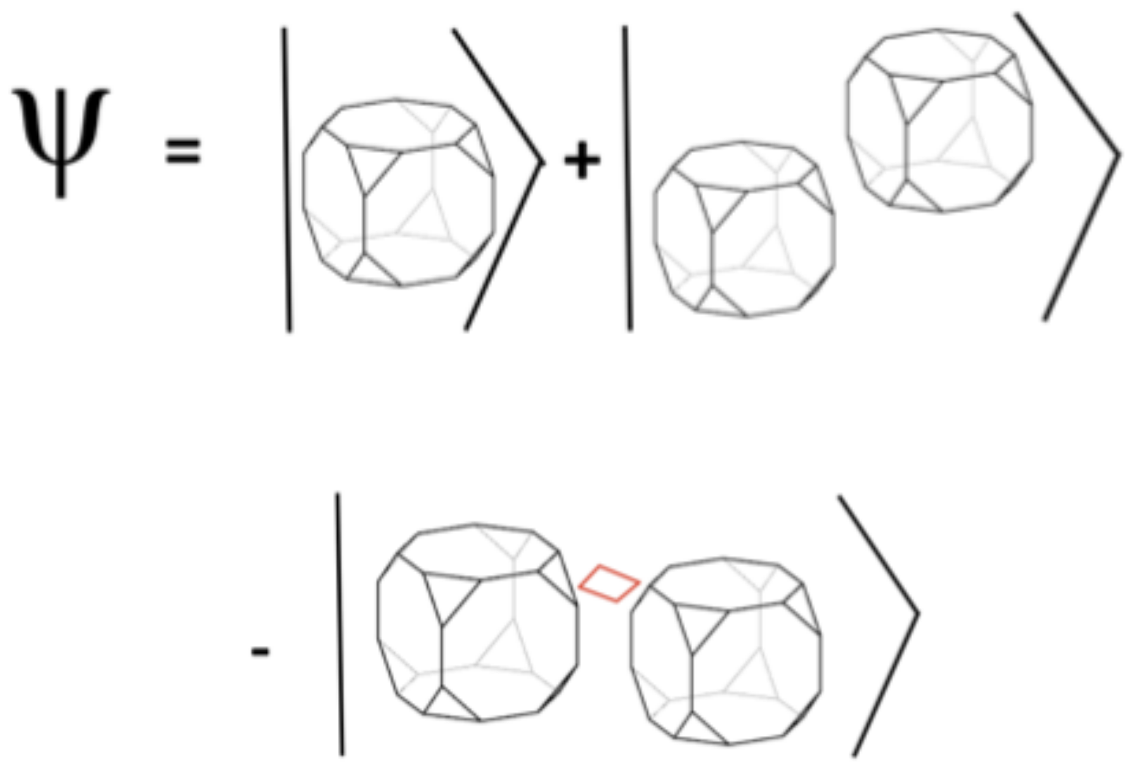
$$B_c = (i)^n \prod_i \sigma_i^x$$

Why the phases?

$$B_p^z = (i)^n \prod_i \sigma_i^x \qquad B_c = (i)^n \prod_i \sigma_i^x$$

- A similar adaptation of the toric code gives a twisted Z_2 gauge theory
- Connections to distinct phases of matter with no topological order and the same Z_2 symmetry

Ground state: cage net

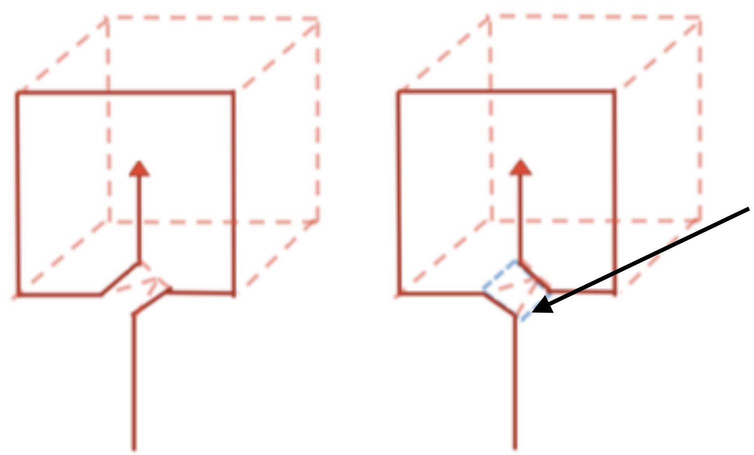


- (Here the relative - sign is important!)

Twisted X-cube excitations

- As before, we get membrane operators creating fractons, and line operators creating lineons, mobile in only 1 dimension

- New kind of “statistical” process:

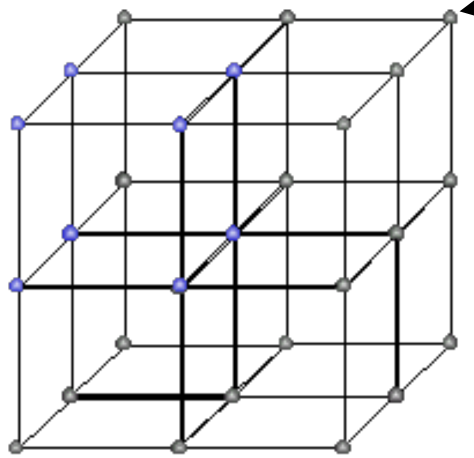


Difference between the two processes: right and left twists differ by a sign!

Twisting lineon left **Twisting lineon right**

Haah 2011

Model 3: Haah code

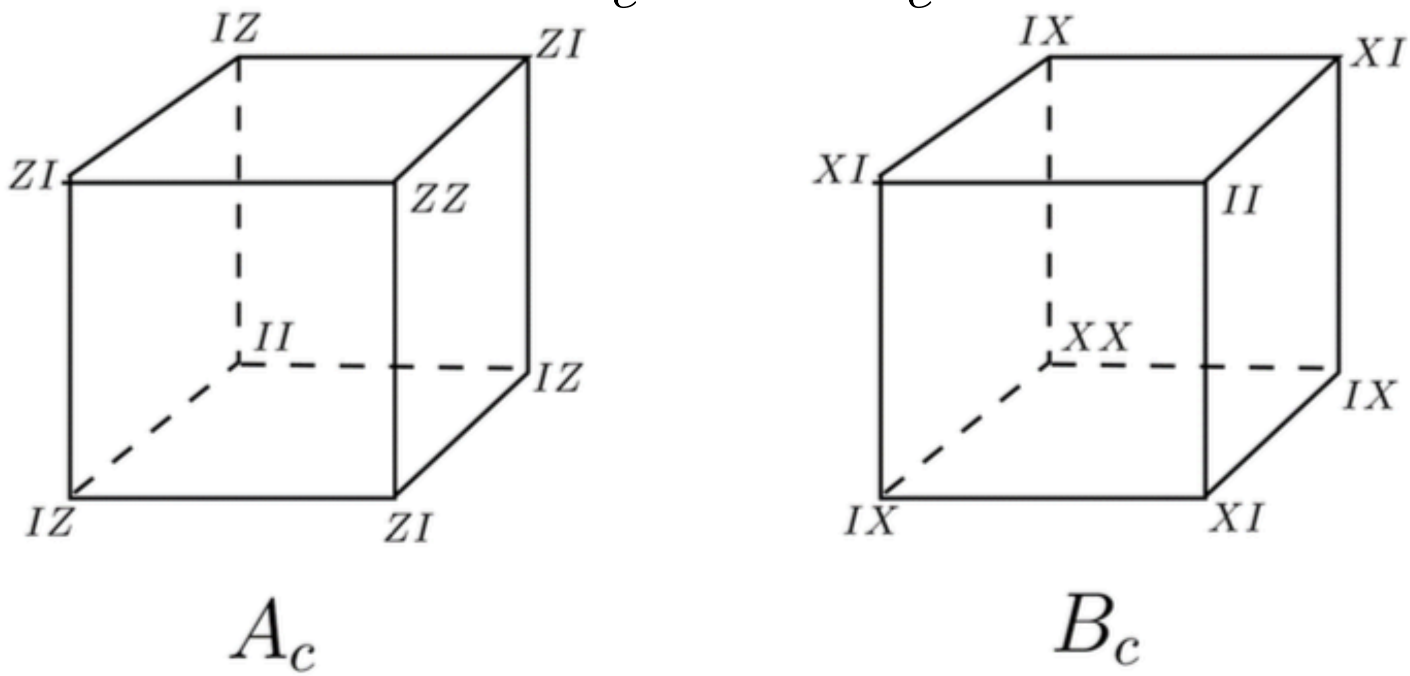


$\sigma_i^z \in \{-1, 1\}$
$\tau_i^z \in \{-1, 1\}$

- 4 states at each site

Model 3: Haah code

$$H = -\sum_c A_c - \sum_c B_c$$

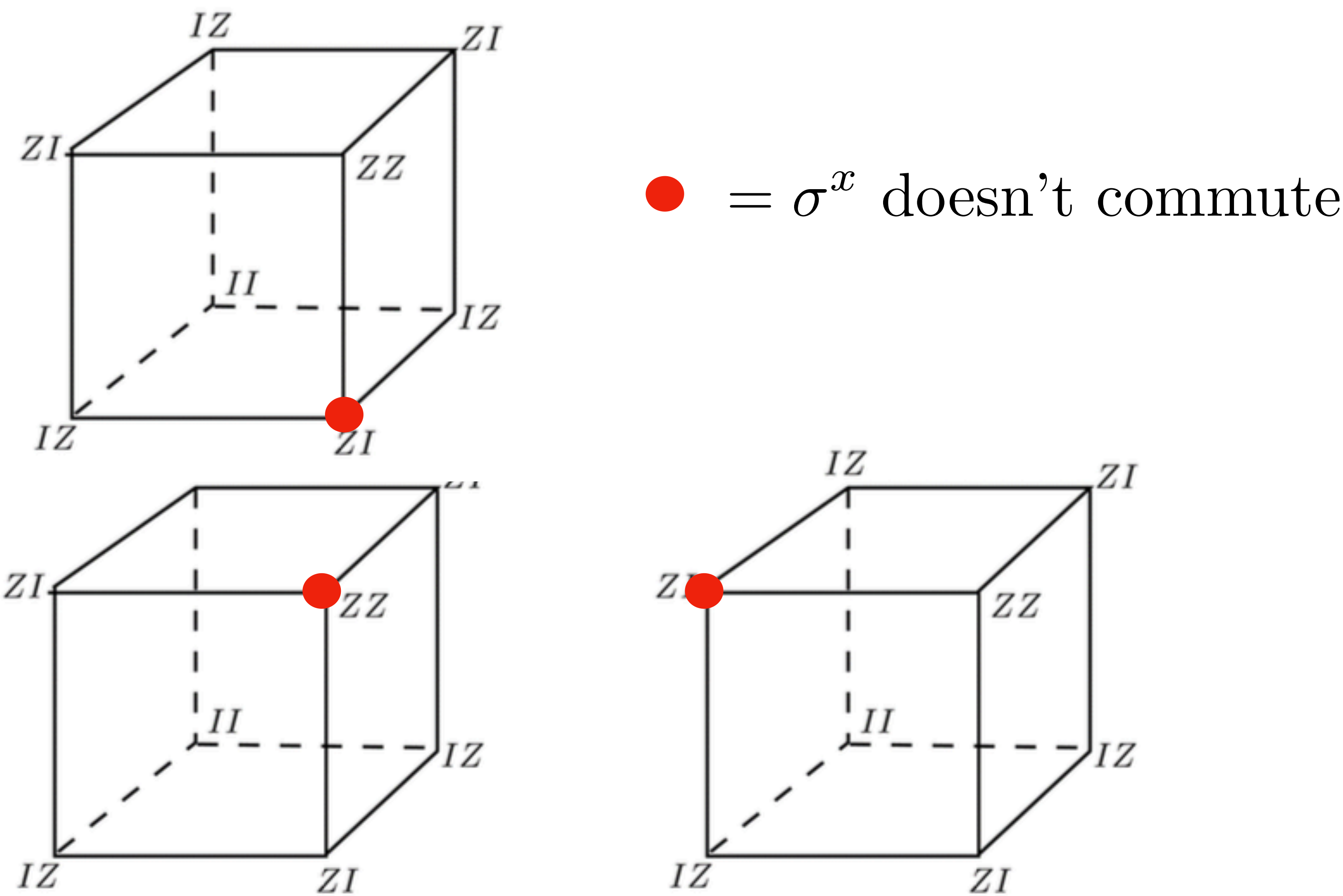


$$IZ \equiv \mathbf{1} \times \tau^z$$

$$XI \equiv \sigma^x \times \mathbf{1}$$

etc.

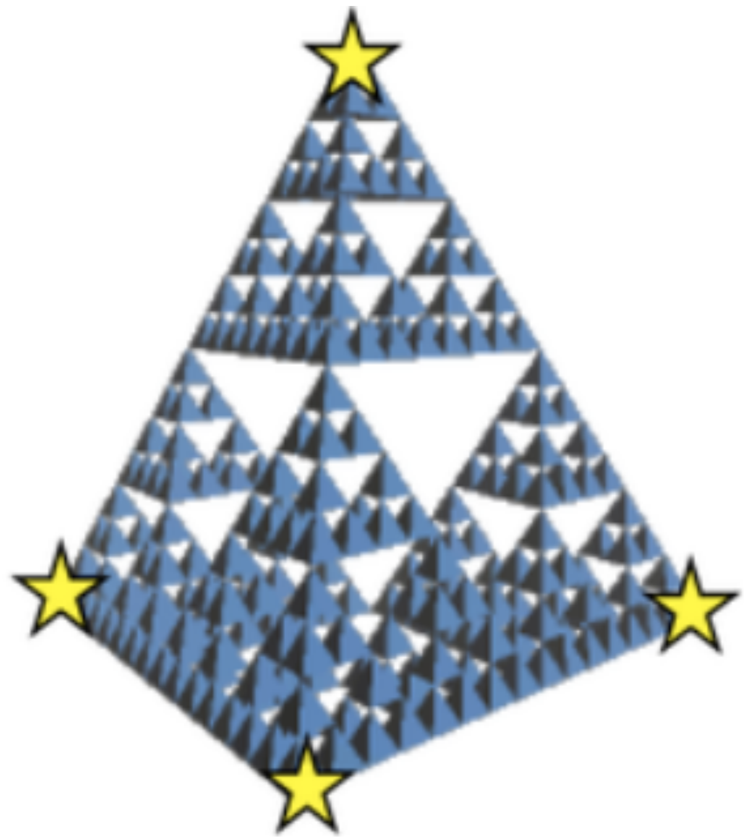
Excitations of Haah's code



- Makes a tetrahedron of 4 cubes no longer in ground state

Excitations of Haah's code

- Only immobile particles (fractons)
- These occur at the "boundary" of a fractal-like structure
- "statistics": (2 kinds of excitations, made by fractal arrays of σ^x, σ^z)
- Fusion: distinct particle for each site

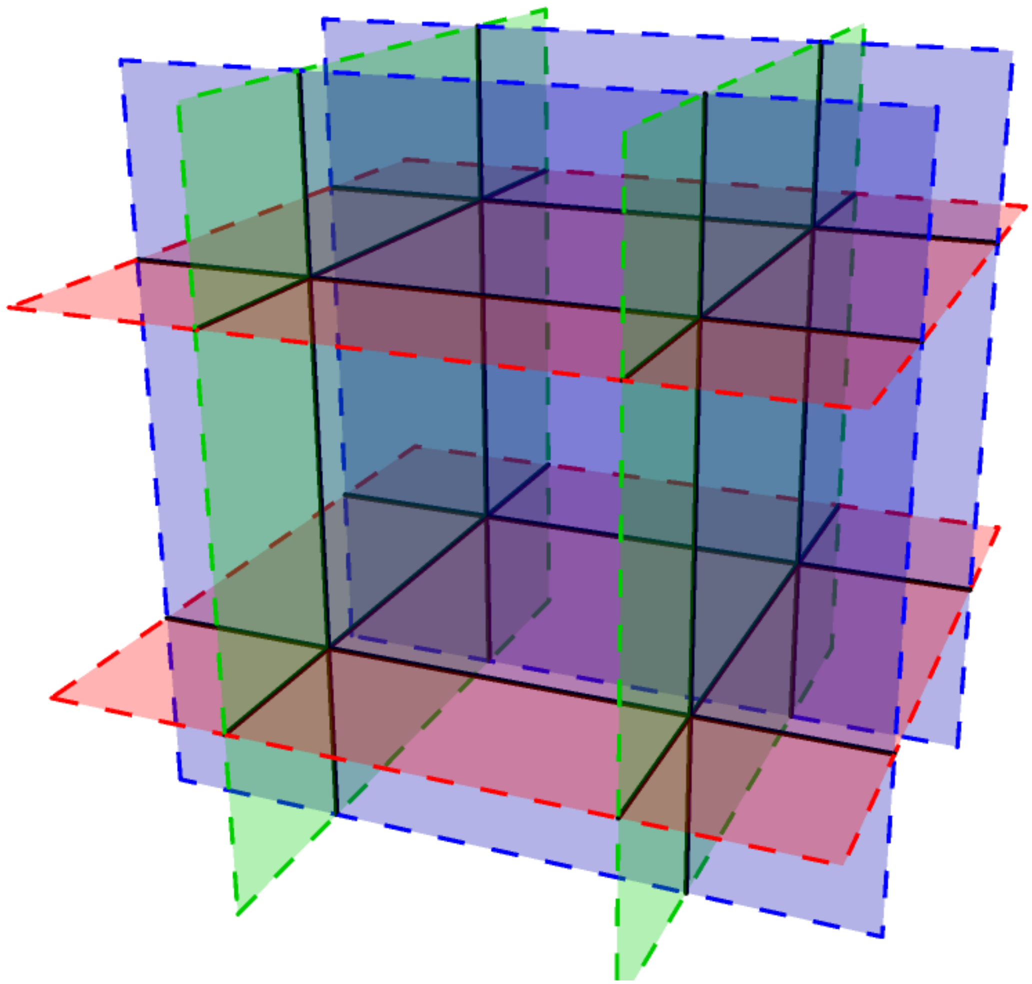


Summary: models

- X-cube, twisted X-cube:
  - immobile “fracton” particles at corners of membranes, and 1-d “lineon” particles at ends or corners of lines
  - Some generalized notion of statistics (braiding-like process) between these; possible in 3+1 d due to restricted mobility
  - “Type I” fracton model
- Haah code:
  - Fractons at boundary of fractal structure
  - “Type II” fracton model

Type I models: hints at the underlying structure

- How to make X-cube from the Toric code:

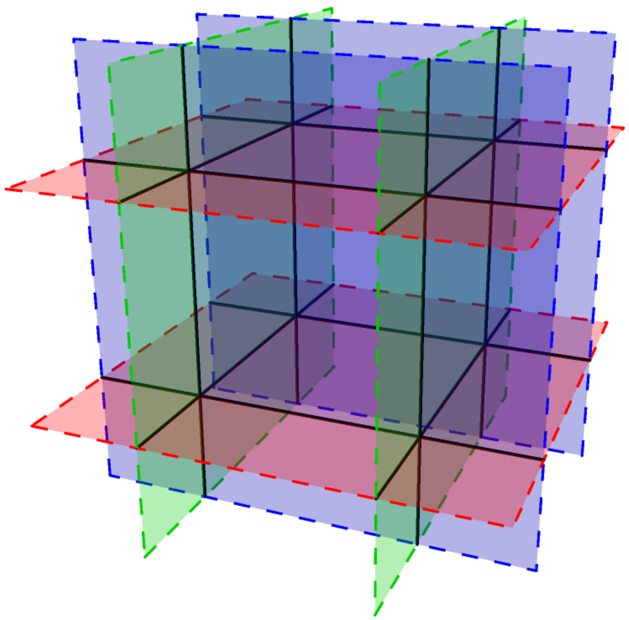


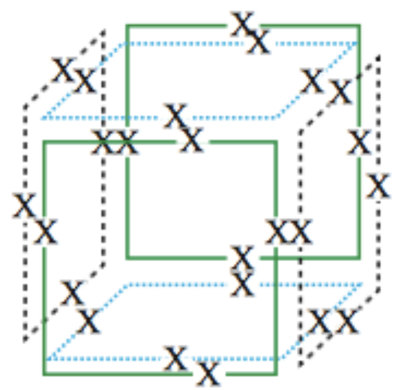
- 3 stacks of decoupled, 2 (spatial) dimensional Toric code models

Ma, Hermele;
Slagle, Aasen, Williamson

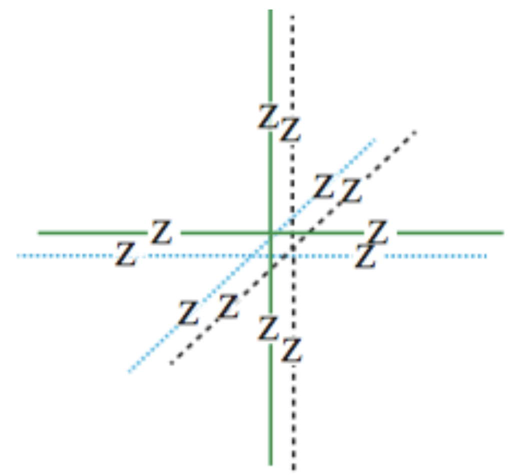
Type I models: hints at the underlying structure

- How to make X-cube from the Toric code:





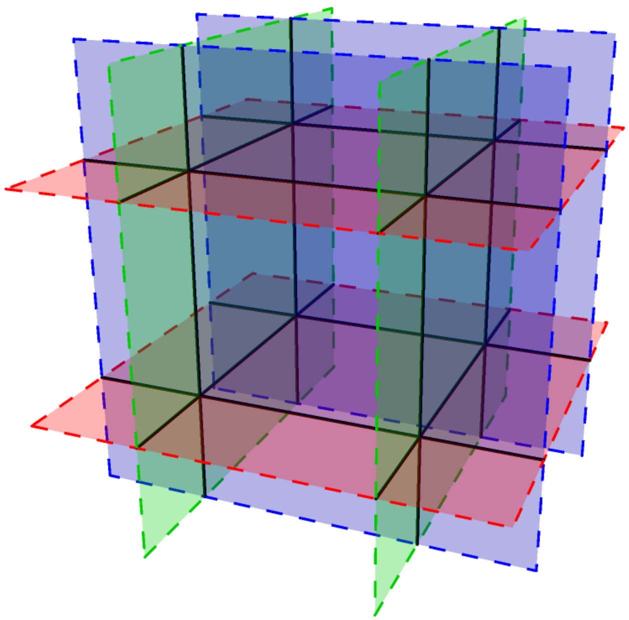
$$B_c = \prod_{p \in \partial_c} B_p$$

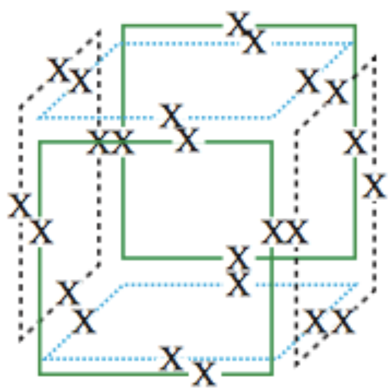


$$A_v$$

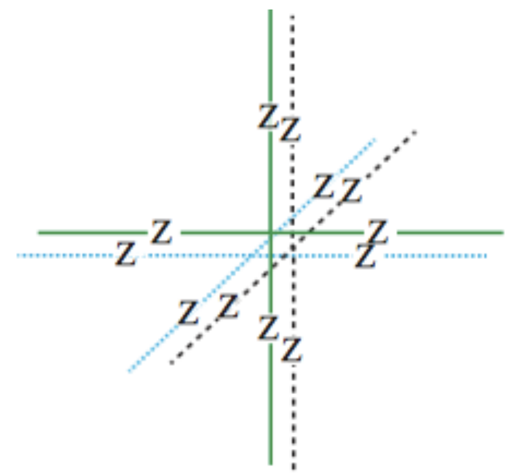
Type I models: hints at the underlying structure

- How to make X-cube from the Toric code:





$$B_c = \prod_{p \in \partial_c} B_p$$

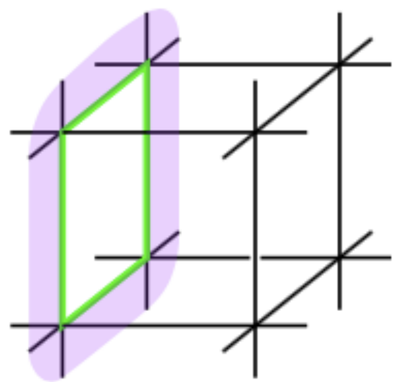
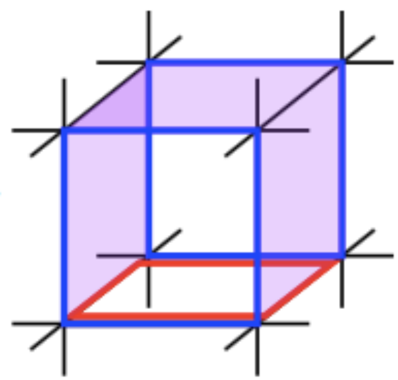
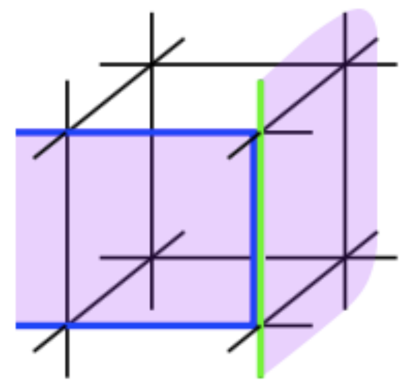


$$A_v$$

- If we can get rid of terms with a single X operator (make them cost a lot of energy), we can get from decoupled stacks to X-cube

Alternative picture of the constraint “no single X operators”:

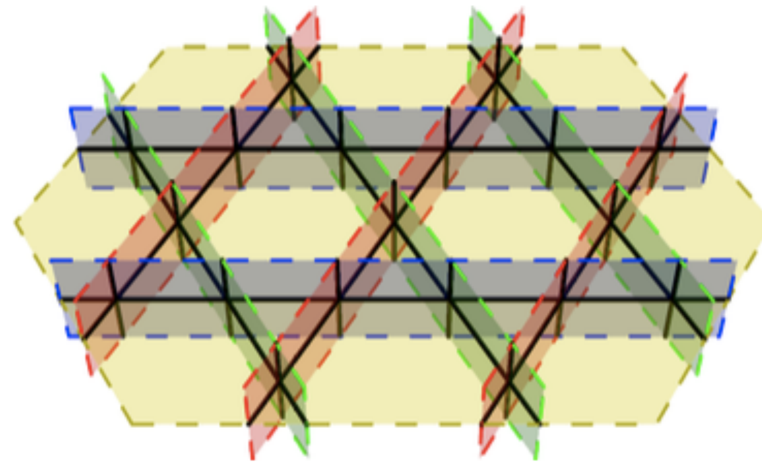
[Slagle, Aasen, Williamson]

$|\Psi\rangle = \cdots +$  $+$  $+$  $+ \cdots,$

- TQFT’s in different planes are “sewn together” by membranes
- Membranes are closed, or can end on edges with odd numbers of Z= -1

Type I models: hints at the underlying structure

- General construction:
  - Stacks of topological phases along various directions
  - Impose some conditions linking them together (energetically)
  - Obtain a Type-I fracton model



Ma, Hermele;
Slagle, Aasen, Williamson

Type I models: many things missing in the general picture

- How to do the coupling correctly for general topological order (UMTC)? This is important for understanding non-abelian examples.

- We know of examples [Devakul, Shirley, Wang] that cannot be obtained in this way. This suggests an underlying more general structure we don't yet understand.

- Do we have a complete list of possible statistical processes?

- Lattice geometry (choice of stacks) is very important. Are these models necessarily defined on discrete geometries? Why?

Other amusing directions …

- Topological order can come from gauging a symmetry. Different symmetry actions (3-cocycles) yield different answers. Can also make new topological orders from old ones by introducing a symmetry action and gauging it (if there are no obstructions).

- Fracton orders can come from gauging a subsystem symmetry. (We do not know whether examples not of this form exist). Different symmetry actions (3-cocycles) lead to different fracton orders. Can we similarly “extend” fracton orders through symmetry?

Fracton phases of matter: let's discuss!

- Fracton order: a new type of mathematical structure, with simple objects that have fusion properties and some statistical interactions (distinct from braiding)

- Some can be obtained from stacks of topologically ordered phases, by trivializing certain excitations

- Geometry plays an important role, key to restricted mobility and statistics