SARAH WITHERSPOON: HOPF ALGEBRAS, I

Our perspective on Hopf algebras, their actions on rings and modules, and the structures on their categories of rings and modules, will be to think of them as generalizations of group actions and representations; groups actions are symmetries in the usual sense, and Hopf algebra actions are often related to "quantum symmetries."

We're not going to give the full definition of a Hopf algebra, because it would require drawing a lot of commutative diagrams, but we'll say enough to give the picture.

Throughout this talk we work over a field k; all tensor products are of k-vector spaces.

Definition 0.1. A *Hopf algebra* is an algebra A together with k-linear maps $\Delta\colon A \to A \otimes A$, called *comultiplication*; $\varepsilon\colon A \to k$, called the *counit*; and $S\colon A \to A$, called the *coinverse*. These maps must satisfy some properties, including that ε is an algebra homomorphism and that S is an *anti-automorphism*, i.e. that $S(xy) = S(y)S(x)$.

The definition is best understood through examples.

Example 0.2.

(1) Let G be a group. Then the group algebra $k[G]$ is a Hopf algebra, where for all $g \in G$, $\Delta(g) \coloneqq g \otimes g$, $\varepsilon(g) \coloneqq 1$, and $S(g) \coloneqq g^{-1}$. This is a key example that allows us to generalize ideas from group actions to Hopf algebra actions: whenever we define a notion for Hopf algebras, when we implement it for $k[G]$ it should recover that notion for groups.

(2) Let $\mathfrak{g}$ be a Lie algebra over k. Then its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra, where for all $x \in \mathfrak{g}$, $\Delta(x) \coloneqq x \otimes 1 + 1 \otimes x$, $\varepsilon(x) \coloneqq 0$, and $S(x) \coloneqq -x$. Since ε is an algebra homomorphism, $\varepsilon(1_{\mathcal{U}(\mathfrak{g})}) = 1$.

For example,

$$\mathcal{U}(\mathfrak{sl}_2) = k\langle e, f, h \mid ef - fe = h, he - eh = 2e, hf - fh = -2f\rangle, \tag{0.3}$$

given explicitly by the basis of $\mathfrak{sl}_2$

$$e \coloneqq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f \coloneqq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h \coloneqq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{0.4}$$

◀

Both of these examples are classical, in that they've been known for a long time. But more recently, in the 1980s, people discovered new examples, coming from quantum groups.

Example 0.5 (Quantum $\mathfrak{sl}_2$)**.** Let $q \in k^\times \setminus \{\pm 1\}$. Then, given a simple Lie algebra $\mathfrak{g}$, we can define a "quantum group," $\mathcal{U}_q(\mathfrak{g})$, which is a Hopf algebra. For example, for $\mathfrak{sl}_2$,

$$\mathcal{U}_q(\mathfrak{sl}_2) = k\left\langle E, F, K^{\pm 1} \mid EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, KE = q^2 EK, KF = q^{-2}EK\right\rangle, \tag{0.6}$$

with comultiplication

$$\Delta(E) \coloneqq E \otimes 1 + K \otimes E \tag{0.7a}$$

$$\Delta(F) \coloneqq F \otimes K^{-1} + 1 \otimes F \tag{0.7b}$$

$$\Delta(K^{\pm 1}) \coloneqq K^{\pm 1} \otimes K^{\pm 1} \tag{0.7c}$$

and counit $\varepsilon(E) = \varepsilon(F) = 0$ and $\varepsilon(K) = 1$. This generalizes to other simple $\mathfrak{g}$, albeit with more elaborate data. ◀

Example 0.8 (Small quantum $\mathfrak{sl}_2$)**.** Let q be an n^{th} root of unity. Then, as before, given a simple Lie algebra $\mathfrak{g}$, we can define a Hopf algebra $u_q(\mathfrak{g})$, called the *small quantum group* for $\mathfrak{g}$ and q, which is a finite-dimensional vector space over k; for $\mathfrak{sl}_2$, this is

$$u_q(\mathfrak{sl}_2) = \mathcal{U}_q(\mathfrak{sl}_2)/(E^n, F^n, K^n - 1). \tag{0.9}$$

◀

Before we continue, we need some useful notation for comultiplication, called *Sweedler notation.* Let A be a Hopf algebra and $a \in A$; then we can symbolically write

$$\Delta(a) = \sum_{(a)} a_1 \otimes a_2. \tag{0.10}$$

Comultiplication in a Hopf algebra is *coassociative*, in that as maps $A \to A \otimes A \otimes A$,

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta. \tag{0.11}$$

Therefore when we iterate comultiplication, we can symbolically write

$$(\mathrm{id} \otimes \Delta) \circ \Delta(a) = \sum_{(a)} a_1 \otimes a_2 \otimes a_3 \tag{0.12}$$

without worrying about parentheses.

Actions on rings. Hopf algebra actions on rings generalize group actions on rings by automorphisms and actions of Lie algebras on rings by derivations. If a group G acts on a ring R, then for all $g \in G$ and $r, r' \in R$,

$$g(rr') = (gr)(gr') \tag{0.13a}$$

$$g(1_R) = 1_R. \tag{0.13b}$$

In $k[G]$, our Hopf algebra avatar of G, $\Delta(g) = g \otimes g$, and $\varepsilon(g) = 1$.

If a Lie algebra $\mathfrak{g}$ acts on a ring R by derivations, then for all $x \in \mathfrak{g}$ and $r, r' \in R$,

$$x \cdot (rr') = (x \cdot r)r' + r(x \cdot r') \tag{0.14a}$$

$$x \cdot (1_R) = 0. \tag{0.14b}$$

In $\mathcal{U}(\mathfrak{g})$, our Hopf algebra avatar of $\mathfrak{g}$, $\Delta(x) = x \otimes 1 + 1 \otimes x$, and $\varepsilon(x) = 0$. These two examples suggest how we should implement a general Hopf algebra action on a ring: comultiplication tells us how to act on the product of two elements, and the counit tells us how to act on 1.

Definition 0.15. Let A be a Hopf algebra and R be a k-algebra. An *A-module algebra structure* on R is data of an A-module structure on R such that for all $a \in A$ and $r, r' \in R$,

$$a \cdot (rr') = \sum_{(a)} (a_1 \cdot r)(a_2 \cdot r) \tag{0.16a}$$

$$a \cdot (1_R) = \varepsilon(a) 1_R. \tag{0.16b}$$

Thus a group action as in (0.13) defines an action of the Hopf algebra $k[G]$, and a Lie algebra action as in (0.14) defines an action of the Hopf algebra $\mathcal{U}(\mathfrak{g})$.

Example 0.17. The quantum analogue of the $\mathfrak{sl}_2$-action on $k[x, y]$, thought of as (functions on the) plane, there is an action of $\mathcal{U}_q(\mathfrak{sl}_2)$ on the *quantum plane*

$$R := k\langle x, y \mid xy = qyx\rangle. \tag{0.18}$$

This is a deformation of $k[x, y]$, which is the case $q = 1$. The explicit data of the action is

$$E \cdot x = 0 \qquad F \cdot x = y \qquad K^{\pm 1} \cdot x = q^{\pm 1} x \tag{0.19}$$

$$E \cdot y = x \qquad F \cdot y = 0 \qquad K^{\pm 1} y = q^{\mp 1} y. \tag{0.20}$$

One has to check that this extends to an action satisfying Definition 0.15, but it does, and R is an A-module algebra. Here E and F act as *skew-derivations*, e.g.

$$E \cdot (rr') = (E \cdot r)r' + (K \cdot r)(E \cdot r') \tag{0.21}$$

for all $r, r' \in R$. ◂

Given a Hopf algebra action of A on R in this sense, we can construct two useful rings: the *invariant subring*

$$R^A \coloneqq \{r \in R \mid a \cdot r = \varepsilon(a) \cdot r \text{ for all } a \in A\}, \tag{0.22}$$

and the *smash product ring* $R \# A$, which as a vector space is $R \otimes A$, with multiplication given by

$$(r \otimes a)(r' \otimes a') \coloneqq \sum_{(a)} r(a_1 \cdot r') \otimes a_2 a'. \tag{0.23}$$

The smash product ring knows the A-module algebra structure on R. Often, rings we're interested in for other reasons are smash product rings of interesting Hopf algebra actions, and identifying this structure is useful.

Example 0.24. The *Borel subalgebra* of $\mathcal{U}_q(\mathfrak{sl}_2)$ is $k\langle E, K^{\pm 1} \mid KE = q^{-2}K\rangle$. This is isomorphic to the smash product $k[E] \# k\langle K\rangle$, where $k\langle K\rangle$ is the group algebra of the free group on the single generator K.

In fact, there's a sense in which $\mathcal{U}_q(\mathfrak{sl}_2)$ is a deformation of $k[E, F] \# k\langle K\rangle$: in this smash product ring, E and F commute, and we deform this to $\mathcal{U}_q(\mathfrak{sl}_2)$, in which they don't commute. ◀

Modules. Given a Hopf algebra A, what is the structure of its category of modules? The first thing we can do is take the tensor product of A-modules U and V using comultiplication: for $a \in A$, $u \in U$, and $v \in V$,

$$a \cdot (u \otimes v) = \sum_{(a)} a_1 \cdot u \otimes a_2 \cdot v. \tag{0.25}$$

Moreover, k has a canonical A-module structure via the counit: $a \cdot x \coloneqq \varepsilon(a)x$ for $a \in A$ and $x \in k$. Finally, if U is an A-module, its vector space dual $U^* \coloneqq \mathrm{Hom}_k(U, k)$ has an A-module structure via S: for all $a \in A$, $u \in U$, and $f \in U^*$, $(a \cdot f)(u) \coloneqq f(S(a)u)$.

The existence of tensor products, duals, and the ground field in the world of Hopf algebra modules is a nice feature: these aren't always present for a general associative algebra. Moreover, these constructions interact well with each other.

(1) Coassociativity of Δ implies the tensor product is associative: for A-modules U, V, and W, we have a natural isomorphism $U \otimes (V \otimes W) \xrightarrow{\cong} (U \otimes V) \otimes W$.
(2) In any Hopf algebra A, we have the condition

$$\sum_{(a)} \varepsilon(a_1) a_2 = \sum_{(a)} a_1 \varepsilon(a_2) \tag{0.26}$$

for any $a_1, a_2 \in A$. This implies k, as an A-module, is the unit for the tensor product: we have natural isomorphisms $k \otimes U \cong U \cong U \otimes k$ for an A-module U.
(3) Suppose U is an A-module which is a finite-dimensional k-vector space. Then it comes with data of a *coevaluation map* $c\colon k \to U \otimes U^*$ sending

$$1 \longmapsto \sum_i u_i \otimes u_i^*, \tag{0.27}$$

where $\{u_i\}$ is a basis for U over k and $\{u_i^*\}$ is its dual basis; this map turns out to be independent of basis. We also have an *evaluation map* $e\colon U^* \otimes U \to k$ sending $f \otimes u \mapsto f(u)$. Now, not only are these A-module homomorphisms, but the composition

$$U \xrightarrow{c \otimes \mathrm{id}_U} U \otimes U^* \otimes U \xrightarrow{\mathrm{id}_U \otimes e} U \tag{0.28}$$

is the identity map.

Definition 0.29. A *tensor category*, or *monoidal category* is a category $\mathcal{C}$ together with a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $\mathbf{1} \in \mathcal{C}$ called the *unit*, and natural isomorphisms $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$ and $\mathbf{1} \otimes U \cong U \cong U \otimes \mathbf{1}$ for all objects U, V, and W in $\mathcal{C}$, subject to some coherence conditions.

Our key examples of tensor categories are the category of modules over a Hopf algebra A, as well as the subcategory of finite-dimensional modules.

If the coinverse of A is invertible, which is always the case when A is finite-dimensional over k, then $\mathcal{C} = \mathcal{M}od_A$ is a *rigid* tensor category, meaning that every object U has a *right dual* ${}^*U \coloneqq \mathrm{Hom}_k(U, k)$, which means the composition (0.28) is the identity.

Remark 0.30. Notations for left and right duals differ. We're following [EGNO15], but Bakalov-Kirillov [BK01] use a different convention; be careful! ◀

Some Hopf algebras' categories of modules have additional structure or properties: they might be semisimple, or braided, or even symmetric. This amounts to additional information on the Hopf algebra itself.

References

[BK01] Bojko Bakalov and Alexander Kirillov, Jr. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001. 4

[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015. 4

MSRI Introductory Workshop: Quantum Symmetries
Monday, Jan. 27, 2020, 9:30–10:30 am

Hopf Algebras I

Thank organizers, MSRI – great to see so many people here at the opening of the spring program

Some motivating remarks about symmetry groups & q. symm. Hopf algs (part of q. symm.)

Defn A Hopf algebra is an algebra A (over a field k — or other commut. ring) together with k-linear maps

$\otimes = \otimes_k$

$$\Delta : A \to A \otimes A, \qquad \varepsilon : A \to k, \qquad S : A \to A$$ satisfying some properties (that mimic properties of groups – can look them up)

motivated by many examples for which these properties have important implications for their actions

Examples (1) $A = kG$, group algebra, $\forall g \in G$:

$$\Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1, \qquad S(g) = g^{-1}$$

(2) $A = \mathcal{U}(\mathfrak{g})$, univ. env. alg. of Lie algebra $\mathfrak{g}$, $\forall x \in \mathfrak{g}$:

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad \varepsilon(x) = 0, \qquad S(x) = -x$$

e.g. $\mathcal{U}(\mathfrak{sl}_2) \cong k\langle e, f, h \mid ef - fe = h,\ he - eh = 2e,\ hf - fh = -2f \rangle$

$\mathfrak{sl}_2$: 2×2 matrices of det 1 $\quad e \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(3) $A = U_q(\mathfrak{g})$, quantum group $\quad q \in k^{\times}$, $q \neq \pm 1$

e.g. $U_q(\mathfrak{sl}_2) = k\langle E, F, K^{\pm 1} \mid EF - FE = \frac{K - K^{-1}}{q - q^{-1}},\ KE = q^2 EK,\ KF = q^{-2} FK \rangle$ $\quad KK^{-1} = 1 = K^{-1}K$

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}$$

(4) $A = u_q(\mathfrak{g})$, small quantum group, $q^n = 1$

" $U_q(\mathfrak{g}) / (E^n, F^n, K^n - 1)$ $\qquad$ (finite dim as v.s., generalizing fin. gp.)

Sweedler notation: $a \in A$ $\qquad$ (one of properties of Δ is coassociativity; quick eyeball example)

$$\Delta(a) = \sum_{(a)} a_1 \otimes a_2 \quad \text{or simply } a_1 \otimes a_2 \text{ (symbolically)}$$

$(\Delta \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \Delta)\Delta(a) = \sum a_1 \otimes a_2 \otimes a_3$

Actions on rings (see Susan Montgomery's book)

Groups act by automorphisms: $\overset{\text{group}}{G}$ acts on $\overset{\text{ring}}{R}$

$g\cdot(rr') = (g\cdot r)(g\cdot r'), \quad g\cdot 1_R = 1_R \quad \forall g\in G, r,r'\in R$

(symmetry groups) cf $\Delta(g) = g\otimes g \qquad \varepsilon(g)=1$

group gradings correspond to action of dual of group algebra

Lie algebras act by derivations: $\overset{\text{Lie alg}}{\mathfrak{g}}$ acts on $\overset{\text{ring}}{R}$

$x\cdot(rr') = (x\cdot r)r' + r(x\cdot r'), \quad x(1_R)=0 \quad \forall x\in\mathfrak{g}, r,r'\in R$

cf. $\Delta(x) = x\otimes 1 + 1\otimes x, \qquad \varepsilon(x)=0$

More generally:

Defn let A be a Hopf algebra and let R be a k-algebra.

R is an A-module algebra if

(i) R is an A-module

(ii) $a\cdot(rr') = \sum (a_1\cdot r)(a_2\cdot r') \qquad \forall a\in A, \; r,r'\in R$

(iii) $a\cdot(1_R) = \varepsilon(a)\cdot 1_R \qquad \forall a\in A$

Example $A = U_q(sl_2) \qquad R = k\langle x,y \mid xy - qyx\rangle$ (quantum plane)

$E\cdot x = 0 \qquad F\cdot x = y \qquad K\cdot x = qx$

$E\cdot y = x \qquad F\cdot y = 0 \qquad K\cdot y = q^{-1}y$ (can check relation ideal gen by $xy-qyx$ preserved)

(K acts as an automorphism and E, F act as skew derivations)

e.g. $E(rr') = (E\cdot r)r' + (K\cdot r)(E\cdot r') \qquad \forall r,r'\in R$

Discuss analogy between symmetry groups and actions of Hopf algebras $\rightsquigarrow$ quantum symmetry

alg geom and NC geom, actions on function algebras on (NC) spaces

<u>Modules</u>

Let A be a Hopf algebra, let U, V be (left) A-modules.

$U \otimes V$ is an A-module via Δ:

$$a \cdot (u \otimes v) := \sum_{(a)} (a_1 \cdot u) \otimes (a_2 \cdot v) \qquad \forall a \in A,\ u \in U,\ v \in V$$

k is an A-module via ε:

$$a \cdot c = \varepsilon(a) c \qquad \forall a \in A,\ c \in C$$

<u>left dual</u> (EGNO) $U^* = \mathrm{Hom}_k(U, k)$ is an A-module via S:
(k-lin. fcns from A to k)

$$(a \cdot f)(u) = f(S(a) \cdot u) \qquad \forall a \in A,\ u \in U,\ f \in \mathrm{Hom}_k(A, k)$$

<u>right dual</u> (EGNO) ${}^*U = \mathrm{Hom}_k(U, k)$ is an A-module via S^{-1} when S is invertible: (which it always is for f.d. Hopf algs)

$$(a \cdot f)(u) = f(S^{-1}(a) \cdot u)$$

<u>Properties</u> (i) $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$ (by coassociativity) $\forall$ A-mods U, V, W

(ii) $k \otimes U \cong U \cong k \otimes U$ $\forall$ A-mods U (by another property: $\sum \varepsilon(a_1) a_2 = a = \sum a_1 \varepsilon(a_2)$ $\forall a \in A$,

[§5 of 1st paper w/ Julia] (iii) and a property for U^* (rigidity) for U f.d.,

$$U \cong k \otimes U \xrightarrow{coev_U \otimes id_U} U \otimes U^* \otimes U \xrightarrow{id_U \otimes ev_U} U$$

Composition is is id. map on U

$$coev_U(1) = \sum_i u_i \otimes u_i^* \qquad ev_U(f \otimes u) = f(u)$$

$\leftarrow$ basis, dual basis

<u>Defn</u> A <u>tensor category</u> (or <u>monoidal category</u>) is a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $\mathbb{1}$, and natural isomorphisms

$$U \otimes (V \otimes W) \cong (U \otimes V) \otimes W, \qquad \mathbb{1} \otimes U \cong U \cong U \otimes \mathbb{1} \quad \text{for all objects } U, V, W$$

satisfying some properties ((triangle and pentagon axioms)).
(<u>rigid</u> if left and right duals exist)

(sometimes w/ additional properties such as a braiding)

<u>Example</u> The category of (f.d.) modules of a Hopf algebra A is a tensor category. (rigid when S invertible)

MSRI Intro. Workshop
Wed, Jan 29, 2020
9:30 - 10:30 am

Motivating Problems

Classify some types of tensor categories

Classify some types of Hopf algebras

Understand objects in tensor categories, modules of Hopf algebras

Hopf Algebras II

- We've been hearing a lot about (s.s.) tensor cats of various types in other talks such as Victor Ostrik + Eric Rowell
- I'm most interested in non-s.s. ones, particularly those that are mods of non-s.s. Hopf algebras such as small $q\mathfrak{g}$'s (we saw in V.O.'s lectures that one construct a s.s. cat fro, but -
- How to understand - may be wild type etc
- Connect to classification results of various types (cf other talks) - classify some types of tensor cats $\leftrightarrow$ classify mods of Hopf algs $\leftrightarrow$ classify Hopf algs

Hopf algebra cohomology

Defn An n-extension of A-modules U, V is a short exact sequence of A-module homs:

$$0 \to V \to M_n \to \cdots \to M_2 \to M_1 \to U \to 0$$

((i.e. A-mod homs, image of each map = kernel of next))

((non-s.s. case: an exact seq does not always split so you want to understand mods via how they're put together ... n-exts))

Defn Hopf algebra cohomology: $H^n(A,k) := \mathrm{Ext}_A^n(k,k) :=$ all n-extensions k, k modulo an equivalence relation (describe) - OTHER WAYS TO DEFINE IT!

Yoneda splice m-ext, n-ext $\rightsquigarrow$ $(m+n)$-ext

$$0 \to k \to M_m \to \cdots \to M_1 \to k \to 0, \quad 0 \to k \to N_n \to \cdots \to N_1 \to k \to 0$$

((can check exact))

Cohomology ring $H^*(A,k) := \bigoplus_{n \geq 0} H^n(A,k)$

under Yoneda splice, is a graded ring

(in fact, it is graded commutative by the Eckmann-Hilton argument)

OTHER WAYS TO DEFINE THIS PRODUCT!

More generally Let $\mathcal{C}$ be a tensor category

$H^*(\mathcal{C}, \mathbb{1})$ defined similarly, as a graded commutative ring.

Conjecture (Friedlander-Suslin '97, Etingof-Ostrik '04, ...)

Let A be a f.d. Hopf algebra. Then $H^*(A,k)$ is finitely generated.

More generally let $\mathcal{C}$ be a finite tensor category. Then $H^*(\mathcal{C}, \mathbb{1})$ is fin. gen.

and $\mathrm{Ext}_A^*(U,V)$ (resp. $\mathrm{Ext}_{\mathcal{C}}^*(X,Y)$) is always f.g. as mod over $H^*(A,k)$ (resp. $H^*(\mathcal{C}, \mathbb{1})$)

(I am not defining finite tensor cats)

(e.g. Bergh - Plavnik - W arXiv '19)

<u>Why is this important:</u>

There is a theory of "varieties for modules" that is most useful when Conjecture is true.

(can say more about later - but for now just take as motivation)

e.g. recent work w/ Petter Bergh, Julia Plavnik on supp vars for ftc's on arXiv and earlier work w/ Julia Plavnik

<u>Warning</u>: An analogous conjecture for Hoch. coh. of fin. dim. alg. is false.

<u>20th century results</u>: <u>Conjecture is true for</u>

(1) $A = kG$, G a finite group, char $k = p > 0$ (o.w. trivial by Maschke's Thm)

(Golod; Venkov; Evens ~1960)

Evens proof outline - reduce to G a p-group, $|G| = p^n$
- induction on n: $Z < Z(G)$, $|Z| = p$
LHS s.s. for $Z \to G \to G/Z$

(p. 132 Benson II)

(2) A - restricted env. alg., char $k = p > 0$ (f.d. quotient of $U(\mathfrak{g})$)

(Friedlander - Parshall 1983)

(see Oberwolfach talk notes 2019 for summary of proof)

(3) $A = u_q(\mathfrak{g})$, small quantum group (char $k = 0$)

(Ginzburg - Kumar 1993) (odd $n \geq$ Cox #) (restrictions on parameter)

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"Like all major historical events, I remember exactly where I was."

i.e. comultiplication same if order switched

(4) A - f.d. cocommut. Hopf alg., char $k = p > 0$ (includes (1) + (2), not (3))

(Friedlander - Suslin 1997)

- Why can't we just prove it in general? - well, back to Classification! We just don't yet know enough about Hopf algebras / tensor categories generally!

(11)

21st century results (see p. 4 of Oberwolfach notes)
on many many more types

- Gordon '00 (fin. quotients of gr. fcn algs at roots of 1) not div by 2, 3, 5, 7
- Mastnak - Pevtsova - Schauenburg - W '10 (ptd abelian gps, restriction on primes dividing order)
- Bendel - Nakano - Parshall - Pillen '14 (removed some restrictions on parameters in GK '93 (ℓ=4))
- Nguyen - W '14 (some skew gr. alg's in pos char)
- Drupieski '16 (fin. supergp schemes)
- Vay - Stefan '16 · Fomin Kirillov algebra
- Friedlander - Negron '18 – D.d.'s of cocommut Hopf algs
- Nguyen - Wang - W · dim p^3, more general results - 2 papers
- Erdmann - Solberg - Wang
- Negron - Plavnik – some results for f.t. C's - (duality and centers) (perhaps the 1st paper w/ results in this generality)

(2 papers with Van Nguyen Xingting Wang)

APOLOGIES IF I HAVE INADVERTENTLY MISSED SOMEONE

(Also work of Benson, Etingof, Ostrik constructing new nonss. f.t.C's in pos char, not on cohomology - but -)

– Lots ongoing projects –

ONE IS MY WORK w/ Nicolas Andruskiewitsch, Ivan Angiono, and Julia Pevtsova

(Nicolas + Julia will be here at MSRI for a couple of months each)

started many years ago but helped by more recent classification results of Nicolas + Ivan (see that part of p. 5) So Hopf subalg that is a group alg — all simple subcoalgebras are one-dimensional

proof conjecture for f.d. pointed Hopf algs w/ abel gp of gplike elts

Andruskiewitsch - Angiono - Pevtsova - W joint work IN PROGRESS

come back to

(Let A be a f.d. pointed Hopf algebra char 0 with abelian gp of gplike elts G)

G-fin gp.

A Yetter-Drinfeld kG-module is a kG-module V that is also a

(i.e. kG-comodule) or $(kG)^*$-module

G-graded module: $V = \bigoplus_{g \in G} V_g$ s.t. $h \cdot V_g \subset V_{hgh^{-1}}$

$T(V)$ is a braided Hopf algebra, i.e. a Hopf algebra in the category ${}^{kG}_{kG}YD$

Fact There is a largest ideal J of $T(V)$, $J \subset \bigoplus_{n \geq 2} T^n(V)$,

that is also a coideal, i.e. $\Delta(J) \subset J \otimes T(V) + T(V) \otimes J$

Nichols algebras are Hopf algebras in categories in YD

Defn Nichols algebra $T(V)/J$ (there are more general and completely different definitions)

Ex $u_q(sl_2)^+ = k\langle E \mid E^n = 0 \rangle$ is a Nichols algebra $(q^n = 1)$

$u_q(sl_2)^{\geq 0} = k\langle E, K \mid E^n = 0, K^n = 1, KE = q^2 EK \rangle \cong u_q(sl_2)^+ \# k\langle K \rangle$ is its bosonization

($u_q(sl_2)$ is quotient of Drinfeld double, deformed by a cocycle)

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NONABEL. GPS ARE THE WILD WEST (Usually not f.d.)

More generally: f.d. pointed Hopf algebras with abelian groups of gplike elts are cocycle deformations of bosonizations of Nichols algebras (of diagonal type))

So go backwards to get:

Outline of proof of conjecture for these ptd Hopf algs A f.d. (of diagonal type) (char 0)

(1) Prove f.g. of coh. of the Nichols algebra (Anick res, s.c., graded etc assoc)

(2) Bosonization

(3) cocycle deformations via Drinfeld double results

(take advantage of D.d. construction, recent work of others)

$$\left(A \text{ fgc} \Leftarrow D(A) \text{ fgc} \underset{\text{Angiono, Heckenberger}}{\Leftarrow} D(\operatorname{gr} A) \text{ fgc} \right)$$

$\operatorname{gr} A$, $(\operatorname{gr} A)^*$ fgc $\Downarrow$

Mention Dynkin diagrams etc classification in Nichols (of type)